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Note

A sharpening of Fisher's inequality

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Abstract

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It is proved that in every linear space on v points and b lines the number of intersecting line-pairs is at least $\binom{v}{2}$. This clearly implies $b \geq v$.

1. Definitions

A hypergraph $\boldsymbol{H}$ is a pair $(V, \mathscr{E})$, where V is a finite set, called vertices, and $\mathscr{E}$, the edges, is a family of non-empty subsets of V. It is called *linear* (or 0-1 intersecting) if $|E \cap E'| \leq 1$ holds for all pairs $\{E, E'\} \subset \mathscr{E}$. $\boldsymbol{H}$ is λ-intersecting if $|E \cap E'| = \lambda$ for all pairs. For a set $S \subset V$ let $\mathscr{E}[S]$ denote the family of edges containing S. The *degree* of the vertex x is $\deg(x) = |\mathscr{E}[\{x\}]|$. $\boldsymbol{H}$ is *k-uniform* if for every edge $E \in \mathscr{E}$, $|E| = k$. The *dual* of the hypergraph $\boldsymbol{H}$, $\boldsymbol{H}^*$, is obtained by interchanging the roles of vertices and edges keeping the incidences, i.e. $V(\boldsymbol{H}^*) = \mathscr{E}(\boldsymbol{H})$ and $\mathscr{E}(\boldsymbol{H}^*) = \{\mathscr{E}[x]: x \in V\}$.

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A *linear space* $\boldsymbol{L} = (P, \mathscr{L})$ is a linear hypergraph consisting of at least 2-element sets such that $|\mathscr{L}[x, y]| = 1$ hold for all pairs. In this case the vertices are called *points*, the edges are called *lines*. It is called *trivial* if $|\mathscr{L}| = 1$, i.e. $\mathscr{L} = \{P\}$. A *near pencil* is a linear space having a line with $|P| - 1$ points. A finite *projective plane* (of order q) is a linear space over $q^2 + q + 1$ points, the same number of lines, each line having $q + 1$ points.

2. Preliminaries, results

In 1948 de Bruijn and Erdős [4] proved that for every nontrivial finite linear space $\boldsymbol{L} = (P, \mathscr{L})$, one has

$$|\mathscr{L}| \geq |P|. \tag{2.1}$$

Moreover here equality holds if and only if $\boldsymbol{L}$ is either a finite projective plane or a near pencil. This result is called sometimes the non-uniform Fisher's inequality, as the proof of the uniform case is due to him [6]. (His inequality applies to general intersection size.) The dual of (2.1) says that if $(V, \mathscr{E})$ is a 1-intersecting family consisting of at least 2-element sets then

$$|\mathscr{E}| \leq |V|. \tag{2.2}$$

Because of its simplicity, the de Bruijn–Erdős theorem has plenty of applications. There is a growing number of different proofs, whose methods and applicability go far beyond the theory of designs and finite geometries. (We mention e.g. the books by Crawley and Dilworth [5], Lovász [11].) Varga [18] proved that for every line $L_m \in \mathscr{L}$ of maximal cardinality there are at least $|P| - 1$ lines intersecting it. Ryser [16] gave a complete characterization of 0-1-intersecting families, in which every set is intersected by all but one edge. Seymour [17] proved that every 0-1-intersecting family $(V, \mathscr{E})$ contains at least $|\mathscr{E}|/|V|$ pairwise disjoint members. (This generalization is related to the Erdős–Faber–Lovász conjecture, see [7].) A weighted version was proved by Kahn and Seymour [10]. Füredi and Seymour (see in [10]) proved that for an intersecting hypergraph $(V, \mathscr{E})$ one can find a pair $\{x, y\} \subset V$ such that $|\mathscr{E}[x, y]| \geq |\mathscr{E}|/|V|$. Another version of (2.1) and (2.2) became known as Motzkin's lemma [13].

The most interesting and fruitful proof was given by Majumdar [12] and Ryser [15]. Using linear algebra they proved (2.2) for λ-intersecting families. Their method was greatly generalized by Ray-Chaudhuri and Wilson [14], Frankl and Wilson [9]. For recent developments see Alon, Babai and Suzuki [1], Babai [2], Babai and Frankl [3], Wilson [19].

In this note another sharpening of (2.2) is proven.

Theorem. *Suppose that $E_1, \ldots, E_m$ is a 1-intersecting family (i.e. $|E_i \cap E_j| = 1$ for all $i \neq j$) of sets having at least 2 elements, moreover $\bigcap E_i = \emptyset$. Then the number of pairs covered by the E_i's is at least $\binom{m}{2}$.*

There is already an application of this theorem (see [8]).

3. Proof

We return to the original proof given in [4]. Let $V = \{x_1, x_2, \ldots, x_n\}$ denote the underlying set of the 1-intersecting family. Denote the cardinality of the edge E_i by e_i, and the degree of x_j by d_j. Without loss of generality we may suppose that

$$e_1 \geqslant \cdots \geqslant e_m, \tag{3.1}$$

and

$$d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n. \tag{3.2}$$

Obviously, we have

$$\sum e_i = \sum d_j. \tag{3.3}$$

We do know that $m \leqslant n$. The main point in the original proof is that for every i if $e_i > 0$, then

$$e_i \geqslant d_i. \tag{3.4}$$

holds. For the reader's convenience a proof of (3.4) is given in the Appendix.

Let $v(N, n)$ be the set of vectors $x = (x_1, \ldots, x_n)$ with nonnegative integer coordinates such that $\sum x_i = N$ and $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$. We say $\boldsymbol{y}$ *covers* $\boldsymbol{x}$ if there exist coordinates $1 \leqslant u < v \leqslant n$ such that

$$y_i = \begin{cases} x_i + 1 & \text{for } i = u, \\ x_i - 1 & \text{for } i = v, \\ x_i & \text{otherwise.} \end{cases}$$

Define the partial ordering of $v(N, n)$ as follows. $\boldsymbol{y} > \boldsymbol{x}$ if there exists a sequence $\boldsymbol{x} = \boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_s = \boldsymbol{y}$ such that $\boldsymbol{x}_{i+1}$ covers $\boldsymbol{x}_i$ $(i = 0, 1, \ldots, s-1)$. This is the usual notion of majorization in $v(N, n)$.

Define the function $f(x_1, \ldots, x_n) = \sum_i \binom{x_i}{2}$. Then the following is trivial.

Lemma 3.5. *If* $\boldsymbol{y} > \boldsymbol{x}$ *then* $f(\boldsymbol{y}) \geqslant f(\boldsymbol{x})$.

Proof. Lemma 3.5 holds for any convex function $g(x): \mathbb{R} \to \mathbb{R}$ whenever $f(x) = \sum_i g(x_i)$. $\square$

Proof of the Theorem. Let $N = \sum e_i = \sum d_j$. Then we have that

$$\boldsymbol{e} = (e_1, e_2, \ldots, e_m, 0, \ldots, 0) > \boldsymbol{d} = (d_1, d_2, \ldots, d_m, \ldots, d_n).$$

Then Lemma 3.5 implies that

$$\sum_i \binom{e_i}{2} \geqslant \sum_j \binom{d_j}{2}.$$

Here the left-hand side is the number of pairs covered by $\{E_1, \ldots, E_m\}$, and the right-hand side is the number of intersections, i.e. $\binom{m}{2}$. □

Appendix. Here we recall the proof of (3.4). For $x \notin E$ we have

$$\deg(x) \leq |E|. \tag{3.6}$$

Let E be any edge not containing $\{x_1, \ldots, x_i\}$. Then (3.6) gives that $|E| \geq d_i$. So (3.4) follows if we have at least i such edges. This settles the case $i = 1$. For $i > 1$ suppose that there are only at most $i - 1$ such edges. All the other edges contain $\{x_1, \ldots, x_i\}$, so we have $m = i$. Then $\min_{j \leq i} d_j = 1$, yielding $e_i \geq 2 > d_i = 1$.

4. Remarks, problems

Conjecture. Suppose that $\boldsymbol{H}$ is a (nontrivial) λ-intersecting family with m edges. Then the number of covered pairs is at least $\binom{m}{2}$.

Can we obtain in this way a purely combinatorial proof for the Majumdar–Ryser theorem? Can we have in this way a new approach to the λ-design conjecture? (See [15].) As a first step, is there a linear algebraic proof for the Theorem?

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