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The Densest Packing of Equal Circles into a Parallel Strip*

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Abstract. What is the densest packing of points in an infinite strip of width w, where any two of the points must be separated by distance at least 1? This question was raised by Fejes-Tóth a number of years ago. The answer is trivial for $w \le \sqrt{3}/2$ and, surprisingly, it is not difficult to prove [M2] for $w = n\sqrt{3}/2$, where n is a positive integer, that the regular triangular lattice gives the optimal packing. Kertész [K] solved the case $w < \sqrt{2}$. Here we fill the first gap, i.e., the maximal density is determined for $\sqrt{3}/2 < w \le \sqrt{3}$.

1. Preliminaries, Results

The open circular disc on the Euclidean plane with radius r and center p is denoted by $C(p, r)$. A family $\mathscr{C}$ of circular discs is called a *packing* if its members are pairwise disjoint. Instead of looking for the largest number of pairwise disjoint circles of diameter 1 contained in a (bounded) region A we can consider the following equivalent form: let R be the region consisting of points p of A with $C(p, \frac{1}{2}) \subset A$, and then we are looking for the largest number of points in R where any two of them are separated by distance at least 1. Such a set of points is called a *point-packing* (in R), and the size of the largest is denoted by $p(R)$. We use this latter terminology. The density, $d(P, R)$, of a point-packing P in R is defined as $|P|/\text{area } R$. Let $d(R) := p(R)/\text{area } R$. We use the fact that (see [F3])

$$d(\text{whole plane}) = \frac{2}{\sqrt{3}}. \tag{1.1}$$

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Formula (1.1) means that for every point-packing P on the plane and (bounded) convex region R we have $\lim_{\lambda\to\infty} d(P\cap\lambda R)\le 2/\sqrt{3}$. Here $2/\sqrt{3}=$ (area of the regular hexagon of width $1)^{-1}$, the density of the regular triangular lattice.

For $w, x\ge 0$ let $R_{w,x}$ denote a (closed) rectangle with sides w and x. For brevity $d(w,x)$ and $p(w,x)$ stand for $d(R_{w,x})$ and $p(R_{w,x})$, respectively. It is easy to see that the following limit exists:

$$\lim_{x\to\infty} d(w,x)=: d(w). \tag{1.2}$$

Indeed, $p(w,x)$ is a positive, monotone nondecreasing function of x and

$$p(w,x)+p(w,y)\ge p(w,x+y)$$

and

$$p(w,x)+p(w,y)\le p(w,x+y+1).$$

These properties imply (1.2) and, even more, the following bounds:

$$d(w)wx\le p(w,x)\le d(w)w(x+1). \tag{1.3}$$

Example 1.1. Let $n=\lfloor 2w/\sqrt{3}\rfloor$ and let T be a set of $\frac{1}{2}(n+1)(n+2)$ points of the triangle lattice packing of points onto a regular triangle of side n. Then let us place triangular blocks of points congruent to T touching alternately the two bordering lines of the strip (of width w), such that the distance between the closest points of two neighboring blocks is 1. (See Fig. 1.1. The segments indicate unit distances.)

This example is due to Molnár [M1]. Denote the density of Example 1.1 by $d_0(w)$, i.e.,

$$d_0(w)=\frac{(n+1)(n+2)}{w(n+\sqrt{4-(2w-n\sqrt{3})^2})}. \tag{1.4}$$

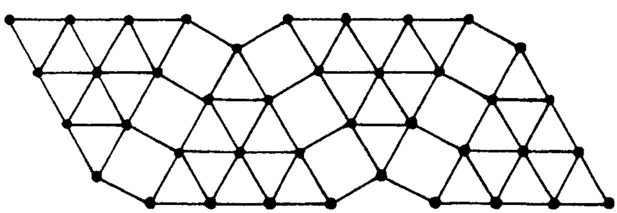


Fig. 1.1

If $2w/\sqrt{3}$ is an integer, then $d_0=(n+1)/w$, and Example 1.1 turns out to be a part of the triangular lattice. Clearly, $d_0(w)\le d(w)$ holds. Our aim is to prove

Theorem 1.2. *For $w\le\sqrt{3}$ we have $d_0(w)=d(w)$.*

This is an improvement of a result of Kertész [K]. The proof is postponed until Section 2. The case $w\le\sqrt{3}/2$ is trivial. Indeed, consider the strip

$$S(w)=:\{(x,y)\colon 0\le y\le w\},$$

and a point-packing P in $S(w)$. If $p_i=(x_i,y_i)\in P$, then $|x_i-x_j|\ge\sqrt{1-w^2}$, so $R_{w,x}$ contains at most $1+(x/\sqrt{1-w^2})$, points of P.

Conjecture 1.3 [M1]. *$d_0(w)=d(w)$ holds for all w.*

Molnár [M1] observed that the validity of his conjecture in the case $w=n\sqrt{3}/2$ (n is an integer) can be derived from the following theorem of Groemer [Gro]: let $\mathscr{C}$ be a circle-packing in the convex region R, then

$$|\mathscr{C}|\frac{\sqrt{3}}{2}\le\operatorname{area}(R)-\frac{2-\sqrt{3}}{4}\operatorname{per}(R)+\frac{\sqrt{12}-\pi(\sqrt{3}-1)}{4},$$

where per(R) denotes the length of the boundary of R.

Another proof can be obtained from the following inequality due to Folkman and Graham [FG]. Let K be a simplical complex in the plane with p vertices, and with Euler characteristic $\gamma(K)$. Suppose that the vertex set of K forms a point-packing. Then

$$p\le\frac{2}{\sqrt{3}}\operatorname{area}(K)+\tfrac{1}{2}\operatorname{per}(K)+\gamma(K).$$

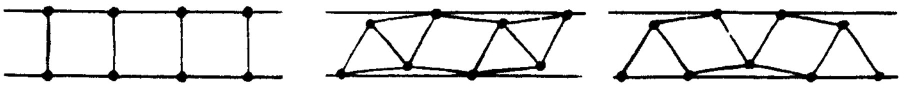
Here we give another proof which uses only (1.1).

Proposition 1.4. *For all $w>0$ we have*

$$d(w)\le\frac{2}{\sqrt{3}}+\frac{1}{w}.$$

Corollary 1.5 [M1]. *If n is a positive integer, $w=n\sqrt{3}/2$, then $d(w)=d_0(w)=(n+1)/w$.*

Proof of (1.4). Let P be a point-packing in $S(w)$. Reflect $S(w)$ (together with P) to the line $y=w+\sqrt{3}/4$, and translate it with the vector $(0,\sqrt{3}/2)$. We obtain another strip S' parallel to S and with distance $\sqrt{3}/2$. It is easy to see that $P\cup P'$ is a point-packing of $S(2w+\sqrt{3}/2)$. Continue this process, now with $S\cup S'$ instead of S. Finally we obtain a point-packing P^* in the upper half-plane. The density of P^* is at most $2/\sqrt{3}$ by (1.1) and, on the other hand, $d(P^*)=d(w)w/(w+\sqrt{3}/2)$. □



(a) (b) (c)

Fig. 1.2

Historical Remark. Originally Fejes-Tóth [F2] raised only the question of determining $d(1)$. (This is the so-called problem of parasites on dimension 2.) He had the wrong conjecture (see Fig. 1.2(b)). The right conjecture was proposed by Graham [G]. Kertész [K] determined the maximum packing density up to $w \leq \sqrt{2}$, but, unfortunately, he did not publish it.

There are two natural generalizations of this question in higher dimension. Packing spheres into a cylinder or into a d-dimensional strip, i.e., a *layer*. Both questions seem to be hopeless in general, but, surprisingly, for a small diameter the exact results were known earlier for $d \geq 3$ and then for $d = 2$. Horváth [H1] described the densest point-packing into a d-dimensional cylinder of diameter at most 1 for $d \geq 3$. The case $d = 3$ was a Schweitzer competition problem in 1966 proposed by Heppes and Fejes-Tóth [HF]. Molnár [M1] determined the maximum density in a three-dimensional layer of width at most $1/\sqrt{2}$. Horváth [H2] has some estimates for the case $d = 4$. More problems and further background can be found in the survey paper of G. Fejes-Tóth [F1], and in the problem book of Moser and Pach [MP].

2. Sketch of the Proof

We will prove four lemmas about the densest point-packings in $S(w)$. The first lemma says that in an extremal configuration most of the points must lie on the boundary of the strip. So we can describe the structure of those point sets. The second step is to partition the strip into rectangular regions (but not necessarily into rectangles), which contain at most six points. The third one is a technical lemma which enables us to eliminate almost all the tiring calculations when checking configurations having a few points. It says that an extremal configuration must be rigid. The last lemma says that the density of the packing in each rectangular region is at most $d_0(w)$.

Most of the standard arguments and calculations are omitted, the unproved propositions are indicated by a ■.

Instead of an infinite strip we deal with the shortest rectangle containing given number, say s, of points. Then let s tend to infinity. Define

$$l_w(s) =: \inf\{x: p(w, x) \geq s\}.$$

For example, $l_1(1) = l_1(2) = 0$, $l_1(3) = \sqrt{9}/2$, $l_1(4) = 1$, $l_1(5) = \sqrt{3}$, $l_1(6) = 2$. By (1.3) we have

$$d(w) \leq \frac{s}{w l_w(s)}. \tag{2.1}$$

We will prove that for $\sqrt{3}/2 < w \leq \sqrt{3}$ we have

$$l_w(s) \geq \frac{s}{6}(1 + \sqrt{4 - (2w - \sqrt{3})^2}) - O(1). \tag{2.2}$$

Then (2.1) and (2.2) imply the theorem.

As $l_w(s)$ (and so $d(w)$) are continuous functions of w we can assume that $w < \sqrt{3}$.

Pushing the Points to the Boundary

For a point p on the plane, $x(p)$ and $y(p)$ denote its coordinates. Let P, $|P| = s$, be a point-packing in the region $S(w, l_w(s))$, i.e., $0 \leq x(p) \leq l_w(s)$, and $0 \leq y(p) \leq w$. Choose P such that

$$\sum_{p \in P} \min\{y(p), w - y(p)\} \tag{2.3}$$

is minimal. (Such a P exists.)

Split S into two half-strips; the lower half-strip $S_l = \{(x, y) \in S: 0 \leq y \leq w/2\}$ and the upper half-strip $S_u = \{(x, y) \in S: w/2 < y \leq w\}$. Clearly, if $p, p' \in P$ belong to the same half-strip, then

$$|x(p) - x(p')| \geq \sqrt{1 - (w/2)^2} > \tfrac{1}{2}. \tag{2.4}$$

We call p and p' *consecutive* in S_l if the region $S_l \cap \{(x, y): x(p) < x < x(p')\}$ does not contain further elements of P (and, of course, $x(p) < x(p')$, $p, p' \in P$). We define consecutiveness analogously in S_u. A point $p = (x, y) \in P$ is an *inner* point if $0 < y < w$.

Lemma 2.1. *Two inner points of P in the same half-strip are never consecutive.*

Proof. We deal with only S_l. Assume that $a, b, c \in P \cap S_l$ are consecutive points. The following two propositions are easy.

Proposition 2.2. *Suppose that b lies below (or on) the line ac. Then b is on the boundary.* ■

Proposition 2.3. *Suppose b lies (strictly) above the line ac. Then b', the image of b after reflecting it to ac, is below the real axis.* ■

Proof of Lemma 2.1. Assume on the contrary that $b, c \in S_l \cap P$ are consecutive inner points, $y(b) \leq y(c)$. Denote by a the point which precedes b. (If such a point does not exist, then b is the first point, and it is easy to see that $y(b) = 0$, a contradiction.) If b is below or on the line ac, then we can apply Proposition

2.2. So we may assume that b is above the line ac. Then $y(a)<y(b)\le y(c)$. For the image of b, b', we have $y(b')\ge y(a)\ge 0$. This contradicts Proposition 2.3. □

The following fact is trivial.

Proposition 2.4. *Let a, b, c be three consecutive points, where b is an inner point. Then the length of ab or bc (or both) is* 1. ■

Decomposition of the Strip

First define S_l^0, the essential part of S_l, by $S_l^0 =: \{(x, y)\in S_l$: there exist $p_1, p_2 \in P$ on the real axis such that $x(p_1)\le x< x(p_2)\}$. That is, we do not consider the points of S_l at the very ends. By Lemma 2.1 we have

$$|P\cap S_l^0|\ge |P\cap S_l|-3. \tag{2.5}$$

Next, decompose S_l^0 into rectangles using the following cuts. If $p\in P$ lies on the boundary (i.e., on the real axis), then cut S_l^0 into two pieces $\{(x, y): x<x((p)\}$ and $\{(x, y): x\ge x(p)\}$. Continuing this process we arrive at a rectangle partition, $\mathcal{R}_l$, of S_l^0. Let $B(R) =: \{(x, 0): (x, y)\in R\}$ be the projection of the rectangle R. We call it the *base* of R. For all $R\in\mathcal{R}_l$ the base $B(R)$ is an interval $[l(R), r(R))$. Denote the lower left (right) corner of R by $p(R)$ ($p^+(R)$, resp.). For all R we have $p(R)\in R\cap P$, but note that $p^+(R)\notin R$ (see type II in Fig. 2.1). Each R has one of the following properties:

(i) The only point of P contained in R is the lower left corner $p(R)$.
(ii) $|P\cap R|=2$, and the second element of P, denoted by $v(P)$, is an inner point.

Analogously we can define a partition, $\mathcal{R}_u$, of the essential part of the upper half-strip, S_u^0. (The base of an $R\in\mathcal{R}_u$ is also a segment of the real axis, but $p(R)$ denotes its upper left corner.) Let $\mathcal{R}$ denote $\mathcal{R}_l\cup\mathcal{R}_u$. We call two rectangles in $\mathcal{R}$ *neighboring* if their union is connected. We say that $R\in\mathcal{R}$ *adjacent* to $R'\in\mathcal{R}$, or $R\to R'$ for short, if $|P\cap R|=2$ and $v(R)\in B(R')$. Note that this relation is not necessarily symmetric, it defines a directed graph, $\mathcal{G}$, with the vertex set $\mathcal{R}$. Moreover, if R is adjacent to R', then they are lying in different half-strips.

The aim of this section is to define the crucial notion of the proof, the *cell-decomposition* of $S^0 = S_l^0\cup S_u^0$. First we define the set of cells, $\mathcal{C}$. Every cell C is the union of at most three rectangles from $\mathcal{R}$. These cells cover S_0, but a point is not necessarily covered by only one cell. Later we see how to construct a $\mathcal{C}^0\subset\mathcal{C}$ which is a partition of S^0. This $\mathcal{C}^0$ is called the cell-decomposition.

Definition 2.5. There are six types of cells in $\mathcal{C}$. See Fig. 2.1.

Type I: $C=R$, where $R\in\mathcal{R}$ has property (i).

Type II: $C=R\cup R'$,where R' has property (i), and $R\to R'$.

Type III: $C=R_1\cup R_2\cup R$, where R has property (i) and $R_1\to R$, $R_2\to R$. Moreover, we will see in Proposition 2.7 that R_1 and R_2 are neighbors, and $B(R)\subset B(R_1)\cup B(R_2)$.

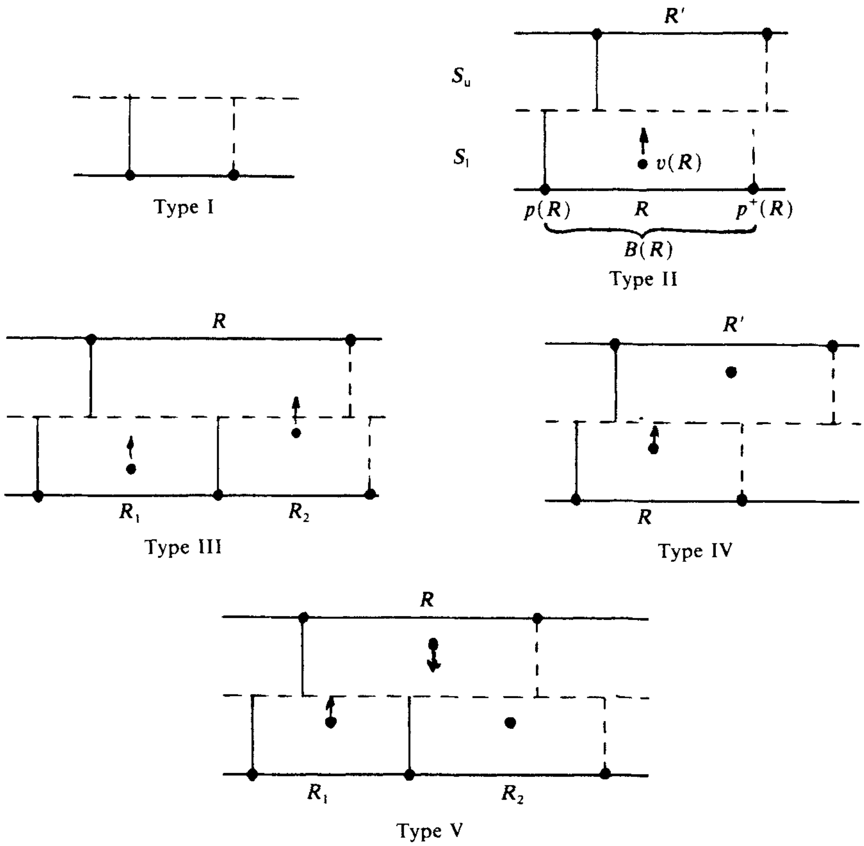


Fig. 2.1

Type IV: $C = R \cup R'$, both have property (ii) and $R \to R'$.
Type V: $C = R_1 \cup R_2 \cup R$, where all have property (ii) and $R_1 \to R$, $R \to R_2$. Moreover, we will see in Proposition 2.9 that R_1 and R_2 are neighbors, and $x(v(R))$ lies between $x(v(R_1))$ and $x(v(R_2))$.
Type VI: $C = R$, where R has property (ii), and lies at the end of its half-strip.

Lemma 2.6. *There exist a cell-decomposition $\mathscr{C}^0$.*

The above six types contain 1, 3, 5, 4, 6, or 2 points of P, resp. In the next section we see that in types I-V the density of P is at most $d_0(w)$, which together with Lemma 2.6 implies Theorem 1.2. The rest of this section is devoted to the proof of Lemma 2.6 via a series of propositions. Our main tool in the proofs about the structure of P is (2.3).

Proposition 2.7. *Suppose that the rectangles R_1 and R_2 have property* (ii) *and $x(v(R_1)) < x(v(R_2))$. Suppose further that P does not have any element p in the* *other half-strip with* $x(v(R_1)) < x(p) < x(v(R_2))$. *Then* R_1 *and* R_2 *are neighbors. Moreover, there exists an* R *having property* (i) *such that* R, R_1, *and* R_2 *form a cell of type* III. ■

Proposition 2.8. *Suppose that* $R_2 \to R$, *where* R *has property* (ii). *Then*

$$|x(R_2) - x(R)| < 1. \quad ■$$

Proposition 2.9. *Suppose that* R_1, R_2, *and* R *have property* (ii), *and* $R_1 \to R \to R_2$. *Then* R_1 *and* R_2 *are neighbors and* $x(v(R))$ *lies between* $x(v(R_1))$ *and* $x(v(R_2))$, *i.e.,* $R_1 \cup R_2 \cup R$ *form a cell of type* V. ■

Proposition 2.10. *Suppose that* $R \in \mathscr{R}$ *has property* (ii) *and* $R_1 \to R$, $R_2 \to R$. *Then* R_1 *and* R_2 *are neighbors and* $x(v(R))$ *lies between* $x(v(R_1))$ *and* $x(v(R_2))$, *i.e.,* $R_1 \cup R_2 \cup R$ *form a cell of type* V. ■

Consider now the undirected adjacency graph $\mathscr{G}^0$ over $\mathscr{R}$, which can be obtained by deleting the directions of the edges of $\mathscr{G}$, and identifying the multiple edges created.

Proposition 2.11. $\mathscr{G}^0$ *consists of a* (*vertex*) *disjoint union of paths and isolated vertices.* ■

Proof of Lemma 2.6. We define $\mathscr{C}^0$ using $\mathscr{G}^0$. If $R \in \mathscr{R}$ is an isolated vertex of $\mathscr{G}^0$, then it forms a one-element cell of type I or VI. If a path of $\mathscr{G}^0$ consists of rectangles of property (ii) only, then it can be easily decomposed into vertex disjoint subpaths of lengths 2 and 3, which correspond to cells of type IV and V, resp. Finally, if the path $R_1, \ldots, R_n$ in $\mathscr{G}^0$ contains a rectangle R_t having property (i), then it is the only member of this type. Moreover, $R_i \to R_{i+1}$ for $i < t$ and $R_{i+1} \to R_i$ for $i \geq t$. Then starting at any end we can chop a subpath of length 2 which does not contain R_t. This subpath corresponds to a cell in $\mathscr{C}^0$ of type IV. We keep doing this until the rest of the original path is either R_t alone or with one or two of its neighbors. Then the remainder of the path forms a cell of type I, II or III, resp. □

The Rigidity of the Extremum

Assume that, given a set of vectors in the plane, $\mathbf{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}$. Assume further that $\mathbf{P}$ satisfies (finitely many) linear inequalities

$$\sum_{j=1}^{n} (\mathbf{p}_j, \mathbf{a}_j^i) \geq b^i \qquad \text{for} \quad i \in I \tag{2.6}$$

and also satisfies some quadratic inequalities of the form

$$\|\mathbf{p}_i - \mathbf{p}_j\| \geq b_{ij} \tag{2.7}$$

for $1 \leq i, j \leq n$. Such a system is called a $\{\mathbf{a}_j', b', b_{ij}, l, n\}$-packing, or briefly an *L-packing*. We call the system of vector functions $\{\mathbf{p}_j(t)\}$, where $1 \leq j \leq n$, a *smooth motion* of $\mathbf{P}$ if:

(α) There exists an $\varepsilon > 0$ such that $\mathbf{p}_j(t): (-\varepsilon, +\varepsilon) \to \mathbf{R}^2$ is a continuous function, for all j.
(β) $\mathbf{p}_j(0) = \mathbf{p}_j$.
(γ) $\mathbf{p}_j(t)$ has a first derivative at $t = 0$, i.e., there exists a vector $\mathbf{v}_j$ such that $\mathbf{p}_j(t) = \mathbf{p}_j + t\mathbf{v}_j + o(t)$.
(δ) The system $\{\mathbf{p}_j(t): 1 \leq j \leq n\}$ is an L-packing for all $|t| < \varepsilon$.

Proposition 2.12. *Suppose that* $\{\mathbf{p}_j(t)\}$ *is a smooth motion of the L-packing* $\mathbf{P}$. *Then the linear system* $\{\mathbf{p}_j + t\mathbf{v}_j\}$ *is also a smooth motion.* ■

The set of all L-packings is denoted by Π. From now on we assume that the set Π is nonempty and compact (in $\mathbf{R}^{2n}$). An inequality of (2.6) or (2.7) is called $\mathbf{P}$-*exact* (or briefly *exact*), for some L-packing $\mathbf{P} \in \Pi$, if it holds with the equality for $\mathbf{P}$. A smooth motion $\mathbf{P}(t)$ is called a *proper motion* if it keeps all the exact inequalities of $\mathbf{P}$ of type (2.6), and there exists an exact inequality of type (2.7), for which $\mathbf{v}_i \neq \mathbf{v}_j$. For example, a translation of $\mathbf{P}$ is never a proper motion. A proper motion always rotates some of the exact segments $\mathbf{p}_i\mathbf{p}_j$. If $\mathbf{P}$ has no proper motion, then it is called *rigid*. Let $f(\mathbf{P})$ be a linear function, i.e., $f(\boldsymbol{P}) = \sum_j (u_j x(\mathbf{p}_j) + v_j y(\mathbf{p}_j))$ for some real constants u_j, v_j.

Lemma 2.13. *f takes its minimum over* Π *in a rigid* $\mathbf{P}^0$. ■

Each Cell has a Low Density

This heading is slightly misleading, since it is true only for types I–V. The *lower* (upper) base of a cell $C \in \mathscr{C}$ is the union of bases of the rectangular components of $C \cap S_\mathrm{l}$ ($C \cap S_\mathrm{u}$, resp.). It is denoted by $B_\mathrm{l}(C)$ ($B_\mathrm{u}(C)$ resp.). If C is of type I or VI then one of its bases is empty. We have

$$\text{area } C = \frac{w}{2}(|B_\mathrm{l}(C)| + |B_\mathrm{u}(C)|).$$

We prove that the density of a cell is at most $d_0(w)$ in the following form.

Lemma 2.14. *Let C be a cell of type* I, II, III, IV *or* V. *Then*

$$|B_\mathrm{l}(C)| + |B_\mathrm{u}(C)| \geq |P \cap C| \tfrac{1}{3}(1 + \sqrt{4 - (2w - \sqrt{3})^2}). \tag{2.8}$$

Proof. Denote the coefficient of $|P \cap C|$ in inequality (2.8) by $\rho(w)$, or briefly by ρ. In the proof we have to check separately all five cases.

Case I. $C = R$, where R has type I, then $|B(R)| \geq 1$. For $(2/\sqrt{3}) < w < \sqrt{3}$ we have

$$1 > \rho(w) > 2/3. \tag{2.9}$$

In the next four cases our main tool is Lemma 2.13, because the problem of minimizing $|B_l(C)| + |B_u(C)|$ for all C with a given type can be reformulated as the problem of minimizing a linear function over the set of all L-packings for some special L.

Case II. $C = R \cup R'$, where $R \to R'$. Let $\mathbf{p}_1 = p(R)$, $\mathbf{p}_2 = p(R')$, $\mathbf{p}_3 = v(R)$, $\mathbf{p}_4 = p^+(R')$, $\mathbf{p}_5 = p^+(R)$, and $\mathbf{p}_i = (x_i, y_i)$. Then we have

$$\begin{gathered} \|\mathbf{p}_i - \mathbf{p}_j\| \geq 1 \qquad \text{for all} \quad i \neq j, \\ y_1 = y_5 = 0, \qquad 0 \leq y_3 \leq w/2, \qquad x_1 \leq x_3 \leq x_5, \\ y_2 = y_4 = w, \qquad x_2 \leq x_4, \\ x_2 \leq x_3 \leq x_4, \\ x_3 = 0, \qquad |x_i| \leq 3 \qquad \text{for all } i. \end{gathered} \tag{2.10}$$

The first line of these inequalities says that $\{\mathbf{p}_1, \ldots, \mathbf{p}_5\}$ is a point-packing, the second and third describe the structures of R and R' (resp.), the fourth says that $R \to R'$, and the last one makes the domain compact. It is only a technical constraint because we would like to minimize the linear function

$$x_4 - x_2 + x_5 - x_1. \tag{2.11}$$

We claim that the minimum value of the function in (2.11) over the constraints of (2.10) is at least 3ρ, which finishes Case II. Before we proceed with the proof of this claim, note that instead of (2.10) we can state somewhat stronger constraints (e.g., we know that strict inequalities hold in the second and the third rows) but the above inequalities obviously describe the structure of the cell C, and we wanted to keep the compactness of the domain of the feasible solutions in order to make use of the possible application of Lemma 2.13. Further, to reduce the number of local minimums we can weaken the first line of (2.10) as follows:

$$\|\mathbf{p}_i - \mathbf{p}_j\| \geq 1 \qquad \text{for} \quad (i, j) \in \{(1,3), (3,5), (2,3), (3,4), (2,4)\}. \tag{2.10a}$$

As $x_4 - x_2 \geq 1$, we may add to the constraints of (2.10) that

$$x_5 - x_1 \leq 3\rho - 1 < 2. \tag{2.12}$$

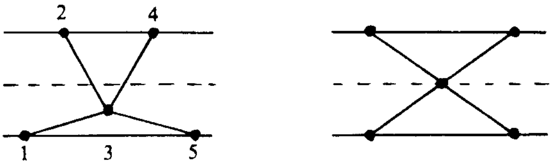


Fig. 2.2

Lemma 2.13 implies that there exists a rigid optimum of system (2.10)-(2.12). If $y_3 = w/2$, then $|y_i - y_3| \geq \sqrt{1-(w/2)^2}$ for all $i \neq 3$. Hence

$$x_4 - x_2 + x_5 - x_1 \geq 4\sqrt{1-(w/2)^2} > 3\rho(w). \tag{2.13}$$

So we may assume that $y_3 < w/2$. For the point $\mathbf{p}_1 = (x_1, 0)$ the only constraint is the following: $\|\mathbf{p}_1 - \mathbf{p}_3\| \geq 1$ and $x_1 \leq 0$. Therefore in the optimal solution we have that $\|\mathbf{p}_1 - \mathbf{p}_3\| = 1$. Similarly, $\|\mathbf{p}_5 - \mathbf{p}_3\| = 1$. Then the only rigid configuration is obtained with $y_3 = w - \sqrt{3}/2$ (see Fig. 2.2). For this configuration the value of (2.11) is just 3ρ.

The rest of the cases require more calculations but the arguments are essentially the same. □

Proof of the Theorem. By (2.5) all but at most six points of P are contained in a cell of $\mathscr{C}^0$. Moreover, by Lemma 2.6 all but at most four cells of $\mathscr{C}^0$ have types I–V. Then Lemma 2.14 implies that

$$\text{area}\left(\bigcup C^0\right) \geq \tfrac{1}{2} w\rho(w)(|P| - 14).$$

This implies (2.2). □

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