Multi. Val. Logic, 1998, Vol. 3, pp. 39–53
Reprints available directly from the publisher
Photocopying permitted by license only

Orders Admitting an Isotone Majority Operation

Z. FÜREDI[a] and I. G. ROSENBERG[b]

[a]*Math. Inst. Hungarian Acad. Sci., 1364 Budapest P.O.B. 127, Hungary;*
[b]*Dept. mathématiques et stat., Université de Montréal,*
C. P. 6128. Succ. "Centre-ville", Montréal, Québec, H3C 3J7, Canada

Recommended for publication by Corina Reischer

(Received 20 December 1996; In final form 30 January 1997)

Let $\leqslant$ be an order on P. A ternary operation $m:P^3 \to P$ is isotone if $m(x,y,z) \leqslant m(x',y',z')$ whenever $x \leqslant x'$, $y \leqslant y'$, $z \leqslant z'$ and a majority operation if $m(y,x,x) = m(x,y,x) = m(x,x,y) = x$ for all $x, y \in P$. Orders admitting an isotone majority operation are called majority orders and are important for clones. The characterization of majority orders seems to be rather difficult and the purpose of this paper is to draw attention to this interesting combinatorial problem. The rather incomplete results are the following. We show that majority orders are partial lattices with a Helly type property. We produce a large class of majority orders, called forest-like, which are not lattices. Finally we derive certain results for majority orders with a crown $p_1 > p_2 < p_3 > \cdots > p_8 < p_1$ and exhibit three types of majority orders with a unique isotone majority operation.

Keywords: (partial) Orders; posets; isotone (monotone or order preserving) operation; majority operation; partial lattice; clones

MR Classification: 06A, 06F, 08A

0. INTRODUCTION

0.1 Let $\underline{P} := (P, \leqslant)$ be a fixed *order* (i.e. $\leqslant$ is a reflexive, transitive and antisymmetric relation on P which is also called a (partial) ordering and $\underline{P}$ is referred to as an ordered set or poset). For n positive integer an n-ary *operation* on P is a map $f: P^n \to P$. The image of $(x_1, \ldots, x_n) \in P^n$ is denoted by $f(x_1, \ldots, x_n)$ or shortly by $fx_1, \ldots, x_n$. The operation f is *isotone* (also *monotone* or *order preserving*) if $fx_1, \ldots, x_n$

$\leqslant fy_1, \ldots, y_n$ whenever $x_1 \leqslant y_1, \ldots, x_n \leqslant y_n$. Let Pol $\leqslant$ denote the set of isotone operations. It is known and easy to see that Pol $\leqslant$ is a clone; i.e., it is composition closed and contains all projections e_i^n, $i \leqslant i \leqslant n$ (defined by setting $e_i^n x_1, \ldots, x_n = x_i$ for all $x_1, \ldots, x_n \in P$). Thus Pol $\leqslant$ is a multi-variable analog of the set End $\leqslant$ of order endomorphism (= unary isotone selfmaps) and so it presents a certain interest. We list certain properties of Pol $\leqslant$.

A clone is *maximal* (also precomplete or preprimal) if it is a dual atom or coatom in the lattice $(\underline{L}, \underline{C})$ of clones on P (i.e. if it is covered exactly by the clone of all operations). For P finite the clone Pol $\leqslant$ is maximal if and only if $\leqslant$ is a bounded order (i.e. $\leqslant$ has at least and greatest element ([10, 6,7] see also [12, 13]). Similarly for P infinite the clone Pol($\leqslant$) is locally maximal if and only if $\leqslant$ is directed and down-directed (i.e. each pair of elements has an upper and lower bound [16–18]). For $P = \{0, 1\}$ and the natural order $\leqslant$ on P the lattice of clones of isotone non-constant operations (ordered by $\subseteq$) form an important part of the lattice of clones [14]; for hypergraph connections see [2–4])

0.2 A ternary operation $m\colon P^3 \to P$ is a **majority operation** if for all $x, y \in P$

$$m(y, x, x) = m(x, y, x) = m(x, x, y) = x. \tag{1}$$

Clones containing a majority operation have remarkable properties. If P is finite then such a clone C is both a compact and co-compact element of the lattice of clones; in particular, C is finitely generated and the lattice of superclones of C is atomic with a finite number of atoms. There are finitely many $\rho_i \subseteq P^2$ $(i = 1, \ldots, n)$ such that $f \in C$ iff $\rho_1, \ldots, \rho_\ell$ are subuniverses of $\langle P; f \rangle^2$. Moreover, the variety generated by the algebra $\langle P; C \rangle$ is congruence distributive [1, 8].

Thus it is natural to investigate orders admitting an isotone majority operation which we call **majority orders**. For finite bounded orders this problem (in a more general context) was raised by Demetrovics *et al.*, in [5]. Somewhat surprisingly, the characterization of majority orders seems to be rather complex. The problem is also interesting because for a finite majority order $\leqslant$ the clone Pol $\leqslant$ is finitely generated. The clone of the form Pol $\leqslant$ where $\leqslant$ is a finite bounded order are the only maximal clones which may be not finitely generated. In this context we mention the following problem from [15] (cf [14]).

Problem Let $\underline{\underline{P}}$ be (fintie) majority order. What are the isotone majority operations on P such that the clone generated by m is an atom of the lattice of clones?

0.3 In §1 we start with some obvious properties of majority orders. We show that they are partial lattices with the Helly property (if pairwise joins (meets) of x, y and z exist then the joint (meet) of x, y and z exists) and so locally bounded majority orders coincide with lattices.

In §2 we look at W-paths i.e. sequences $\langle p_i \rangle$ such that p_{2i} is the meet of p_{2i-1} and p_{2i+1} and p_{2i+1} is the join of p_{2i} and p_{2i+2}. We derive a condition on an isotone majority operation for orders with a W-path and such that p_{2i} is the meet of each $x \geqslant p_{2i-1}$ and each $y \geqslant p_{2i+1}$ and p_{2i+1} the join of every $x \leqslant p_{2i}$ and every $y \leqslant p_{2i+2}$. From this we obtain that a crown is not a majority order. An order is **tree-like** if it is obtained from a discrete order whose diagram is a tree by replacing each interval $[q, q']$ such that q' covers q by a lattice $F_{qq'}$ with a least element q and greatest element q' so that two such lattices intersect in at most one bound. We show that the tree-like orders form a large class of majority orders that need not be lattices. A majority order is *stiff* if it has a unique isotone majority operation. A lattice is *stiff* if it is distributive. We show that an order $p_1 > p_2 > p_3 > \cdots$ is stiff but a simple tree-like order on 7 elements which is not of this form has already a huge number of isotone majority operations.

Finally we turn to majority orders containing a W-path $\{p_1, \ldots, p_8, p_1\}$. If, moreover p_1 p_3 p_5 and p_7 are maximal elements then the meet of p_1, p_3, p_5 and p_7 are maximal elements then the meet of p_1, p_3, p_5 and p_7 exists, $p_1, \ldots, p_8$ are the unique elements greater than θ and the values of a majority isotone operation on $\{\theta, p_1, \ldots, p_8\}$ are unique. Finally we show that the 9-element order consisting of a crown $\{p_1, \ldots, p_8\}$ and a least element θ is stiff.

Finite majority orders can also be characterized in terms or zigzags, see [20] and [21] Remark 2.4, and undoubtedly some of our results could be derived by this technique but for simplicity's sake we shall neither define nor use zigzag here.

The results of this paper are certainly far from definitive. The main purpose of this paper is to draw attention to the problem of majority orders, which, beside the above mentioned motivation, are of their own combinatorial interest. The financial support provided by NSERC Canada operating grant A-5407, NATO grant RG-09782 and FCAC Québec Subvention *d'*équipe Eq-0539 is gratefully acknowledged. The authors would like to thank Dr. B. Larose for very helpful suggestions.

1.1. Preliminaries

Let $\underline{P} := (P; \leqslant)$ be a fixed order. Let m be a ternary operation on P i.e. a map from P^3 into P which assigns the value $m(x, y, z)$ or, briefly, (xyz) to each $(x, y, z) \in P^3$.

For a $\in P$ put $[a) := \{x \in P : x \geqslant a\}$ and $(a] := \{x \in P : x \leqslant a\}$. We say that $\underline{A}$ is a *partial lattice* if for all $x, y \in P$ we have (i) $[x) \cap [y) = \emptyset$ or $[x) \cap [y)$ has a least element which is called the *join* of x and y and denoted by $x + y$ and (ii) $(x] \cap (y] = \emptyset$ or $(x] \cap (y]$ has a greatest element which is termed the *meet* of x and y and denoted xy. Clearly $\langle A; +, \cdot \rangle$ is a partial algebra satisfying the partial version of lattice axioms (i.e. if one side of an axiom or law exists so does the other side and they are equal), in particular the associative law for $+$ is: (i) if both $u := x + y$ and $u + z$ exist then both $v := y + z$ and $x + v$ exist and

$$[x + y] + z = x + [y + z] \tag{2}$$

and (ii) if both $v := y + z$ and $x + v$ exit, then both $u := x + y$ and $u + z$ exist and (2) holds. The common value in (2) is denoted by $x + y + z$. The situation for "$\cdot$" is quite analogous.

For the simplicity of notation we use the arithmetical convention that "$\cdot$" takes precedence over "$+$" e.g. $xy + z$ stands for $[xy] + z$ (and is defined iff x, y, z have a common upper bound). Note that for $a < b$ the interval $[a) \cap (b]$ is a lattice.

We say that a partial lattice has the *Helly property* if (i) $x + y + z$ exists if $x + y$, $x + z$ and $y + z$ exist and (ii) xyz exists whenever xy, xz and yz exist. Note that (i) and (ii) are statements about upper and lower bounds. We have the following result ([18] Lemma 5.2).

1.2 PROPOSITION *Let $\underline{P}$ have an isotone majority operation m. Then $\underline{P}$ is a partial lattice with the Helly property satisfying for all*

$x, y, z \in P$

$$z \geqslant x + y \Rightarrow (xyz) = (yxz) = (xzy) = x + y \tag{3}$$

$$z \leqslant xy \Rightarrow (xyz) = (yxz) = (xzy) = xy; \tag{4}$$

if $x + y + z$ *exists, then*

$$(xyz) \leqslant [x + y][x + z][y + z]; \tag{5}$$

if xyz *exists, then*

$$(xyz) \geqslant xy + xz + yz. \tag{6}$$

1.3 *Remark* ([18] Cor. 5.3). Let $\underline{P}$ have an isotone majority operation m. Then for $x, y, z, u, \ell \in P$ we have

$$x, y, z \in (u] \text{ or } x, y, z \in [u) \Rightarrow ((xyu)zu) = (x(yzu)u), \tag{7}$$

$$x, y \in [\ell) \cap (u] \Rightarrow (x(xyu)\ell) = (x(xy\ell)u) = x \tag{8}$$

(indeed these are the associative and absorptive identities combined with (3) and (4)).

As usual $\underline{P}$ is *directed* (*down-directed*) if for all $x, y \in P$ the set $[x) \cap [y)$ (the set $(x] \cap (y]$) is nonempty. We have ([18] Cor. 5.4 and for a finite bounded order [5]):

1.4 Corollary *If $\underline{P}$ is a directed (down-directed) majority order then $\langle P; + \rangle$ $\langle P; \cdot \rangle$ is a semilattice. If $\underline{P}$ is both directed and down-directed then $\underline{P}$ is a majority order if and only if P is a lattice*

Proof It is well known that a lattice is a majority order (choose m equal to $xy + xz + yz$ or to $[x + y][x + z][y + z]$). □

It is known [5] and easy to prove that the class M of majority orders is an order variety (i.e. closed under direct products and retracts). The *cardinal sum* of orders $(P_i; \leqslant_i)$ $(i \in I)$ is the order $\leqslant$ on $P := \bigcup_{i \in I} P_i \times \{i\}$ defined by setting $(x, i) \leqslant (y, j)$ if and only if, $i = j$ and $x \leqslant_i y$. We have [19]:

1.5 LEMMA *The order variety M of majority orders is closed under cardinal sums.*

Note that the order variety M is not closed under lexicographic products.

1.6 *Remark* Let m be an isotone majority operation on $\underline{P}$ and let $p \in P$. Clearly, if $x_i \geqslant p$ $(i = 1, 2, 3)$, then $(x_1 x_2 x_3) \geqslant p$ and so the restriction of m to $[p)$ is an isotone majority operation on $([p); \leqslant)$. The same holds for $(p]$ and therefore for $p, p' \in P, p < p'$ the restriction of m to the interval $I := [p, p']$ is a majority operation of the lattice $(I; \leqslant)$ and so e.g. its values on I are between the two lattice medians. Clearly, if for some $p \in P$ the order $([p); \leqslant)$ is not a majority order, then $\underline{P}$ is not a majority order.

2. *W*-PATHS AND FOREST-LIKE ORDERS

2.1 A *W-path* is a sequence Q of elements of P of the form $\langle p_0, p_1, \ldots, p_n \rangle$, $\langle p_1, p_2, \ldots, p_n \rangle$, $\langle p_0, p_1, \ldots \rangle$, $\langle p_1, p_2, \ldots \rangle$ or $\langle \cdots, p_{-1}, p_0, p_1, \cdots \rangle$ such that

$$p_{2i} = p_{2i-1}\ p_{2i+1}, \quad p_{2i+1} = p_{2i} + p_{2i+2} \tag{9}$$

for all i (such that all the elements in (9) belong to the sequence; as usual, (9) also stipulates that the meets and joins exist and are equal to the indicated elements).

The next proposition gives bounds on the values of a majority isotone operation in a special case.

2.2 PROPOSITION *Let m be an isotone majority operation and let Q be a W-path such that for all i*

$$x \geqslant p_{2i-1},\ y \geqslant p_{2i+1} \Rightarrow x \cdot y = p_{2i}, \tag{10}$$

$$x \leqslant p_{2i},\ y \leqslant p_{2i+2} \Rightarrow x + y = p_{2i+1}. \tag{11}$$

Let $q_i \geqslant p_i$ if i is odd and $q_i \leqslant p_i$ if i is even. Then for all $1 \leqslant i \leqslant j \leqslant k$ and arbitrary permutation π of $\{i, j, k\}$

$$(q_{\pi(i)}\ q_{\pi(j)}\ q_{\pi(k)}) = \begin{cases} \geqslant p_{\pi(j)} & \text{if } j \text{ is odd,} \\ \leqslant p_{\pi(j)} & \text{if } j \text{ is even.} \end{cases} \tag{12}$$

Moreover, if $k \leqslant i+2$ *then* $(p_{\pi(i)}p_{\pi(j)}p_{\pi(k)}) = p_{\pi(j)}$.

Proof By induction on $\ell := k - i$. Let $\ell \leqslant 1$. Then we have $j = i$ or $j = k$ and the statement holds by the majority property.

Suppose the statement holds for some $1 \leqslant \ell$. Let $1 \leqslant i \leqslant j \leqslant k$ satisfy $k - i = \ell + 1$. We start with a particular case.

A. Let i, j and k be all even. Then

$$a := (q_i q_j q_k) \leqslant x := (q_i q_{j-1} q_{k-1}),\ a \leqslant y := (q_{i+1} q_{j+1} q_k).$$

By the inductive assumption $x \leqslant p_{j-1}$ and $y \geqslant p_{j+1}$.

By (10) we have $a \leqslant x \cdot y = p_j$. Obviously the same holds for any permutation of i, j and k.

B. Suppose that j is odd. If k is odd we have $(q_i q_j q_k) \geqslant (q_i q_j q_{k-1}) \geqslant p_j$ by the induction hypothesis and the same holds for any permutation of i, j and k. Thus assume that k is even. By the same argument i may be assumed even as well. Now

$$a := (q_i q_j q_k) \geqslant x := (q_i q_{j-1} q_k),\ a \geqslant y := (q_{i+1} q_{j+1} q_k).$$

By A above $x \leqslant p_{j-1}$ and $y \leqslant p_{j+1}$ whence applying (10) we get the required $(q_i q_j q_k) = a \geqslant p_j$. Evidently the same holds for each permutation of i, j and k.

C. If j is even then (12) holds by duality.

This concludes the induction step and thus the proof of (12). For the last statement note that for $i+1$ odd by (12) and (11)

$$p_{i+1} \leqslant (p_i p_{i+1} p_{i+2}) \leqslant (p_{i+1} p_{i+1} p_{i+1}) = p_{i+1}$$

and similarly for $i+1$ even. □

A sequence $\langle p_1, \ldots, p_{2n} \rangle$ of distinct elements of P is a *W-cycle* if $n > 1$ and $\langle p_1, \ldots, p_{2n}, p_1 \rangle$ is a w-path.

The following are special cases of a more general statement proved in [16] by slightly different methods.

2.3 COROLLARY *A majority order has no W-cycle satisfying (10) and (11).*

Proof Suppose $\langle p_1, \ldots, p_{2n} \rangle$ is a W-cycle satisfying (10) and (11) in a majority order.

By Proposition 2.2 we have $(p_1 p_2 p_3) \leqslant p_2$. Now $\langle p_3, \ldots, p_{2n}, p_1, p_2 \rangle$ is also a W-cycle satisfying (10) and (11) and by the same taken $(p_1 p_2 p_3) \geqslant p_1$, a contradiction. □

2.4 COROLLARY *If $\underline{P}$ has a W-cycle $\langle p_1, \ldots, p_{2n} \rangle$ such that $p_1, p_3, \ldots, p_{2n-1}$ are maximal elements and $p_2, p_4, \ldots, p_{2n}$ are minimal elements of $\underline{P}$, then $\underline{P}$ is not a majority order.*

2.5 *Example A crown is not a majority order.* (A crown is a W-cycle $p_1 > p_2 < p_3 > \cdots < p_{2n} < p_1$ with $p_1, \ldots, p_{2n}$ pairwise distinct and $p_i \not< p_j$ otherwise).

2.6 As usual $x \in P$ covers $y \in P$, in symbols $x \sqsupset y$, if $x > y$ but $x > z > y$ for no $z \in P$. The *diagram* of $\underline{P}$ is the unoriented graph $G = (P, E)$ where $\{x, y\}$ is in the edge set E iff $x \sqsupset y$ or $y \sqsupset x$ (i.e. E is the symmetric hull of the covering relation). A graph G is a *tree* if each pair of vertices is connected by exactly one path. We need the following:

2.7 FACT *Let G be a tree and let $e_i := \{v_{i1}, v_{i2}\}$ $(i = 1, 2, 3)$ be 3 pairwise distinct edges of G. Denote by $\Pi_{ij}^{k\ell}$ the unique path from v_{ik} to $v_{j\ell}$ $(1 \leqslant i < j \leqslant 3,\ 1 \leqslant k, \ell \leqslant 2)$. Then either (i) there exists a unique vertex v common to all $\Pi_{ij}^{k\ell} (1 \leqslant i < j \leqslant 3,\ 1 \leqslant k, \ell \leqslant 2)$, or (ii) one edge, say e_i, is between the others (e.g. for $i = 2$ we have $e_2 \subseteq \bigcap_{1 \leqslant k, \ell \leqslant 2} \Pi_{13}^{k\ell}$)*

Proof We can choose the notation so that $v_{11}, v_{21} \in \Pi_{12}^{22}$. If $v_{22} \in \Pi_{23}^{2j}$ for some $j \in \{1, 2\}$ then we have (ii). Thus we may choose the names of the two vertices on e_3 so tht $v_{21}, v_{31} \in \Pi_{23}^{22}$. Let v be the last vertex on $\Pi_{21}^{22} \cap \Pi_{23}^{22}$ (going from v_{22} to v_{12}). Since G is cycle-free, the path Π from v_{12} to v_{32} through v is Π_{13}^{22} and (i) follows. □

2.8 An order $\underline{P}$ is *tree-like* if it is the transitive hull of the relation obtained from a tree $G = (Q; E)$ by replacing each edge $\{q, q'\}$ by a bounded lattice with bounds q and q' so that for two distinct edges from E the corresponding lattices intersect in at most a singleton from Q. In other words, $\underline{P}$ is tree-like iff there is $Q \subseteq P$ such that (i) the diagram of $(Q; \leqslant)$ is a tree on Q, (ii) for all $q, q' \in Q$ such that q' covers q in $(Q; \leqslant)$ the interval $I_{qq'} := [q, q']$ in $\underline{P}$ is a lattice and (iii) each $x \in P \setminus Q$ belongs to exactly one $I_{qq'}$ $(q, q' \in Q, q'$ covers q in $(Q; \leqslant))$.

We say that $\underline{\underline{P}}$ is *forest-like if* $\underline{\underline{P}}$ is the union of vertex disjoint tree-like orders.

The proof of the next proposition is easy but tedious and therefore it is omitted; however it can be obtained upon request from the second author.

2.9 PROPOSITION *A forest-like order is a majority order.*

We say that a majority order $\underline{\underline{P}}$ is *stiff* if it has a unique isotone majority operation. Obviously the unique isotone majority operation of a stiff order is totally symmetric. We have:

2.10 FACT *A directed and down-directed order* $\underline{\underline{P}}$ *is stiff if and only if* $\underline{\underline{P}}$ *is a distributive lattice.*

Proof By Corollary 1.4 the order $\underline{\underline{P}}$ is a lattice $\langle P, +, \cdot \rangle$. By the proof of Corollary 1.4 both $m_\ell(x,y,z) := x \cdot y + x \cdot z + y \cdot z$ and $m_u(x,y,z) := [x+y][x+2][y+2]$ are isotone majority operations. It is well known and easy to prove that every isotone majority operation m satisfies

$$m_\ell(x,y,z) \leqslant m(x,y,z) \leqslant m_u(x,y,z)$$

for all $x, y, z \in P$. It follows that $\underline{\underline{P}}$ is stiff iff $m_\ell = m_u$. It is well known that the latter is equivalent to $\underline{\underline{P}}$ distributive. □

The following example shows a tree-like majority order wich is not stiff.

2.11 *Example* Let $P = \{1, 2, \ldots, 7\}$ be ordered by $1 > 2 < 3 > 4 < 5$, $3 > 6 < 7$. Let m be an isotone majority operation. Proposition 1.6 determines all the values of (xyz) except for (246), (146), (247), (256), (147), (257), (156), (157) and those obtained from the above by permuting the variables. it is easy to establish that (246) is a lower bound of (146), (247) and (256), 3 and (157) are upper bounds for (147), (257) and (156) and

$$(147) \geqslant (247) \leqslant (257) \geqslant (256) \leqslant (156) \geqslant (146) \leqslant (147).$$

while otherwise these values are independent from the others.

In particular, we can choose all these values equal 3 or all equal 4 and so the order is not stiff. An easy calculation shows that there are exactly $51^6 \approx 1.76 \times 10^{10}$ isotone majority operations. □

13. ORDERS WITH A W-CYCLE $\{p_\perp,\ldots,p_8\}$

3.1 Lattices may contain W-cycles and so we should look at orders containing a W-cycle $\{p_1,\ldots,p_{2n}\}$ such that either the join of $p_1, p_3, \cdots, p_{2n-1}$ or the meet of $p_2, p_4, \cdots, p_{2n}$ does not exist. By the Helly property we may assume $n>3$. The first case is $n=4$ i.e. in this section we assume that a partial lattice P contains a W-cycle $\{p_1,\cdots,p_8\}$. By duality we assume that $p_1+p_3+p_5+p_7$ does not exist. We show that neither p_1+p_5 nor p_3+p_7 exist. Indeed, if this does not hold, then by symmetry we may assume that $q:=p_1+p_5$ exists. Clearly then q is an upper bound for p_2 and p_4 and hence $p_3=p_2+p_4\leqslant q$. Similarly $p_7=p_6+p_8\leqslant q$ and we have the contradiction $q=p_1+p_3+p_5+p_7$. We use the following notation. Write i instead of $p_i (i=1,\cdots,8)$ and put $p_1^1:=(137)$, $p_2^1:=(248)$, $p_3^1:=(351)$, $p_4^1:=(246)$, $p_5^1:=(357)$, $p_6^1:=(468)$, $p_7^1:=(157)$, $p_8^1:=(268)$. For $p_i^1=(abc)$ put $p_i^2=(bca)$ and $p_i^3=(cab)$ $(i=1,\ldots,8)$. Further for $p_i^j=(abc)$ put $q_i^j=(bac)$ $(i=1,\ldots,8,\ j=1,2,3)$. We have:

3.2 LEMMA *Let $\underline{P}$ be an order with a majority isotone operation m and W-cycle $\langle p_1,\ldots,p_8\rangle$. Then the p_i^j defined above satisfy: (i) For all $i=1,\ldots,4$ and $j=1,2,3$*

(i)

$$r_{2i-1}:=p_{2i-1}^1 p_{2i-1}^2 p_{2i-1}^3 \geqslant p_{2i-1},$$

$$r_{2i}:=p_{2i}^1+p_{2i}^2+p_{2i}^3 \leqslant p_{2i}, \tag{13}$$

(ii)

$$p_2^j \leqslant r_1 r_3 p_5^j p_7^j,\quad p_4^j \leqslant p_1^j r_3 r_5 p_7^j,$$

$$p_6^j \leqslant p_1^{j+1} p_3^j r_5 r_7,\quad p_8^j \leqslant r_1 p_3^j p_5^j r_7 \tag{14}$$

$$p_1^j \geqslant r_2+p_4^j+p_6^{j-1}+r_8,\quad p_3^j \geqslant r_2+r_4+p_6^j+p_8^j,$$

$$p_5^j \geqslant p_2^j+r_4+r_6+p_8^j,\quad p_7^j \geqslant p_2^j+p_4^j+r_6+r_8 \tag{15}$$

(where $p_1^4:=p_1^1$ and $p_1^0:=p_1^3$). Similar relations hold for the q_i^j s.

(as usual, (13) (15) also stipulate the existence of the meets and joins involved).

Proof It is easy to verify that due to the monotonicity of m we have

$$p_1^j \geqslant p_4^j \leqslant p_7^j \geqslant p_2^j \leqslant p_5^j \geqslant p_8^j \leqslant p_3^j \geqslant p_6^j \leqslant p_1^{j+1} \tag{16}$$

($j = 1,2,3$; e.g. $p_1^1 := (137) \geqslant (246) = p_2^1$).

Next $p_1^1 := (137) \geqslant (227) = 2$, $p_1^1 \geqslant (838) = 8$ shows that $p_1^1 \geqslant 2 + 8 = 1$. The same argument shows $p_1^2 = (371) \geqslant 1$, $p_1^3 = (713) \geqslant 1$. It follows that $r_1 := p_1^1 p_1^2 p_1^3$ exists and satisfies $r_1 \geqslant 1$. The proof of the remaining cases in (13) is quite similar.

From (13) we get $p_2^j \leqslant 2$. In the W-cycle $\{1, \ldots, 8\}$ we have $2 \leqslant 1$ and $2 \leqslant 3$, by (13) we get $1 \leqslant r_1$ and $3 \leqslant r_3$ and so $p_2^j \leqslant r_1$ and $p_2^j \leqslant r_3$. Finally from (16) $p_2^j \leqslant p_5^j$ and $p_2^j \leqslant p_7^j$. Thus p_2^j is a lower bound of r_1, r_2, p_5^j and p_7^j and so in the partial lattice $\underline{P}$ the meet $s := r_1 r_3 p_5^j p_7^j$ exists and $s \geqslant p_2^j$. The remaining relations are proved in a similar fashion. □

From (14)–(15) we obtain:

3.3 COROLLARY *Under the assumptions of Lemma 3.2 we have for all $j = 1, 2, 3$*

$$(iii) \quad p_3^j p_5^j p_7^j \geqslant p_2^j + p_4^j + p_6^j + p_8^j, \quad p_2^j + p_4^j + p_8^j \leqslant p_1^j p_3^j p_5^j p_7^j, \tag{17}$$

$$(iv) \quad p_1^{j+1} p_3^j p_5^j p_7^j \geqslant p_6^j, \quad p_2^j + p_4^j + p_6^{j-1} + p_8^j \leqslant p_1^j. \tag{18}$$

We consider a special case:

3.5 COROLLARY *Let $\underline{P}$ have an isotone majority operation m and let $\underline{P}$ contain a W-cycle $\langle 1, \ldots, 8 \rangle$ such that $1, 3, 5, 7$ are pairwise distinct maximal elements of $\underline{P}$. Then*

(*i*) *The meet $0 := 1.3.5.7$ exists, $1.5 = 3.7 = 0$ and $[0) = \{0, 1, \ldots, 8\}$.*
(*ii*) *m is totally symmetric and unique on $\{0, 1, \ldots, 8\}$.*
(*iii*) *If $\{x, y, z, t\} = \{1, 3, 5, 7\}$, then $(xyz) \equiv t + 4$ (all congruences are* mod *8 and the right sides $\neq 0$).*
(*iv*) *If $x, y, z \in \{2, 4, 6, 8\}$ are pairwise distinct, then $(xyz) = 0$.*
(*v*) *Let $x, y \in \{1, 3, 5, 7\}$ and $z \in \{2, 4, 6, 8\}$.*

1) If $y \equiv x + 2$ then $(xyz) \equiv x, y$ provided $z \equiv x - 1,\ y + 1$ and $(xyz) = x + 1$ otherwise.
2) If $y \equiv x + 4$ then $(xyz) = z$.

(vi) *Let* $x \in \{1,3,5,7\}$ *and* $y, z \in \{2,4,6,8\}$, *1) Let* $z \equiv y+2$. *Then* $(xyz) = y,\ y+1,\ y+2,\ 0$ *provided* $x \equiv y-1,\ y+1,\ y+3,\ y+5$. *2) Let* $z \equiv y+4$. *Then* $(xyz) = y$ *provided* $x \equiv y \pm 1$ *and* $(xyz) \equiv z$ *provided* $x \equiv y+3$ *or* $x \equiv y+5$.

(vii) $(0yz) = yz$ *(where* $y \cdot z$ *is the meet in* $[0)$*).*

Proof Clearly in Lemma 3.2 we have

$$p^j_{2i-1} = q^j_{2i-1} = 2i-1\,(i = 1,\dots,4, j = 1,2,3). \tag{19}$$

(i) By (14) the meet 0 exists and so [0) is a meet semilattice.

Since $0 \leqslant 4.6$, from (4), the isotony and (14) we get $4.6 = (046) \leqslant (246) = p^1_4 \leqslant 0$ providing $4.6 = 0$. By a similar argument $2i \cdot 2j = 0$ for all $1 \leqslant i < j \leqslant 4$. Put $a = 1.5$. By the Helly property $b := 2+4+a$ exists. Since $b \geqslant 2+4 = 3$ and 3 is maximal, we have $b = 3$ and $3 \geqslant a$ i.e. $a \leqslant 1.3.5 \leqslant 2.4 = 0$. By symmetry $3.7 = 0$ and so (i) holds

(iii) From (19) for $i = 1$

$$(137) = (731) = (713) = (317) = (371) = (173) = 1 \tag{20}$$

and similarly for $i = 2,3,4$.

(iv) By (14) we have $(246) = p^1_4 \leqslant 0$ and similarly in the other cases.

(v) and (vi). Direct check using isotony, majority property and (iii)

(vii) As $0 \leqslant y \cdot z$ it suffices to apply (4).

(ii) Follows from (iii)–(vii). □

We conclude with the following example of a stiff order.

3.6 *Example* Let $P = \{0,\dots,8\}$ and let $\leqslant$ be defined by $0 < i$ $(i = 1,\dots,8)$ and $1 > 2 < 3 > 4 < 5 > 6 < 7 > 8 < 1$ (i.e. $\underline{P}$ is a crown with an appended least element). The totally symmetric majority operation m defined in Corollary 3.5 is a unique isotone majority operation on $\underline{P}$.

To check this let $x, y, z,\ x', y', z' \in P$, $x \leqslant x'$, $y \leqslant y'$, $z \leqslant z'$ and $(xyz) = t$, $(x'\,y'\,z') = t'$.

A. Let $x = y$. Then $t = x$. If $x = 0$ or $x' = y'$ we have $t \leqslant t'$. Thus let $0 \neq x = y \leqslant x' \neq y'$. Suppose $0 \neq x = x' = y < y'$. By symmetry we may assume $x=2$ and $y'=1$. from (v) we get

$$(213)=(132)=2, \quad (215)=(152)=2, \quad (217)=(712)=2.$$

Similarly applying (vi) we obtain

$$(214)=(124)=2, \quad (216)=(126)=2, \quad (218)=(128)=1$$

and finally $(210)=(012)=1.2=2$ by (viii). Thus for all z' we have $t' \geqq t$. Similarly if $y'=x'$ and so by symmetry we may assume that $x=y=2$, $x'=1$, $y'=3$. Now by (v) 1) we have $t'=(13z') \in \{1,2,3\}$ and $t' \geqq t$.

B. Let $x \neq y \neq z \neq x$. We distinguish 5 cases according to which of (iii)–(vii) in Corollary 1.14 applies to $\{x,y,z\}$.

(iii) If $x,y,z \in \{1,3,5,7\}$ then by maximality $x'=x$, $y'=y$ and $z=z'$ proving $t=t'$.

(iv) If $x,y,z \in \{2,4,6,8\}$, then $t=0 \leqslant t'$.

(v) Let $x,y \in \{1,3,5,7\}$ and $z \in \{2,4,6,8\}$. Then $x'=x$ and $y'=y$. If $z'=z$ then clearly $t=t'$. Thus let $z<z'$. 1) Let $y \equiv x+2$, e.g. $x=1$, $y=3$. If $z=8$ then $t=(138)=1$. For $z'=1$ we have $t'=(131)=1$ while for $z'=7$ also $t'=(137)=1$ (by (iii)). Let $z=4$. Then $t=(134)=3$ while t' is either $(133)=3$ or $(135)=3$. For $z \in \{2,6\}$ we have $t=(13z)=2$. On the other hand $t'=(13z')$ is one of $(131)=1$, $(133)=3$, $(135)=3$ and $(137)=1$ i.e. $t'>t$.

2) If $y \equiv x+4$ then by (v)2) we have $t=(xyz)=z<z'=(xyz)=t'$.

(vi) Let $x \in \{1,3,5,7\}$ and $y,z \in \{2,4,6,8\}$.

1) Let $z \equiv y+2$ e.g. $y=2$, $z=4$. If $x=7$ we have $t=0$ and we are done. Note that again by maximality $x'=x$. a) For $x=1$ we have $t=(124)=2$. If $y'=1$ we have $t'=1$. If $y'=3$ then t' is one of (133), (134), (135) which are all equal to 3. Thus let $v'=2$. Moreover, $(123)=(125)=2$. b) Let $x=3$. Then $(324)=3$. If $y'=3$ clearly $t'=(33z')=3$. If $y'=1$, then t' is one of (313), (314), (315) which all equal 3. Finally, if $y'=2$, then t' is among (323), (324) and (325) that all equal 3. c) Let $x=5$. Then $t=(524)=4$. If $y'=1$, then t' is one of (513), (514), (515), i.e. $t' \in \{3,4,5\}$ as required. If $y'=2$ then t' is among (523), (524) and (525), i.e. $t' \in \{3,4,5\}$. Finally if $y'=3$ then t' is one of (533), (534) and (535), i.e. $t' \in \{3,4,5\}$.

2) Let $z \equiv y+4$ e.g. $y=2$, $z=0$. A) Let $x \in \{1,3\}$. Then $t=\{x26)=2$ and $t'=(xy'z')$ is one of the values of m on $\{1,3\} \times \{1,2,3\} \times \{5,6,7\}$. A direct check shows that these are all in $\{1,2,3\}$. b) Let $x \in \{5,7\}$. Then $t=(x26)=6$ and $t'=(xy'z')$ is one of the values of m on $\{5,7\} \times \{1,2,3\} \times \{5,6,7\}$. It may be verified that those are $\{5,6,7\}$.
(vii) Let $x=0$. Then $t=yz \leqslant y'z' \leqslant t'$. □

References

[1] Baker, K. A. and Pixley, A. F. (1975). Polynomial interpretation and the Chinese remainder theorem for algebraic systems, *Math. Z.*, **143**, 165–174.

[2] Benzaken, C. (1978). The Post's closed systems and the weak chromatic number of hypergraphs, *Discrete Math.*, **23**, 77–84.

[3] Benzaken, C. (1980). Critical hypergraphs for the weak chromatic number, *J. Combin. Theory*, **29**, 328–330.

[4] Banzaken, C. (1980). Hypergraphes critiques pour le nombre chromatique et conjecture de Lovász, Combinatorics 79 (Deza, M.; Rosenberg, I. G. eds.), *Annals of Discrete Math.*, North Holland, 91–100.

[5] Demetrovics, J., Hannák, L. and Rónyai, L. (1984). Near unanimity functions of partial orderings, Proceedings of 15th Internat. Sumpos. Multiple-valued Logic, Winnipeg, May, IEEE, 52–56.

[6] Jablonskiĭ, S. V.(1958). Functional constructions in k-valued logic, *Trudy Mat. Inst. Steklov*, **51**, 5–142.

[7] Martynjuk, V. V. (1960). Investigation of a certain class of functions in many-valued logics (Russian), *Problemy Kibernet*, **3**, 49–60.

[8] McKenzie, R., McNulty, G. and Taylor, W. F. (1987). Algebras, lattices, varieties, **1**, Wadsworth & Brooks/Cole.

[9] Pöschel, R. and Kaluznin, L. A. (1979). *Funktionen- und Relationen–Algebren. Ein Kapitel der Diskreten Mathematik*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1979. Birkhauser, V. Basel, Stuttgart, Math. R. B. **67**.

[10] Post, E. L. (1921). Introduction to a general theory of elementary proposition, *Am. J. Math.*, **43**, 163–185.

[11] Post, E. L. (1941). The two-value iterative systems of mathematical logic, *Annals of Math. Studies*, **5**, Princeton Univ. Press.

[12] Quackenbush, R. W., Rival, I. and Rosenberg, I. G. (1990). Clones, order varieties, near unanimity functions and holes, *Order*, **7**, 239–48.

[13] Rosenberg, I. G. (1970). Über die funktionale Vollständigkeit in den mehrwertigen Logiken (Struktur der Funktionen von mehreren Veränderlichen auf endli-chen Mengen), *Rozpr. Česk, Akad, Věd*, *Řada Mat. Přirod. Věd*, **80**, 3–93.

[14] Rosenberg, I. G. (1984). Completeness properties of multipe-valued logic algebras, in: *Computer Science and Multiple-valued Logic* (ed. D. C. Rine), North Holland, Amsterdam-New York-Oxford (1977), Second (revised) edition, North Holland Elsevier, 150–192.

[15] Rosenberg, I. G. (1986). Minimal clones, I: The five cases, Lectures in Universal Algebra (Szabo L., Szendrei, A. eds). *Colloq. Math. Soc. J. Bolyai*, **43**, 405–427.

[16] Rosenberg, I. G. and Schweigert, D. (1982). Local clones, *Elektron. Informationsverarb. u. Kybernetik*, **18**(7–8), 389–401.

[17] Rosenberg, I. G. and Szabó, L. (1984). Local completeness I, *Algebra Universalis*, **18**, 308–326.

[18] Rosenberg, I. G. and Schweigert, D. (1984). Compatible orderings and tolerances of lattices, in Orders: Description and Roles (Pouzet, M. Richard, D., eds.) *Annals of Discrete Mathematics*, **23**, North Holland, pp. 119–150.
[19] Rosenberg, I. G. (April 1985). Near unanimity orders, Preprint CRM-1274 31 pp.
[20] Tardós, G. (1986). A maximal clone of monotone operations which is not finitely generated. *Order*, **3**(3), 211–218.
[21] Zádori, L. (1995). Monotone Jónsson operations and near unanimity functions. Algebra Universalis, **33**, 216–36.