Journal of Geometry 0047-2468/93/020055-11$1.50+0.20/0
Vol. 46 (1993)

THE SECOND AND THE THIRD SMALLEST DISTANCES ON THE SPHERE

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Let $s_1(n)$ denote the largest possible minimal distance among n distinct points on the unit sphere $\mathbb{S}^2$. In general, let $s_k(n)$ denote the supremum of the k-th minimal distance. In this paper we prove and disprove the following conjecture of A. Bezdek and K. Bezdek: $s_2(n) = s_1(\lceil n/3 \rceil)$. This equality holds for $n > n_0$ however $s_2(12) > s_1(4)$.

We set up a conjecture for $s_k(n)$, that one can always reduce the problem of the k-th minimum distance to the function s_1. We prove this conjecture in the case $k = 3$ as well, obtaining that $s_3(n) = s_1(\lceil n/5 \rceil)$ for sufficiently large n.

The optimal construction for the largest second distance is obtained from a point set of size $\lceil n/3 \rceil$ with the largest possible minimal distance by replacing each point by three vertices of an equilateral triangle of the same size ε. If $\varepsilon \to 0$, then s_2 tends to $s_1(\lceil n/3 \rceil)$. In the case of the third minimal distance, we start with a point set of size $\lceil n/5 \rceil$ and replace each point by a regular pentagon.

1. INTRODUCTION, RESULTS

Let $\mathcal{P}$ be a finite point set on the 2-dimensional unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. The spherical distance between the points $x, y \in \mathbb{S}^2$ is denoted by $d(x,y)$. Consider the set of distances between the points of $\mathcal{P}$, $D(\mathcal{P}) = \{d(x,y) : x, y \in \mathcal{P}, x \neq y\}$. Order the elements in $D(\mathcal{P}) = \{d_1, \ldots, d_t\}$ such that $d_1 < d_2 < \cdots < d_t$. Then d_t is the *diameter* of $\mathcal{P}$ and the k-th smallest distance, d_k is denoted by $s_k(\mathcal{P})$. (If $t > k$, define $s_k(\mathcal{P}) = \infty$.) So $s_1(\mathcal{P})$ is the minimum distance.

Let $s_k(n)$ denote the supremum of the k-th smallest distance in n-point sets on the sphere, $s_k(n) = \sup\{s_k(\mathcal{P}) : |\mathcal{P}| = n, \ \mathcal{P} \subset \mathbb{S}^2\}$. The problem of finding $s_1(n)$ was raised by

Tammes [11] in 1930. The exact value of $s_1(n)$ and the extremal arrangements are known for a couple of small values of n. (The cases $n = 5, 7, 8$ by Schütte and Van der Waerden [10], n=10, 11 by Danzer [5,6], $n = 11$ by Böröczky [3], $n = 24$ by Robinson [9] and $n = 3, 4, 6, 12$ by L. Fejes Tóth [8].) Here we will use the asymptotic result

$$s_1(n) = (1 + o(1))\sqrt{\frac{8\pi}{\sqrt{3}\,n}}\,. \tag{1.1}$$

This formula means that $\lim_{n\to\infty} ns_1(n)^2 = 8\pi/\sqrt{3}$. L. Fejes Tóth proved the following

$$s_1(n) \le \arccos\frac{1}{2}\left(\cot^2\left(\frac{n}{n-2}\frac{\pi}{6}\right) - 1\right) \tag{1.2}$$

which yields the upper bound in (1.1). The lower bound can be obtained from a hexagonal like packing of circles.

The problem of $s_2(n)$ was proposed by A. Bezdek and K. Bezdek [1]. They showed that

$$s_2(n) \ge s_1(\lceil n/3 \rceil). \tag{1.3}$$

The construction giving (1.3) is obtained from an s_1-extremal arrangement $\mathcal{P}'$ with $|\mathcal{P}'| = \lceil n/3 \rceil$, i.e., $s_1(\mathcal{P}') = s_1(|\mathcal{P}'|)$. Then, replace each point $p \in \mathcal{P}'$ by an equilateral triangle of side length ε and with a vertex in p. Finally, let ε tend to 0.

In [1] an upper bound (twice the right hand side of (1.2)) was proved. Obviously, $s_2(1) = \ldots = s_2(4) = \infty$, and $s_2(5) = s_2(6) = \pi = 180°$, because it cannot be larger. A. Bezdek and K. Bezdek proved that $s_2(9) = s_1(3) = 2\pi/3$, and they asked whether equality holds in (1.3) for all $n > 4$. Here we determine $s_2(n)$ for all $n \le 12$ showing that $s_2(12) = 116.56\ldots° > s_1(4) = 109.47\ldots°$. However, the conjecture is true for large n.

THEOREM 1.1. (1) $s_2(7) = s_2(8) = s_2(9) = 120° = s_1(4)$.
(2) *Let $u = \tan^{-1} 2 = 63.43\ldots°$ be the minimum distance in the vertex set of a regular inscribed icosahedron. If $\mathcal{P} \subset \mathbb{S}^2$ is a point set of size 10, 11, or 12, then $s_2(\mathcal{P}) \ge \pi - u$. Equality holds if and only if $\mathcal{P}$ is a subset of the vertex set of the icosahedron.*
(3) $s_2(17) = s_2(18) = 90° = s_1(6)$.

THEOREM 1.2. *For $n > n_0$ one has $s_2(n) = s_1(\lceil n/3 \rceil)$.*

CONJECTURE 1.3. *Equality holds in (1.3) for all $n > 12$.*

Let $f(k)$ denote the largest integer f such that for all $\varepsilon > 0$ there exists a k-distance set in $\mathbb{S}^2$ of size f and of diameter less than ε. We have $f(0) = 1$, $f(1) = 3$, $f(2) = 5$, $f(3) = 7$. For large k the best known upper bound is $O(k^{5/4})$ due to Chung, Szemerédi and

Trotter [4], and it is a challenging problem to decide whether $f(k) = O(k)$, or not. Erdős, Hickerson and Pach [7] proved some results which give support to the conjecture that $\lim f(k)/k = \infty$. Replacing the points of an s_1-extremal set on the sphere by congruent small copies of a $(k-1)$-distance set we obtain $s_k(n) \geq s_1(\lceil n/f(k-1)\rceil)$.

CONJECTURE 1.4. *For $n > n_0(k)$ one has $s_k(n) = s(\lceil n/f(k-1)\rceil)$.*

THEOREM 1.5. *For $n > n_1$ one has $s_3(n) = s_1(\lceil n/5\rceil)$.*

In general we can only prove a weaker upper bound.

THEOREM 1.6. *For $n > n_0(k)$ one has $s_k(n) \leq s_1(\lceil n/6f(k-1)\rceil)$.*

This result can be easily extended in higher dimensions, though obtaining an exact formula for $s_k^d(n)$ looks to be very difficult. The one dimensional case is easy, one has $s_k^1(n) = 2\pi k/n$ for $n \geq 2k$. The only extremal configuration is the regular n-gon.

2. A LEMMA ON THE RATIO OF s_1 AND s_2

Let $\Delta \geq 0$ be an integer, $0 < s < \pi/2$. Define the regular Δ-gon (on the unit sphere $\mathbb{S}^2$) with center c and inscribed radius s as follows:

for $\Delta = 0$ the whole sphere,
for $\Delta = 1$ halfsphere including c such that the distance from c to the boundary is s,
for $\Delta = 2$ a digon with center c whose distance from the sides is s,
for $\Delta \geq 3$ as usual.

We can extend these definitions to the Euclidean plane, in the cases $\Delta = 0, 1, 2$ the regular Δ- gon is the whole plane, a halfplane or an infinite strip of width $2s$. Define the function $A(\Delta, D, s)$ as the area of the intersection of a regular Δ-gon with inscribed radius $s/2$ and a circle of diameter D with the same center. The same function on the plane is denoted by $A_\infty(\Delta, D, s)$. Clearly, $A_\infty(\Delta, D, s) = s^2 A_\infty(\Delta, D/s, 1)$. If Δ and D/s are given, then

$$\lim_{s\to 0} \frac{A(\Delta, D, s)}{s^2} = A_\infty(\Delta, \frac{D}{s}, 1)$$

For brevity we use $A(\Delta, x)$ for $A_\infty(\Delta, x, 1)$. E.g., $A(0, x) = x^2\pi/4$, $A(4, \infty) = 1$, $A(6, \infty) = \sqrt{3}/2$.

Let $\mathcal{P}$ be an n-element set on $\mathbb{S}^2$, $s_i = s_i(\mathcal{P})$, $(n > 4)$. Define the minimum distance graph $\mathcal{G} = \mathcal{G}(\mathcal{P})$ with vertex set $\mathcal{P}$ as follows: two points are connected if their distance is s_1. Obviously, every point has at most 5 neighbors, so for the maximum degree, $\Delta(\mathcal{G})$, of $\mathcal{G}$ we have $\Delta(\mathcal{G}) \leq 5$.

LEMMA 2.1. *Let $0.1 > \varepsilon > 0$ and suppose that $n > n_0(\varepsilon), s_2 < \varepsilon$. Then*

$$s_1(\mathcal{P}) < s_1(n)(1+\varepsilon)\sqrt{\frac{\sqrt{3}/2}{A(\Delta,\ s_2/s_1)}}\ .$$

Proof. By (1.1) for every $\varepsilon > 0$ there exists an $n_0(\varepsilon)$ such that

$$(1+\varepsilon)s_1^2(n)\frac{\sqrt{3}}{2}\,n\ >\ \text{Area}\ \mathbb{S}^2 = 4\pi. \tag{2.1}$$

On the other hand for every $p \in \mathcal{P}$ define its Dirichlet cell, $C(p) = \{q \in \mathbb{S}^2 : d(p,q) = d(\mathcal{P},q)\}$. Let $p_1,\dots,p_t$ be the neighbors of p in $\mathcal{G}$, and let H_i be the half sphere containing p which perpendicularly bisects pp_i. Then $C(p)$ contains the intersection of H_i's and a spherical circle of radius $s_2/2$ around p. Hence Area $C(p) \geq A(\Delta, s_2, s_1)$. Obviously, $A(\Delta, s_2, s_1) > A(\Delta, s_2/s_1)s_1^2/(1+\varepsilon)$. Then

$$4\pi \geq \sum \text{ Area } C(p) > n\ A(\Delta, s_2/s_1)s_1^2/(1+\varepsilon). \tag{2.2}$$

Finally, (2.1) and (2.2) imply the Lemma. □

3. A GENERAL UPPER BOUND

Here we prove Theorem 1.6. Let $\mathcal{P}$ be a finite point-set on $\mathbb{S}^2$. Consider the minimum $(k-1)$-distance graph $\mathcal{G}^{k-1} = \mathcal{G}(\mathcal{P})$, two points x, y in $\mathcal{P}$ are connected if $d(x,y) \leq s_{k-1}(\mathcal{P})$. Let $f(k-1,\varepsilon)$ denote the maximum size of a $(k-1)$-distance set of diameter at most ε.

PROPOSITION 3.1. *Every point in $\mathcal{G}^{k-1}$ is connected by less than $6f(k-1, s_{k-1}) - 6$ other points.*

Proof. Let $p \in \mathcal{P}$, and consider a closed circle C with radius s_{k-1} and center p. Divide C into 6 congruent pieces with 3 diagonals through p, any two of them have an angle $\pi/3$. Then the diameters of each piece is s_{k-1}, so it contains at most $f(k-1, s_{k-1})$ elements of $\mathcal{P}$. □

Proof of Theorem 1.6. There exists an $\varepsilon > 0$ such that $f(k-1,\varepsilon) = f(k-1)$. We have an $n_0(k)$ such that for $n > n_0(k)$ $s_{k-1}(n) < \varepsilon$ holds. Then $\mathcal{G}^{k-1}$ does not contain a complete subgraph of $6f(k-1)-5$ vertices $(k \geq 2)$, but every degree is not larger than $6f(k-1)-6$. One can use Brook's theorem (see, e.g., in Bollobás' book [2]), that the chromatic numbers of $\mathcal{G}^{k-1}$ is at most $6f-6$. So there exists a $\mathcal{P}' \subset \mathcal{P}$ with $|\mathcal{P}'| \geq |\mathcal{P}|/(6f-6)$ such that $s_1(\mathcal{P}') \geq s_k(\mathcal{P})$. □

4. THE SECOND SMALLEST DISTANCE

Here we prove Theorem 1.2. By (1.1) we have an n_2 such that for all $n > n_2$

$$\frac{s_1(\lceil n/3 \rceil)}{s_1(n)} > 1.71 \tag{4.1}$$

holds. Suppose that $\mathcal{P}$ is an arbitrary n-set on the sphere with $n > n_2$. To prove the theorem we have to show that $s_2(\mathcal{P}) < s_1(\lceil n/3 \rceil)$. We may suppose that

$$s_2(\mathcal{P}) > 1.71 s_1(n), \tag{4.2}$$

otherwise (4.1) implies the Theorem.

As $s_1(n) \to 0$ if $n \to \infty$ we have an n_3 such that $s_1(\lceil n/4 \rceil) < 0.01$ holds for all $n > n_3$. Then by Theorem 1.6 we have $s_2(\mathcal{P}) < 0.01$. So we may apply Lemma 2.1 to $\mathcal{P}$ with $n > \max\{n_2, n_3\}$, $\Delta = 5$ and $\varepsilon = 0.01$. We have $A(5, 1.71) = A(5, \infty) = (5/4)\tan 36^\circ \sim 0.908\ldots$ so by Lemma 2.1 $s_1(\mathcal{P}) < s_1(n) \cdot 0.986\ldots$. This inequality and (4.2) imply that

$$s_2(\mathcal{P})/s_1(\mathcal{P}) > 1.733\ldots > \sqrt{3}.$$

CLAIM 4.1. $\Delta(\mathcal{G}) \le 3$.

Proof. Suppose on the contrary that $p \in \mathcal{P}$, $q_1, \ldots, q_4 \in \mathcal{P}$ with $d(p, q_i) = s_1$. If the distances $d(q_i, q_j)$ are all at least $\sqrt{3}\ s_1$ then each angle $q_i p q_{i+1}$ is at least 120°, a contradiction. So we have, say, $d(q_1, q_2) = s_1$. If $d(q_i, q_1)$ $(i = 3, 4)$ is less than $\sqrt{3}\ s_1$ then it is also s_1, but then $s_1 < d(q_2, q_i) < \sqrt{3}\ s_1 < s_2$, a contradiction. Hence $d(q_i, q_j) \ge \sqrt{3}\ s_1$ for $i = 1, 2$, $j = 3, 4$. Then we obtain the contradiction $d(q_3, q_4) < s_1$. □

As $\mathcal{G}$ does not contain a complete graph of four vertices, Brook's theorem implies that its chromatic number is at most 3. So there exists a $\mathcal{P}' \subset \mathcal{P}$, $|\mathcal{P}'| \ge n/3$, such that $s_1(\mathcal{P}') > s_1(\mathcal{P})$. Then we have $s_1(\lceil n/3 \rceil) \ge s_1(\mathcal{P}') \ge s_2(\mathcal{P})$, and the proof of 1.2 is complete. □

5. THE THIRD SMALLEST DISTANCE

Here we prove Theorem 1.5. We are going to use the method of the proof of Theorem 1.2 but we have to investigate more subcases. We will use the following simple facts on 2-distance sets $\mathcal{R}$ on the sphere. Suppose that the distances are $u < v < 0.001$.

FACT 5.1. $|\mathcal{R}| \le 5$. *In case of equality $\mathcal{R}$ is a regular pentagon and*

$$1.6180\ldots = 2\sin 54^\circ < \frac{v}{u} < 1.62. \quad \square \tag{5.1}$$

FACT 5.2. *If $|\mathcal{R}| = 4$, then one of the following six cases holds:* (See Fig. 1)

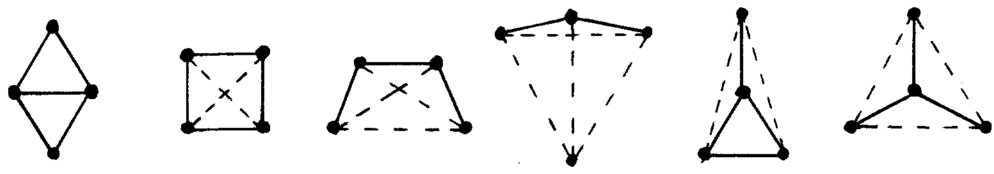

Figure 1

(i) $\mathcal{R}$ consists of 2 equilateral triangles of side length u, with a common side. Then $(1 + \cos u)(1 + \cos v) = 4\cos^2 u$, hence

$$1.732\ldots = \sqrt{3} < \frac{v}{u} < 1.733.$$

(ii) $\mathcal{R}$ is a regular quadrilateral. Then $1.414\ldots = \sqrt{2} < v/u < 1.415$.

(iii) $\mathcal{R}$ consists of four vertices of a regular pentagon. Then (5.1) holds.

(iv) $\mathcal{R}$ is a convex quadrilateral with diagonals of length v, and sides u, u, v, v. Then $\cos u = \cos^2 v + \sin^2 v\sqrt{(1 + 2\cos v)(2 + 2\cos v)}$, hence $(v/u) \sim \sqrt{2 + \sqrt{3}}$, i.e.,

$$1.931 < \frac{v}{u} < 1.932 \tag{5.2}$$

(v) $\mathcal{R}$ is a triangle of side lengths v, v, and u and its center with circumscribed radius u. Then $\cos v = \cos^2 u - \sin^2 u\sqrt{(1 + 2\cos u)(2 + 2\cos u)}$, hence $v/u \sim \sqrt{2 + \sqrt{3}}$, i.e., (5.2) holds.

(vi) $\mathcal{R}$ is an equilateral triangle of side length v and its center. Then $\cos v = 1 - 1.5\sin^2 v$, hence

$$1.732 < \frac{v}{u} < \sqrt{3} = 1.732\ldots \quad \square$$

By (1.1) the limit of $s_1(\lceil n/5 \rceil)/s_1(n)$ is $\sqrt{5} = 2.236\ldots$, so there exists an n_4 such that for all $n > n_4$

$$\frac{s_1(\lceil n/5 \rceil)}{s_1(n)} > 2.236 \tag{5.3}$$

holds. Suppose that $\mathcal{P}$ is an arbitrary n-set on the sphere with $n > n_4$. To prove our theorem we have to show that

$$s_3(\mathcal{P}) < s_1(\lceil n/5 \rceil). \tag{5.4}$$

We may suppose that

$$s_3(\mathcal{P}) > 2.236 s_1(n), \tag{5.5}$$

otherwise (5.3) implies (5.4).

By Theorem 1.6 there exists an n_5 such that $s_3(n) < 0.001$ holds for all $n > n_5$. From now on we suppose that $n > \max\{n_4, n_5\}$. We need a definition. Let $\mathcal{G}^2 = \mathcal{G}^2(\mathcal{P})$ be a graph with vertex set $\mathcal{P}$ defined in the following way: connect two points $p, q \in \mathcal{P}$ with an arc (with the shortest path on $\mathbb{S}^2$) if $d(p,q) = s_1$ or s_2. If these arcs form a planar representation of $\mathcal{G}^2$ then (by the four color theorem) there exists a $\mathcal{P}' \subset \mathcal{P}$, $|\mathcal{P}'| \geq |\mathcal{P}|/4$ such that $s_3(\mathcal{P}) \leq s_1(\mathcal{P}') \leq s_1(|\mathcal{P}'|) \leq s_1(\lceil n/5 \rceil)$.

From now on we suppose that there exist two arcs ac and bd such that $\{a,b,c,d\} \subset \mathcal{P}$, $d(ac)$ and $d(bd) \in \{s_1, s_2\}$ and int $\widehat{ac} \cap$ int $\widehat{bd} \neq \emptyset$. We claim that in this case

$$s_2 \leq 2 s_1. \tag{5.6}$$

Indeed, if three of the points of a, b, c, d, are lying on a great circle, then (5.6) follows easily. Otherwise, we are going to use the following.

FACT 5.3. *If a, b, c, d form a convex quadrilateral with diagonals ac and bd, then $d(ab) + d(cd) < d(ac) + d(bd)$ and $d(ad) + d(bc) < d(ac) + d(bd)$.* □

α) If one of the diagonals is s_1, then Fact 5.3 implies that the sum of any two opposite sides is less than $s_1 + s_2$. All the four sides must be s_1, we get that a, b, c, d is a 2-distance set isomorphic to 5.2 (i).

β) Both diagonals have length s_2. Then at least one of any two opposite sides has length less than s_2. One can find two adjacent sides of length s_1, so (5.6) follows from the triangle inequality. □

We claim that in $\mathcal{G}$ (not in $\mathcal{G}^2$!) every degree is small:

CLAIM 5.4. $\Delta(\mathcal{G}) \leq 3$.

Indeed if $p \in \mathcal{P}$ and $\Gamma(p)$ is the set of its neighbors then $p \cup \Gamma(p)$ is a 2-distance set by (5.5) and (5.6). Then $|p \cup \Gamma(p)| \leq 5$ by Fact 5.1. Moreover, if $|p \cup \Gamma(p)| = 5$ then it is a regular pentagon, but in that case $|\Gamma(p)| = 2$, a contradiction. □

PROPOSITION 5.5. $s_3 > s_1 + s_2$.

Proof. We will prove that for every Dirichlet cell D we have

$$\text{Area } D > 0.1734 (s_1 + s_2)^2. \tag{5.7}$$

This implies the Proposition as follows:

$$1.001\frac{\sqrt{3}}{2}\,\frac{1}{2.236^2}\,(s_1+s_2)^2 < 0.1734(s_1+s_2)^2 \le \frac{4\pi}{n} < 1.001\frac{\sqrt{3}}{2}\,s_1(n)^2\ ,$$

i.e., $s_1+s_2 < 2.236 s_1(n)$. Then (5.5) implies Proposition 5.5. To prove (5.7) we have two cases. Let p be the only point of $\mathcal{P}$ contained in D.

— If $\deg_{\mathcal{G}}(p) \le 2$ (i.e., p has at most 2 neighbors in $\mathcal{G}$), then we can apply Lemma 2.1 with $\Delta = 2$, i.e.,

$$\text{Area } D > A(2, s_2, s_1) > 0.999\left(s_1\sqrt{s_2^2 - s_1^2} + s_2^2 \arcsin\frac{s_1}{s_2}\right).$$

Here, the right hand side is larger than $0.1734(s_1+s_2)^2$ for $0 < s_1 \le s_2 \le 2s_1$.

— If $\deg_{\mathcal{G}}(p) \ge 3$ then $\deg_{\mathcal{G}}(p) = 3$ by Claim 5.4. Let $\Gamma(p) = \{u, v, w\}$. Then every distance in $\{p, u, v, w\}$ is either s_1 or s_2. Then one of the following three subcases holds (by Fact 5.2)

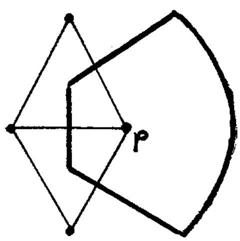

Figure 2

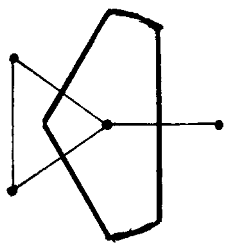

Figure 3

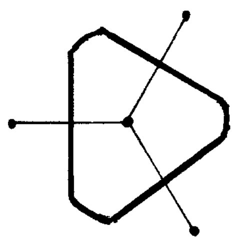

Figure 4

— — $\{p, u, v, w\}$ is isomorphic to 5.2 (i). Then D contains the intersection of 3 halfspheres and a circle of radius $s_2/2$, (see Fig. 2). Hence $\sqrt{3}\ s_1 < s_2 < 1.733 s_1$ and

$$\text{Area } D > 0.999\left(\frac{\sqrt{3}}{6} + \frac{\sqrt{2}}{4} + \frac{3}{8}\left(\frac{4}{3}\pi - 2\arctan\sqrt{2}\right)\right)s_1^2 > 1.48\ s_1^2$$
$$> 0.1734(s_1 + 1.733 s_1)^2 > 0.1734(s_1+s_2)^2.$$

— — $\{p, u, v, w\}$ is isomorphic to 5.2 (v) (Fig. 3). Then $1.931 < s_2/s_1 < 1.932$ and Area $D > 0.999(1.497\ldots s_1^2) > 1.495 s_1^2 > 0.1734(s_1+s_2)^2$.

— — $\{p, u, v, w\}$ is isomorphic to 5.2 (vi). (See Fig. 4). Then $\sqrt{3}\ s_1 > s_2 > 1.732 s_1$, hence Lemma 2.1 yields (with $\varepsilon = 0.001$, $\Delta = 3$, $s_2 = s_1 1.732$) that

$$s_1(\mathcal{P}) < s_1(n) 0.828\ .$$

Thus by (5.5) we have

$$s_3 > 2.7 s_1.$$

We claim that in this case there are no two crossing edges of $\mathcal{G}^2$, a contradiction to our earlier assumptions. If $\{p, u, v, w\}$ is a 2-distance set of type (vi) then there is no other type of 4-element 2-distance set in $\mathcal{G}^2$. Consider the crossing edges ac and bd. The case α is impossible so we have that $s_1 = d(a, b) = d(a, d)$, $s_2 = d(a, c) = d(b, d) \sim 1.73 s_1$. This is not a 2-distance set so we may suppose that, e.g., $d(d, c) \geq s_3 > 2.7 s_1$. Then $d(b, c) \leq s_2$, too. It is easy to check, that such a convex quadrilateral does not exist. The proof of Proposition 5.5 is complete. □

PROPOSITION 5.6. *If ac and bd are two crossing arcs in $\mathcal{G}^2$ then* $\min\{\deg_{\mathcal{G}^2}(x) : x \in \{a, b, c, d\}\} \leq 4$.

Proof. As we have seen above, we may suppose that $d(a, b) = d(a, d) = s_1$, and $d(a, c) = s_1$ or s_2. Then, by Proposition 5.5, $\{a, b, c, d\}$ is a 2-distance set. So its type is among (i)–(iv), by Fact 5.2. We claim that $\deg_{\mathcal{G}^2}(a) \leq 4$. If $e \in \mathcal{P} - \{a, b, c, d\}$ and $d(a, e) = s_1$, then $\{a, b, c, d, e\}$ is a 2-distance set with $\deg_{\mathcal{G}}(a) \geq 3$, which contradicts to Fact 5.1. The same argument verifies the case when $\{a, b, c, d\}$ is similar to (i), so a has already three neighbors of distance s_1. If $d(a, e) = s_2$ then $\{a, b, c, d\}$ and $\{a, b, e, d\}$ are similar 2-distance sets. This is impossible in the cases (ii) and (iv), and in the case (iii) we obtain a regular pentagon. □

PROPOSITION 5.7. *There exists a $\mathcal{P}' \subset \mathcal{P}$, $|\mathcal{P}'| \geq |\mathcal{P}|/5$ such that $s_1(\mathcal{P}') \geq s_3(\mathcal{P})$.*

Proof. Let $\mathcal{P}_0 = \mathcal{P}$ and consider two crossing edges. An endpoint of them has degree at most 4 (in $\mathcal{G}^2$). Denote this point by p_1 and let $\mathcal{P}_1 = \mathcal{P} - \{p\} - \{\Gamma(p)\}$. Repeat this step until we have crossing edges of length at most s_2 in $\mathcal{P}_i$. Finally, we have a set $\mathcal{Q} = \{p_1, \ldots, p_t\}$ such that $d(q, p) > s_2$ for $q \in \mathcal{Q}$, and $p \in \mathcal{P}_t \cup Q$, and $|\mathcal{P}_t| \geq |\mathcal{P}| - 5|\mathcal{Q}|$. Then, by the four color theorem we have a $\mathcal{Q}' \subset \mathcal{P}_t$, $|\mathcal{Q}'| \geq |\mathcal{P}_t|/4$ with $s_1(\mathcal{Q}') > s_2$. Then let $\mathcal{P}' = \mathcal{Q} \cup \mathcal{Q}'$. □

Finally, Proposition 5.7 obviously implies (5.4). □

6. THE CASES OF $n \leq 12$, $n = 17, 18$

The cases $n = 17, 18$. Let $\mathcal{P} \subset \mathbb{S}^2$, $|\mathcal{P}| = 17$ or 18. The chromatic number of $\mathcal{G}$, the minimum distance graph of $\mathcal{P}$, is at most 4. So we have a subset $\mathcal{P}'$ of size at least $\lceil |\mathcal{P}|/4 \rceil = 5$ not containing any distance of $s_1(\mathcal{P})$. We obtain $s_2(\mathcal{P}) \leq s_1(\mathcal{P}') \leq s_1(5) = 90°$. □

To finish Theorem 1.1, it is sufficient to consider the cases $n = 7$ and $n = 10$. Let $\mathcal{P} \subset \mathbb{S}^2$ be an n-element set with $s_2(\mathcal{P}) \geq 120°$ (for $n = 7$), or with $s_2 \geq \pi - u = 116.56\ldots°$ (for $n = 10$). In the first case we will get a contradiction, in the second we will see that $\mathcal{P}$ is a subset of an icosahedron. We will use that $s_1(7) < 90°$, and $s_1(10) < 70°$.

If every degree in $\mathcal{G}$, is at most 3, then, as we have seen in Section 4, there is a subset $\mathcal{P}' \subset \mathcal{P}$ with $|\mathcal{P}'| \geq |\mathcal{P}|/3$, $s_1(\mathcal{P}') > s_1(\mathcal{P})$. In the case of $n = 10$ we get $s_2(\mathcal{P}) \leq s_1(\mathcal{P}') \leq s_1(4) = \cos^{-1}(-1/3) = 109.47\ldots < \pi - u$, a contradiction.

In the case of $n = 7$ we get $s_2(\mathcal{P}) \leq s_1(3) = 120°$. Equality must hold, so we get 3 points $p_1, p_2, p_3 \in \mathcal{P}$ with mutual distances 120°. The open discs of radius 120° and centers p_i cover all other points of the sphere at least twice. So for each $q \in \mathcal{P} \setminus \{p_1, p_2, p_3\}$ there are (at least) two p_i's with $d(q, p_i) = s_1$. There are two points $q_1, q_2 \in \mathcal{P}$ in the same hemisphere determined by the great circle through the p_i's. The distance of q_1 and q_2 is less than 120° hence it is s_1. This implies $s_1 = \cos^{-1}(1/3) = 70.52\ldots°$. There is a pair, say p_1, p_2, such that there are q^1 and q^2 from $\mathcal{P}$ with all four distances $d(p_i, q^j) = s_1$. Then the distance $d(q^1, q^2) = \cos^{-1}(1/9) = 96.37\ldots°$, a contradiction.

From now on, we suppose that there are points $p, q_1, q_2, q_3, q_4 \in \mathcal{P}$ such that $d(p, q_i) = s_1$. Suppose these points lie around p in this order. Then $d(q_1, q_3)$ and $d(q_2, q_4)$ are larger than s_1. There is another distance exceeding s_1, say $d(q_1, q_4) > s_1$.

PROPOSITION 6.1. $d(q_1, q_2) = d(q_2, q_3) = d(q_3, q_4) = s_1$.

Proof. Suppose, on the contrary, that there are at most two more minimum distances. Then there is a pair, say $q_1 q_2$, such that $d(q_1, q_2) > s_1$ but the angle $q_1 p q_2$ is less than 120°. In the case $n = 7$ this implies $s_2 \leq d(q_1, q_2) < 120°$, and we are done. For $n = 10$ we get $\cos(q_1 q_2) \leq (\cos s_1)^2 - (1/2)(\sin s_1)^2$, implying $d(q_1, q_2) \leq 108.93\ldots° < \pi - u$, a contradiction. □

The case $n = 7$. Here we finish the proof of $s_2 < 120°$. Let c be the center of the regular triangle $p q_2 q_3$. The angle $q_1 c q_4 = 120°$ and $d(c, q_1) = d(c, q_4)$. Hence $d(q_1, q_4) \leq 120°$. This distance is not minimal, so we get $s_2 \leq 120°$. In case of equality we get $d(c, q_1) = 90°$ which implies again that $s_1 = \cos^{-1}(1/3) = 70.52\ldots°$. If there is a vertex in $\mathcal{G}$ with degree at most two, then we can find three independent points forming a regular triangle of side length 120°, and we can finish the proof as we did above. If all the degrees in $\mathcal{G}$ are at least three, then we obtain that both points, $\{r_1, r_2\}$ of $\mathcal{P}$ not connected to p have two neighbours among $\{q_1, \ldots, q_4\}$. Only the pairs $q_i q_{i+1}$ qualify, so we get two regular triangles, $q_1 q_2 r_1$ and $q_3 q_4 r_2$. Then $d(r_1, r_2) = \cos^{-1}(1/16) = 86.41\ldots°$, a contradiction.

The case $n = 10$. If $s_1 < u$, then $d(q_1, q_3) < \pi - u$, a contradiction. Suppose that $s_1 \geq u$. There is a vertex $x \in \mathcal{P}$ of degree at most 4. Let us denote the point on the sphere antipodal to x by $-x$, and let D be a closed disc about $-x$ of radius u. We have $|\mathcal{P} \cap D| \geq 5$. We claim that $s_1 = u$. If the center $-x \in \mathcal{P}$, we are done; otherwise there is an angle $r_1(-x)r_2$, $r_i \in (\mathcal{P} \cap D)$, at most 72°, implying $d(r_1, r_2) \leq u$.

We have obtained $s_1 = u$, hence $s_2 = \pi - u$. Consider the icosahedron with vertices $p, q_1, \ldots, q_4$. All neighbours of q_i's are also vertices of this icosahedron. This finishes the proof, since the graph $\mathcal{G}$ is connected. □

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(Eingegangen am 5. April 1988)

(Revidierte Form am 27. Oktober 1992)