On Representing Sylvester–Gallai Designs

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Abstract. A misstated conjecture in [3] leads to an interesting "$(1, 3)$ representation" of the 7-point projective plane in R^4 where points are represented by lines and planes by 3-spaces. The corrected form of the original conjecture will be negated if there is a $(1, 3)$ representation of the 13-point projective plane in R^4 but that matter is not settled.

1. Introduction

A Sylvester–Gallai (SG) design consists of two finite sets P (points) and L (lines) and an incidence relation such that each two points are incident with exactly one line and each line is incident with at least three points. Finite projective planes are examples of such designs. SG designs were introduced and studied in [4].

An SG configuration in a projective or affine space is a finite set of points such that the line joining any pair of points of the set contains at least one more point. *Representations* of SG designs by SG configurations is of obvious interest and we are also interested in more liberal representations where points of the design are represented by subspaces of dimension m and lines of the design are represented by subspaces of dimension n. We refer to such representations as (m, n) representations. SG configurations can thus be regarded as $(0, 1)$ representations.

The well known Sylvester–Gallai theorem asserts that any SG configuration in an ordered projective or affine space is a subset of a single line. However, in complex projective 2-space it is a classic bit of lore that the nine inflection points of a nondegenerate cubic curve are points of an SG configuration. It is not hard to generalize this example to sets in C^2 of cardinality $3k$ for any integer $k > 3$.

In 1966 Serre [5] suggested that SG configurations in C^n might be confined to the plane and this was recently confirmed in [3]. The proof, however, invoked

a presumably deep result in complex manifold theory and prompted speculation that a simpler proof might be contrived if this C^2 theorem were given a conventional real 4-space interpretation in which points of C^2 correspond $(1-1)$ to points in R^4 but lines of C^2 correspond to a (proper) subset of planes in R^4. Since each two lines in C^2 intersect in a single point each two planes of this special family must also have only one point in common. This family, or any family projectively equivalent to it, will be called a set of C-planes of R^4. The dual of a set of C-planes in R^4 is a set of pairwise skew lines which are referred to as a set of C-lines of R^4.

The key fact needed in [3] to establish the Serre result was a corollary of an inequality of Hirzebruch [1] to the effect that a nonlinear SG configuration in C^2 must be intersected by some line in exactly three points. Interpreting this in the R^4 setting we have

Theorem. *If a finite family of C-lines in R^4 is such that the 3-space containing any two of the lines contains at least two more, then the family is in a single 3-space.*

This is obtained by first dualizing in C^2, interpreting the result in R^4, and finally dualizing in that setting. This leads to the obvious conjecture that the theorem might be true in R^4 without the proviso that the family be a subset of C-lines. That is:

Conjecture. *If a finite family of pairwise skew lines in R^4 is such that the 3-space containing any two of the lines contains at least two more, then the family is in a single 3-space.*

A conjecture in [3] was actually a misstatement of this where "at least two more" was inadvertently replaced by "at least one more." That conjecture was clearly false since any nonlinear SG configuration in C^2 gives rise to a counterexample in R^4. For example, the existence of the nine-inflection-point SG configuration in C^2 implies the existence in R^4 of a set of nine pairwise skew lines spanning R^4 such that the 3-space containing any pair contains exactly one more line of the set. In representation language this is a $(1,3)$ representation in R^4 of a 9-point, 12-line SG design.

The misstated conjecture had the desirable effect of prompting the discovery of a $(1,3)$ representation in R^4 of a nonlinear SG design which is not obtainable as a transform of a nonlinear SG configuration in C^2 and suggested a construction technique which might produce a counterexample to the (properly stated) conjecture.

2. A $(1,3)$ Representation of the Seven-point Projective Plane

Let A and B be two different hyperplanes in R^4 intersecting in a plane C, let $\gamma_1, \ldots, \gamma_7$ be seven lines in general position in C, let $\alpha_1, \ldots, \alpha_7$ be seven planes in general position in A, and let $\beta_1, \ldots, \beta_7$ be a similar set of planes in B and assume $\alpha_i \cap \beta_i = \gamma_i$.

Now let $x_1, \ldots, x_7$ be the seven lines of a Fano plane, F. If x_i, x_j, x_k concur in F define $P_{ijk} = \alpha_i \cap \alpha_j \cap \alpha_k$ and $Q_{ijk} = \beta_i \cap \beta_j \cap \beta_k$. The seven lines l_{ijk} in R^4 joining P_{ijk} and Q_{ijk} may or may not be pairwise skew but if they are not, considerations of continuity and the wide latitude available in the choice of α_i, β_i, γ_i should make it clear that judicious choices will produce a pairwise skew set. For the skeptical we produce a specific example below.

Granting this, the other relevant properties of the lines l_{ijk} are immediate. That is they span R^4 and the 3-space containing any two contains exactly one more.

3. Numerical Example

	P_{ijk}					Q_{ijk}		
(−40,	4,	0,	10)		(0,	−6,	−30,	0)
(4,	−24,	0,	16)		(12/11,	72/11,	191/11,	0)
(108,	168,	0,	−72)		(−24,	−24,	−48,	0)
(−36/7,	36/7,	0,	−60/7)		(−12,	0,	36,	0)
(10,	−4,	0,	2)		(7,	−6,	4,	0)
(−12,	30,	0,	−3)		(−12,	30,	6,	0)
(−15,	5,	0,	11)		(−15,	−6,	−33,	0)

A direct calculation shows that the seven lines l_{ijk} joining P_{ijk} and Q_{ijk} are pairwise skew, span R^4, and the hyperplane in R^4 containing any pair contains exactly one more.

4. Remarks

A $(1, 3)$ representation in R^4 of the 13-point projective plane seems possible but so far our efforts to exploit these techniques have been inconclusive. If such a representation exists it would invalidate our conjecture and show that membership in the C-line set was very relevant and that the Serre result is probably intimately involved with deeper algebraic properties of C^2.

Since SG configurations in C^n are confined to the plane and the only nonlinear examples which seem to be known are those of cardinality $3k$ and $3k+1$, $k \geq 3$, a more complete classification could be revealing. Further work on $(1, 3)$ representations of SG designs in R^4 is in progress.

Finally we make an observation correcting an impression given in [3]. All proofs we know of the original Sylvester theorem employ the order structure of the space. Those interested in foundational studies have long wondered whether adjoining the Sylvester assumption to the basic projective axioms would produce a space in which the order axioms could be proved. This was shown not to be the case in 1984 by Joussen [2]. This fact deserves to be better known.

References

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