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ON THE NUMBER OF HALVING PLANES

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Let $S \subset \mathbf{R}^3$ be an n-set in general position. A plane containing three of the points is called a halving plane if it dissects S into two parts of equal cardinality. It is proved that the number of halving planes is at most $O(n^{2.998})$.

As a main tool, for every set Y of n points in the plane a set N of size $O(n^4)$ is constructed such that the points of N are distributed almost evenly in the triangles determined by Y.

1. Halving planes

A point-set $S \subset \mathbf{R}^d$ is *in general position* if no $d+1$ points of it lie in a hyperplane. The plane determined by the non-collinear points a, b, c is denoted by $P(a,b,c)$. In general, the affine subspace spanned by the set A is denoted by aff (A). As usual, conv(A) stands for the convex hull of A.

Assume that S is an n-element point-set in the three-dimensional Euclidean space in general position. A plane $P(a,b,c)$, where $a,b,c \in S$, is called a *halving plane* if it dissects S into two equal parts, that is, on both sides of P there are exactly $(n-3)/2$ points of S. Denote the number of halving planes by $h(S)$, and set

$$h(n) = \max\{h(S) : S \subset \mathbf{R}^3,\ |S| = n,\ S \text{ is in general position}\}.$$

Clearly, $h(n) \le \binom{n}{3}$. The aim of this paper is to improve this trivial bound proving

Theorem 1. $h(n) \le O(n^{2.998})$.

The proof which is postponed to section 7 is similar to that of the 2-dimensional case given in [9], but the crucial step requires new tools (Theorem 2.). Actually, we will prove $h(n) \le O(n^{3-a})$ with $a = 1/343$. (With more effort, one could prove the result with $a = 1/64$.)

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Define $h_2(n)$ as the maximum number of halving lines of a planar n-set. It is well-known [6] that $h_2(n) \geq \Omega(n \log n)$. This result is used in [4] to give an example proving

$$h(n) \geq \Omega(n^2 \log n).$$

2. Covering most of the triangles by crossings

A point-set S in $\mathbf{R}^d$ is said to be in *totally general position* if

$$\dim \left(\bigcap_{i=1}^{s} \operatorname{aff} A_i \right) \leq \max \left\{ -1, \sum \dim(A_i) - (s-1)d \right\}$$

holds for all subsets $A_i \subset S$. From now on we always suppose, if it is not otherwise stated, that the (finite) point-sets are in totally general position. A set F *covers* t triangles from the set $Y \subset \mathbf{R}^2$ if at least t open triangles (y_1, y_2, y_3) (where $y_i \in Y$) contain a point of F. Obviously, no set can cover more than $\binom{|Y|}{3}$ triangles.

Theorem 2. *For every n element set $Y \subset \mathbf{R}^2$ there exists a set F with $|F| < n^{0.998}$ which covers all but at most $O(n^{2.998})$ triangles from Y.*

Two lines determined by four distinct points of Y intersect in a *crossing*. Define $C(Y)$ as the set of crossings. We have

$$|C(Y)| = \frac{1}{2}\binom{n}{2}\binom{n-2}{2} = \Theta(n^4).$$

Let $N(R)$ denote the number of crossings in the interior of the region R, and $N(abc) = N(\operatorname{conv}(a, b, c))$.

It is perhaps instructive to show at this step that the average number of crossings in a triangle with vertices from Y is $\Omega(n^4)$. Our first observation is that every set of nine points, $E \subset Y$, contains a triangle such that at least one of the crossings defined by four of the remaining 6 points lies inside the triangle. Indeed, a theorem of Tverberg [12] (cf. also Reay [11]) states that there is a partition $\{a_1, b_1, c_1\} \cup \{a_2, b_2, c_2\} \cup \{a_3, b_3, c_3\} = E$ such that the intersection of the three triangular regions $\operatorname{conv}(a_i b_i c_i) (1 \leq i \leq 3)$ is non-empty. Then $\cap_i \operatorname{conv}(a_i b_i c_i)$ is a convex polygon. Assume that the line $a_3 b_3$ contains an edge of this polygon. The prolongation of this edge in any direction will leave one of the triangles $\operatorname{conv}(a_1 b_1 c_1)$ or $\operatorname{conv}(a_2 b_2 c_2)$ first; assume it leaves $\operatorname{conv}(a_2 b_2 c_2)$ first, at a point p. Then p is a crossing, defined by four of the points $a_2 b_2 c_2 a_3 b_3$, and it is contained in the triangle $\operatorname{conv}(a_1 b_1 c_1)$.

So every nine-tuple from Y contains an (ordered) seven-tuple $abcxyuv$ such that $(\operatorname{aff}\ xy \cap \operatorname{aff}\ uv) \in \operatorname{int} \operatorname{conv}(abc)$. As every seven-tuple is contained in $\binom{n-7}{2}$ nine-tuples we have that the number of suitable seven-tuples is at least

$\binom{n}{9} \Big/ \binom{n-7}{2} = \frac{1}{36}\binom{n}{7}$. Hence we have

$$\text{(1)} \qquad \underset{a,b,c\,\in Y}{\text{Average}}\ N(abc) = \frac{1}{\binom{n}{3}} \sum_{a,b,c\in Y} \sum_{\substack{x,y,u,v\in Y\setminus\{a,b,c\} \\ \text{aff } xy \cap \text{aff } uv \in \text{int conv}(abc)}} 1$$
$$= \frac{1}{\binom{n}{3}} (\#\text{ suitable seven-tuples}) \geq \binom{n-3}{4} \Big/ 1260.$$

∎

Unfortunately, this computation is not enough to guarantee that most triangles contain $\Omega(n^4)$ crossings. For this we need a colored version of Tverberg's theorem:

Lemma 3. *There is a positive integer t such that the following holds. Assume that $A, B, C \subset \mathbf{R}^2$ are disjoint sets with at least t elements each, such that their union is in general position. Then there exist three disjoint triples $a_i b_i c_i$, $a_i \in A$, $b_i \in B$, $c_i \in C$ $(1 \leq i \leq 3)$ such that $\cap_i \text{conv}(a_i b_i c_i) \neq \emptyset$.*

The smallest value of t for which we managed to prove this lemma is 4, and we do not have a counterexample even for $t = 3$. For brevity's sake we give the proof for $t = 7$.

The other tool of the proof is the following lemma, which strengthens the averaging in (1). This lemma will imply that the number of triangles with vertices from Y containing "few" crossings is "small".

Lemma 4. *Let t satisfy the previous Lemma. Then there exist positive constants c' and c'' such that the following holds. Assume that $1 \leq k \leq c' n^{1/t^2}$, and $\mathcal{H}$ is a set of triples from Y with $|\mathcal{H}| > \binom{n}{3}/k$. Then the average number of crossings in the members of $\mathcal{H}$ is at least $c'' n^4 / k^{t^3-1}$.*

3. Corollaries and a polynomial algorithm

In this section t is a value that satisfies Lemma 3, $k \geq 1$ and $Y \subset \mathbf{R}^2$ is an n-element set in general position. A straightforward application of Lemma 4 yields the following covering theorem, where c is again an absolute constant.

Theorem 5. *There is a set $F \subset \mathbf{R}^2$ of size at most ck^{t^3-1} such that the number of triangles with vertices from Y containing no point of F is at most $\binom{n}{3}/k$.*

It is interesting to compare Theorem 5 to a result from [3], which states that there is a point contained in at least $\frac{2}{9}\binom{n}{3}$ triangles from Y. (For higher dimensions, see [1].) The covering set F in Theorem 5 is obtained by a random process. We have a deterministic, polynomial time algorithm to construct a suitable F as well, but now F will have larger size:

Theorem 6. *There is an algorithm, polynomial in n, which supplies a set F with $|F| \leq \exp(c'k^{9000})$ such that the number of triangles from Y containing no point of F is at most $\binom{n}{3}/k$.*

Here c' is another absolute constant. The following corollary of Theorem 5 concerns the difference between the behavior of a continuous and a discrete measure of the planar convex regions.

Theorem 7. *There is a set $F \subset \mathbb{R}^2$ of size at most $c'k^{3t^3-3}$ such that any convex region R with $|R \cap Y| \geq n/k$ contains a point of F.*

This follows from Theorem 5, as if $|R \cap Y| \geq n/k$, and $F \cap Y = \emptyset$, then F avoids at least $\binom{n/k}{3}$ triangles.

4. The proof of Lemma 3

Lemma 8. *Let $E_1, E_2, E_3 \subset \mathbb{R}^2$ be finite nonempty subsets and p any point. Then p is not contained in any triangle $\operatorname{conv}(e_1e_2e_3)$ with $e_i \in E_i$ if and only if there exist a $k \in \{1,2,3\}$ and two closed halfplanes H, H' such that $p \notin H' \cup H''$, $E_i \subset H' \cap H''$ if $i \neq k$ and $E_k \subset H' \cup H''$.*

Proof. By Theorem 2.3 of [1], p is contained in a triangle $\operatorname{conv}(e_1e_2e_3)$ if it is contained in the convex hull of two of the sets E_1, E_2, E_3. So we may suppose that $p \notin \operatorname{conv} E_i$ for $i = 1, 2$, say. Write C_1 and C_2 for the smallest cone containing E_1 and E_2 and having apex p. It is easy to see that if $C_1 \cup C_2$ contains a line, then p is contained in a triangle $\operatorname{conv}(e_1e_2e_3)$. Then the smallest cone containing $C_1 \cup C_2$ and having apex p is of the form $H_1 \cap H_2$, where H_1 and H_2 are two halfspaces. It follows readily that H_1 and H_2 satisfy the requirements. ∎

Let $U = A \cup B \cup C$, $|U| = 3t \geq 21$, $i \geq 0$. The *i-th convex hull*, $\operatorname{conv}_i(U)$, is the intersection of all the (open) halfplanes containing at least $|U| - i$ elements of U. Then $\operatorname{conv}_i(U)$ is a convex polygonal region for $0 \leq i \leq t-1$. Let p be a point from $\operatorname{int}\operatorname{conv}_{t-1}(U)$, such that $U \cup \{p\}$ is in general position. Then for all open halfplanes H we have that

$$p \in H \text{ implies } |H \cap U| \geq t. \tag{2}$$

We claim that p is contained in at least three vertex-disjoint multicolored triangles of U.

Suppose, to the contrary, that one can find only s ($s = 0, 1$ or 2) triangles $a_ib_ic_i$ ($i = 1, \ldots, s$) such that $p \in \operatorname{conv}(a_ib_ic_i)$. Let $U' = U \setminus \{a_i, b_i, c_i : i \leq s\}$, $A' = A \cap U'$ and so on. We have $|U'| = 3t - 3s$. Apply Lemma 8 to A', B', C' and p. We obtain two halfplanes H', H'' such that (say) $A' \cup B' \subset H' \cap H''$, $C' \subset H' \cup H''$, and $p \notin H' \cup H''$. The complementary halfplanes $\overline{H'}$ and $\overline{H''}$ both contain at most $2s$ points from $\{a_i, b_i, c_i : i \leq s\}$. One of them, say $\overline{H'}$ contains only at most one half of the points of C' from U'. So altogether $\overline{H'}$ contains at most $2s + (t-s)/2$ points of U. This contradicts (2) as $t \geq 7 > 3s$. ∎

5. The proof of Lemma 4.

A *hypergraph* $\mathbf{H}$ is a pair $\mathbf{H}=(V,\mathcal{E})$, where V is a finite set, the set of *vertices*, and $\mathcal{E}$ is a family of subsets of V, the set of *edges*. If all the edges have r elements, then $\mathbf{H}$ is called *r-graph*, or r-uniform hypergraph. The complete r-partite hypergraph $\mathbf{K}(t_1, t_2, \ldots, t_r)$ has a partition of its vertex set $V = V_1 \cup \cdots \cup V_r$, such that $|V_i| = t_i$, and $\mathcal{E} = \{E : |E \cap V_i| = 1\}$ for all $1 \le i \le r$. Erdős [5] proved the following theorem in an implicit form. (More explicit formulations are given in Erdős and Simonovits [7] or in Frankl and Rödl [8]).

Lemma 9. *For any positive integers r and $t_1 \le \ldots \le t_r$ there exist positive constants c' and c'' such that the following holds. If $\mathbf{H}$ is an arbitrary r-graph with n vertices $e > c'n^{r-\varepsilon}$ edges where $\varepsilon = 1/(t_1 \cdots t_{r-1})$, then $\mathbf{H}$ contains at least*

$$c'' \frac{e^{t_1 \cdots t_r}}{n^{r t_1 \cdots t_r - t_1 - \ldots t_r}}$$

copies of $\mathbf{K}(t_1, \ldots, t_r)$. ∎

Now consider the triangle system $\mathcal{H}$, and consider it as a 3-regular hypergraph with vertex set Y. Lemma 9 implies that there is a constant c_1 such that the number of copies of $\mathbf{K}(t,t,t)$ in $\mathcal{H}$ is at least

$$c_1 n^{3t} / k^{t^3} \tag{3}$$

Every copy of $\mathbf{K}(t,t,t)$ contains three multicolored triangles with a common interior point so, as we have seen in the argument leading to (1), it contains a *suitable* seventuple, i.e., seven distinct points $\{a,b,c,x,y,u,v\}$ such that $\{a,b,c\} \in \mathcal{H}$ and $(\operatorname{aff} xy \cap \operatorname{aff} uv) \in \operatorname{conv}(abc)$. Then, by (3), the total number of suitable seven-tuples is at least $(c_1 n^{3t}/k^{t^3}) / \binom{n-7}{3t-7} \ge c_2 n^7 / k^{t^3}$. Then, as in (1), one has

$$\underset{\{a,b,c\} \in \mathcal{H}}{\text{Average}} \quad N(abc) \ge \frac{1}{|\mathcal{H}|}(\#\text{ suitable seven-tuples}) \ge c_3 n^4 / k^{t^3 - 1}.$$ ∎

6. The proofs of Theorem 5 and 2

As Theorem 2 is a trivial corollary of 5 (with $k = cn^{1/t^3}$) we turn to the proof of Theorem 5. Define the triangle system $\mathcal{H}(i)$ as the set of triangles $\{a,b,c\} \subset Y$ satisfying

$$c'' n^4 \frac{i}{4k}^{t^3 - 1} \le N(abc) < c'' n^4 \left(\frac{i+1}{4k}\right)^{t^3-1}$$

for $i = 0, 1, \ldots$. If $k > (2n)^{1/(t^3-1)}$, then the bound on $|F|$ is larger than $2n$, and it is easy to see ([10] or [2]) that $2n$ points are always sufficient to cover all triangles from Y. So we may suppose that $k \le (2n)^{1/(t^3-1)} \le c' n^{1/t^2}$. Then Lemma 4 implies that

$$|\mathcal{H}(0)| + |\mathcal{H}(1)| + \cdots + |\mathcal{H}(i)| \le \binom{n}{3} \frac{i+1}{4k}. \tag{4}$$

Now we are going to give a random construction for the covering set F. A crossing from $C(Y)$ is put into F with probability p, where $p = \alpha k^{t^3-1} n^{-4}$ and α is an absolute constant to be fixed later. The expected number of points in F is

$$E(|F|) = p\frac{1}{2}\binom{n}{2}\binom{n-2}{2} \leq \frac{\alpha}{8} k^{t^3-1}.$$

We estimate the expected number of triangles from $\mathcal{H}(i)$ containing no point of F:

$$(5) \qquad X_i =: E(\#\{a,b,c\} \in \mathcal{H}(i) : F \cap \operatorname{conv}(abc) = \emptyset) = \sum_{abc \in \mathcal{H}(i)} (1-p)^{N(abc)} \leq |\mathcal{H}(i)|(1-p)^{\min N(abc)}$$
$$\leq |\mathcal{H}(i)| \exp(-p \min N(abc)) \leq |\mathcal{H}(i)| \exp(-\alpha c''(i/4)^{t^3-1}).$$

Then (4) and (5) imply that

$$(6) \qquad \sum X_i \leq \binom{n}{3}\frac{1}{4k}\sum_{i\geq 0} e^{-\alpha c'' i/4} < \binom{n}{3}\frac{1}{4k}\left(1 + \frac{4}{\alpha c''}\right).$$

Then the expectation of the random variable

$$\frac{|F|}{\alpha k^{t^3-1}/8} + \frac{\sum X_i}{\binom{n}{3}\frac{1+4/\alpha c''}{4k}}.$$

is less than 2. So there is a choice of F such that $|F| \leq \alpha k^{t^3-1}/4$ and the number of triangles avoiding F is at most $\binom{n}{3}\frac{1+4/\alpha c''}{2k}$. Choosing α properly ($\alpha = 4/c''$), one obtains Theorem 5. ∎

7. The proof of the main Theorem 1

We have to prove the upper bound. Suppose that $S \subset \mathbf{R}^3$ is an n-set and $P(abc)$, and $P(abd)$ are halving planes, $\{a,b,c,d\} \subset S$. Let H_c and H_d be halfspaces with boundary planes $P(abc)$ and $P(abd)$, resp., such that $\{a,b,c,d\} \subset H_c \cap H_d$. Then there is a point $x \in S$ outside of $H_c \cup H_d$ such that abx is again a halving triangle. This can be seen by rotating (around the line ab) $P(abc)$ into $P(abd)$.

Now take a plane Q in general position with respect to S, and consider Y, the image of S on Q under orthogonal projection. Denote the system of images of the halving triangles in S by $\mathcal{H}$. Let χ denote the sum of the characteristic functions of the open triangles in $\mathcal{H}$. By the above observation, χ changes by at most 1 when one crosses a segment uv with $u, v \in Y$. This means that χ is at most $\binom{n}{2}$. To put this differently, every line orthogonal to Q and in general position with respect to S intersects at most $\binom{n}{2}$ halving triangles of S.

Let F be a point set according to Theorem 2. Then

$$|\mathcal{H}| \leq |F|\binom{n}{2} + (\#\text{ empty triangles}) \leq O(n^{3-1/t^3}).$$ ∎

8. Sketch of the algorithm in Theorem 6

Lemma 10. *Assume that A, B and C are sets in general position in the plane. Then there are subsets $A' \subset A, B' \subset B$ and $C' \subset C$ with $|A'| \geq |A|/12, |B'| \geq |B|/12$ and $|C'| \geq |C|/12$ and a point p such that p is contained in all triangles abc whenever $a \in A', b \in B'$ and $c \in C'$.*

Proof. First, for any direction l, one can find two lines l_1 and l_2 parallel to l which divide $\mathbf{R}^2$ into three regions R_0, R_1 and R_2 (where R_0 and R_2 are halfplanes with boundaries l_1 and l_2, resp., and R_1 is a strip), such that each R_i contains one third of the points of some color class. Say, e.g., $A_1 =: A \cap R_0, |A_1| \geq |A|/3$ and $B_1 =: B \cap R_1, |B_1| \geq |B|/3$, finally $C_1 =: C \cap R_2, |C_1| \geq |C|/3$. By the Ham-Sandwich theorem, there exists a line l_3 that divides both A_1 and C_1 into almost equal parts. Denote by H_3 the halfplane with boundary l_3 and containing the larger part of B_1. Then let $B_2 =: B_1 \cap H_3$, we have $|B_2| \geq |B_1|/2 \geq |B|/6$ and $A_2 =: A_1 \setminus H_3, |A_2| \geq |A|/6$, finally $C_2 =: C_1 \setminus H_3, |C_2| \geq |C|/6$.

One can divide A_2 into two equal parts by a halfline h_1 starting from the intersection point of l_2 and l_3. Similarly, a halfline h_2 parallel to l divides B_2 into two equal parts, and finally, a halfline h_3 starting from the point $l_1 l_3$ divides C_2. Then consider the triangle T formed by h_1, h_3 and the continuation of h_2. The sides of T divide the plane into 7 regions. Let A' the part of A_2 contained in the region with 2 sides. The definition of B' and C' are similar. Then every point $p \in T$ satisfies the requirements in the Lemma. ∎

For the proof of Theorem 6 we only need from the above argument that for arbitrary sets A, B and C there is a point p which is contained in at least

$$|A||B||C|/1728 \tag{7}$$

triangles abc with $a \in A, b \in B, c \in C$; and moreover, it is easy to find such a point p algorithmically. We mention here that Lemma 10 implies Lemma 3 with $t = 36$.

Suppose we have an algorithm supplying a cover F, which avoids at most $\binom{n}{3}/k$ triangles of Y. Let $|F| = f(k)$ or briefly f.From any point $x \in F$ one can start halflines $h_1(x), h_2(x), \ldots, h_m(x)$ such that the cone defined by $h_i(x)$ and $h_{i+1}(x)$ contains about n/m points of Y. Let $R_1, R_2, \ldots, R_M$ be the cell-decomposition of the plane defined by the halflines $\{h_i(x) : x \in F, 1 \leq i \leq m\}$. Then $M \leq (fm)^2$. Call a three-tuple of the regions R_a, R_b, R_c *uncovered* if all triangles $\text{conv}(xyz)$ with $x \in R_a, y \in R_b$ and $z \in R_c$ avoid F. They are *covered* if all triangles contain a point from F, and *ambiguous* if both of the above constraints fail.

The number of triangles from Y which are covered by at most 2 regions R_i, R_j is at most $O(1/m)\binom{n}{3}$. It is easy to see that the number of triangles in ambiguous triples is at most $(9f/m)\binom{n}{3}$.

The above Lemma yields a point $p(a,b,c)$ for each uncovered triple $R_aR_bR_c$. Then, the set $F \cup \{p(a,b,c) : 1 \le a,b,c \le M\}$ avoids less than

$$\binom{n}{3}\left(\left(\frac{1}{k}-\frac{10f}{m}\right)\frac{1727}{1728}+\frac{10f}{m}\right) \tag{8}$$

triangles by (7). If we choose $m = 20fk$, then (8) gives $3455\binom{n}{3}/3456k$. This leads to the recursion

$$f\left(\frac{3456}{3455}k\right) \le f + \binom{M}{3} \le (fm)^6 = O(f(k)^{12}k^6). \tag{9}$$

9. Problems

Define $h_d(n)$ as the maximum number of halving hyperplanes in d dimensions. The construction in [4] gives

$$h_d(n) = \Omega(nh_{d-1}(n)) = \Omega(n^d \log n).$$

The above arguments would give that for some $c = c(d)$ one has

$$h_d(n) = O(n^{d-c}). \tag{?}$$

The only thing that is missing for this is a d-dimensional version of Lemma 3. In this d-dimensional version we would need $d+1$ multicolored simplices with a common point.

It would be interesting to find the higher dimensional analogues of Lemma 8 and of the algorithm in Theorem 6.

What is $\varepsilon(f)$, the maximum ratio of the number of covered triangles by f points? The only known value, as it was mentioned, is $\varepsilon(1) = 2/9$. We conjecture that $\varepsilon(\sqrt{n}) = O(1/\sqrt{n})$.

Of course, it would be also interesting to find the best value for the constants in our lemmas, like in Lemma 3.

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