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EXTREMAL PROBLEMS AND THE LAGRANGE FUNCTION FOR HYPERGRAPHS

BY

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Abstract. With every family $\mathscr{G}$ of k-subsets of $\{1, 2, \ldots, n\}$ one associates a polynomial, homogeneous of degree k. An extremal value of this polynomial called the Langrange function of $\mathscr{G}$, has an important combinatorial signification. We describe ways of estimating this function and using it for solving some extremal problems.

1. Introduction. Let $X = \{1, 2, \cdots, n\}$ be a finite set, $n \geq k \geq 2$ and $\mathscr{G}$ a k-graph, that is $\mathscr{G} \subset \binom{X}{k} = \{A \subset X : |A| = k\}$.

One can associate with every k-graph $\mathscr{G}$ a polynomial

$$(1.1) \qquad p(\mathscr{G}) = p(\mathscr{G}, x_1, \cdots, x_n) = \sum_{F \in \mathscr{G}} \prod_{i \in F} x_i.$$

Note that $p(\mathscr{G})$ is homogeneous of degree k and linear in every variable. Also, one has

$$(1.2) \qquad p\left(\mathscr{G}, \frac{1}{n}, \cdots, \frac{1}{n}\right) = |\mathscr{G}|/n^k.$$

Often we will write $p(\mathscr{G}, \vec{x})$ to abbreviate $p(\mathscr{G}, x_1, \cdots, x_n)$.

Let us define the operation called *blow-up*. Suppose that $\mathscr{G} \subset \binom{X}{k}$ and $m_1, \cdots, m_n$ are non-negative integers. Let $X_1, \cdots, X_n$ be pairwise disjoint sets, $|X_i| = m_i$. Set $\vec{m} = (m_1, \cdots, m_n)$. We define

$$\mathscr{G} \otimes \vec{m} = \left\{G \in \binom{X_1 \cup \cdots \cup X_n}{k} : \{i : G \cap X_i \neq \emptyset\} \in \mathscr{G}\right\}.$$

Note that $|G| = k$ implies that $|G \cap X_i| = 0$ or 1 for every i and for every edge G of $\mathscr{G} \otimes \vec{m}$. Set $m = m_1 + \cdots + m_n$.

(1.3) CLAIM. $|\mathscr{G} \otimes \vec{m}| = m^k p(\mathscr{G}, m_1/m, \cdots, m_n/m)$.

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Proof. If $\{i_1, \cdots, i_k\} \in \mathcal{G}$ then this gives $m_{i_1} \cdot \cdots \cdot m_{i_k}$ edges in $\mathcal{G} \otimes \vec{m}$. Summing over $F \in \mathcal{G}$ gives (1.3).

For k-graphs $\mathcal{F}$ and $\mathcal{G}$ one says that the map $\varphi : \cup \mathcal{G} \to \cup \mathcal{F}$ is a *homomorphism* if $\{\varphi(i) : i \in G\} \in \mathcal{F}$ holds for all $G \in \mathcal{G}$.

(1.4) The map $\rho : X_1 \cup \cdots \cup X_n \to X$ defined by $\rho(x) = i \Leftrightarrow x \in X_i$ is a homomorphism.

For example, a graph has a homomorphism into K_s, the complete graph on s vertices, if and only if its chromatic number is at most s.

Let $U = \{\mathcal{A}_1, \cdots, \mathcal{A}_t\}$ be a collection of k-graphs. Define $\operatorname{ex}(n, U)$ as $\max|\mathcal{G}|$ where $\mathcal{G} \subset \binom{X}{k}$ and $\mathcal{G}$ contains no copy of any $\mathcal{A} \in U$. Such an $\mathcal{G}$ is called U-free.

A classical result of Katona, Nemetz and Simonovits [KNS] is the following.

(1.5) THEOREM. $\operatorname{ex}(n, U)/\binom{n}{k}$ *is monotone decreasing and therefore* $\pi(U) = \lim_{n\to\infty} \operatorname{ex}(n, U)/\binom{n}{k}$ *exists.*

Call U *closed under homomorphism* (shortly *closed*), if $\mathcal{A} \in U$ implies that U contains (a copy of) every homomorphic image of $\mathcal{A}$.

E. g. for graphs $U = \{C_3, C_5\}$ is closed but $U = \{C_3, C_4\}$ is not.

By (1.4) and this definition we have.

(1.6) If U is closed and $\mathcal{G}$ is U-free then so is $\mathcal{G} \otimes \vec{m}$ for all $\vec{m} = (m_1, \cdots, m_n)$.

(1.7) Define the *Lagrange function* $\lambda(\mathcal{G})$ as $\max p(\mathcal{G}, x_1, \cdots, x_n)$ where $x_i \geq 0$, $x_1 + \cdots + x_n = 1$.

Note that $\lambda(\mathcal{G}) \geq 1/k^k$ if $\mathcal{G} \neq \emptyset$.

(1.8) THEOREM. *Let* $\mathcal{G}$ *be* U*-free, where* U *is closed. Then* $\pi(U) \geq \lambda(\mathcal{G})k!$ *holds.*

Proof. Let ε be an arbitrarily small positive number and choose non-negative rational numbers $x_1, \cdots, x_n$, $x_1 + \cdots + x_n = 1$ such that $p(\mathcal{G}, \vec{x}) > \lambda(\mathcal{G}) - \varepsilon$ holds. Let m be an arbitrary common multiple of the denominators of $x_1, \cdots, x_n$. Set $m_i = mx_i$. Using (1.3) and (1.6), $\mathcal{G} \otimes \vec{m}$ is U-free and $|\mathcal{G} \otimes \vec{m}|/\binom{m}{k} > k!\, p(\mathcal{G}, \vec{x}) > (\lambda(\mathcal{G}) - \varepsilon)\, k!$. Since ε was arbitrary and m can be arbitrarily large, the statement follows.

(1.9) REMARK. For $n \geq k \geq 2$ define the complete equipartite k-graph $\mathcal{P}(n, k)$ in the following way. Let $X = X_0 \cup \cdots \cup X_{k-1}$ be a partition with $|X_i| = \lfloor (n+i)/k \rfloor$, and set

$$\mathcal{P}(n, k) = \{P \subset X : |P \cap X_i| = 1 \quad \text{for all} \quad 0 \leq i \leq k-1\}.$$

$$|\mathcal{P}(n, l)| = \prod_{0 \leq i < k} \left\lfloor \frac{n+i}{k} \right\rfloor = (1 - o(1))\, n^k/k^k \tag{1.10}$$

whenever $n \to \infty$.

The (strong) chromatic number $\chi(\mathcal{A})$ of a k-graph $\mathcal{A} \subset \binom{X}{k}$ is the minimum integer l, such that there exists a partition $X = X_0 \cup \cdots \cup X_{l-1}$ with $|X_i \cap A| \leq 1$ for all i and $A \in \mathcal{A}$. Obviously, for $\emptyset \subset \mathcal{B} \subset \mathcal{A}$, $k \leq \chi(\mathcal{B}) \leq \chi(\mathcal{A})$ holds. If $\chi(\mathcal{A}) = k$, then $\mathcal{A}$ is called *k-partite.*

It is obvious that if no member of U is k-partite, then $\mathrm{ex}\,(n, U) \geq |\mathcal{P}(n, k)|$, i.e.,

$$\pi(U) \geq \frac{k!}{k^k}$$

holds. Erdös [E] proved that if any member of U is k-partite, then $\pi(U) = 0$, and, even more, there exists a $c = c(U) > 0$ such that

$$\mathrm{ex}\,(n, U) = 0(n^{k-c})$$

holds.

The method of Lagrange function is useless if $\pi(U) = 0$, however it can be applied to determine $\pi(U)$ in various other cases.

2. Computing the Lagrange function. Throughout the rest of the paper $\vec{x}$ will denote vectors $(x_1, \cdots, x_n)$ with $x_1 + \cdots + x_n = 1$ and $x_i \geq 0$, $i = 1, \cdots, n$.

For a family $\mathcal{G} \subset 2^X$ and $E \subset X$ define the *link* of E in $\mathcal{G} : \mathcal{G}(E) = \{F - E : E \subset F \in \mathcal{G}\}$. For $E = \{i\}$ we write simply $\mathcal{G}(i)$.

Simple computation shows the validity of the following statement. We use the notation $\vec{x}_{-j}=(x_1,\cdots, x_{j-1}, x_{j+1},\cdots, x_n)$.

(2.1) PROPOSITION. *Fix a real* δ *and* $1\le i<j\le n$. *Let* $\vec{x}=(x_1,\cdots, x_n)$ *and* $\vec{y}=(y_1,\cdots, y_n)$ *where* $y_i=x_i+\delta$, $y_j=x_j-\delta$ *and* $x_l=y_l$ for *for* $l\ne i, j$. *Let* $\mathcal{G}\subset\binom{X}{k}$ *be a k-graph. Then*

$$\begin{aligned} p(\mathcal{G}, \vec{x})-p(\mathcal{G}, \vec{y}) \\ =\delta(p(\mathcal{G}(j), \vec{x}_{-j})-p(\mathcal{G}(i), \vec{x}_{-i})) \\ +\delta^2 p(\mathcal{G}(\{i, j\}), \vec{x}_{-\{i,j\}}). \end{aligned} \tag{2.2}$$

Let us note that $p(\mathcal{G}(i), \vec{x}_{-i})=\partial p(\mathcal{G}, \vec{x})/\partial x_i$ is the partial derivative. For a vector $\vec{x}$ define its *support* $S(\vec{x})=\{i: x_i>0\}$. For $S\subset X$ define $\mathcal{G}_S=\{F\in\mathcal{G}: F\subset S\}$. Call $\mathcal{G}\subset 2^X$ *2-complete* if every $p\in\binom{X}{2}$ is contained in some edge $F\in\mathcal{G}$.

(2.3) LEMMA. [FR] *Let* $\mathcal{G}\subset\binom{X}{k}$ *be given and choose* $\vec{x}$ *with* $p(\mathcal{G}, \vec{x})=\lambda(\mathcal{G})$ *such that* $S(\vec{x})$ *is the smallest possible. Then* $\mathcal{G}_{S(\vec{x})}$ *is 2-complete.*

Moreover, $p(\mathcal{G}(j), \vec{x}_{-j})=k\lambda(\mathcal{G})$ *holds for all* $j\in S(\vec{x})$.

Proof. It follows from (2.2) that $p(\mathcal{G}(j), \vec{x}_{-j})=p(\mathcal{G}(i), \vec{x}_{-i})$ for all $i, j\in S(\vec{x})$, otherwise by choosing δ very small and either negative or positive we obtain the contradiction $p(\mathcal{G}, \vec{x})<p(\mathcal{G}, \vec{y})$. It there was no set $F\in\mathcal{G}$ with $\{i, j\}\subset F\subset S(\vec{x})$, then choosing $\delta=x_j$ or $-x_i$ according as $x_i>x_j$ or $x_j\ge x_i$ would produce $\vec{y}$ with $p(\mathcal{G}, \vec{y})=p(\mathcal{G}, \vec{x})$ but $|S(\vec{y})<(S(\vec{x})|$, a contradiction.

Fix $l\in S(\vec{x})$. Using the polynominal identity

$$kp(\mathcal{G}, \vec{x})=\sum_i x_i\, p(\mathcal{G}(i), \vec{x}_{-i}),$$

we obtain

$$k\lambda(\mathcal{G})=\sum_i x_i\ p(\mathcal{G}(l), \vec{x}_{-l})=p(\mathcal{G}(l), \vec{x}_{-l})$$

as desired.

(2.4) COROLLARY. (*Motzkin-Straus* [MS]) *Let* $\mathcal{G}\subset\binom{X}{2}$ *and let* $w(\mathcal{G})$ *be the clique number of* $\mathcal{G}$ (*i.e.*, max $s: K_s$ *is a subgraph of* $\mathcal{G}$). *Then*

$$\lambda(\mathcal{G})\le(w(\mathcal{G})-1)/2w(\mathcal{G}).$$

Proof. Choose $\vec{x}$ with $p(\mathcal{G}, \vec{x}) = \lambda(\mathcal{G})$. By (2.3) the graph $\mathcal{G}_{S(\vec{x})}$ is complete. Consequently, $s = |S(\vec{x})| \le w(\mathcal{F})$. Thus

$$p(\mathcal{G}, \vec{x}) = \sum_{\{i,j\} \subset S(\vec{x})} x_i x_j \le \binom{s}{2}\left(\frac{1}{s}\right)^2 \le (w(\mathcal{G}) - 1)/2w(\mathcal{G}).$$

Let $\sigma_k(S) = \sum_{F \subset S} \prod_{i \in F} x_i$ be the k'th elementary symmetric polynomial. Then by known inequalities (or one can use (2.3))

(2.5) $$\sigma_k(S) \le \binom{|S|}{k}\frac{1}{|S|^k} \text{ holds.}$$

Let $K_s(k)$ be the complete k-graph on s vertices, i. e., $\binom{S}{k}$ for some s-element set S.

(2.6) PROPOSITION.

$$\lambda(K_s(k)) = \binom{s}{k}/s^k.$$

Proof. The inequality in one direction follows from (2.5), in the other one by setting $x_i = 1/s$ for every vertex.

(2.7) CONJECTURE. [FF] Suppose that $\mathcal{F}$ is a k-graph, $|\mathcal{F}| = z(z-1)\cdots(z-k+1)/k!$, $z \ge k$ is real. Then $\lambda(\mathcal{F}) \le \binom{z}{k}/z^k$.

Note that the case $k = 2$ is a consequence of 2.4.

3. de Caen's problem.

(3.1) PROBLEM. (de Caen [C]) Determine $\max |\mathcal{F}|$ where $\mathcal{F} \subset \binom{X}{k}$ and $\mathcal{F}$ contains no three sets F_1, F_2, F_3 with $|F_1 \cap F_2| = k-1$ and $F_1 \Delta F_2 \subset F_3$.

Actually, one can formulate (3.1) in terms of $\mathrm{ex}(n, U)$. Define $\mathcal{A}_i = \{\{1, 2, \cdots, k\}, \{1, 2, \cdots, k-1, k+1\}, \{i, i+1, \cdots, i+k-1\}\}$ and set $\mathcal{C} = \{\mathcal{A}_2, \mathcal{A}_3, \cdots, \mathcal{A}_k\}$. Then (3.1) asks for the determination $\mathrm{ex}(n, \mathcal{C})$.

Note that $\chi(\mathcal{A}_i) = k+1$ for $2 \le i \le k$.

(3.2) OBSERVATION. *If $\mathcal{F} \subset \binom{X}{k}$ is 2-complete and $\mathcal{C}$-free then*

$$|F \cap F'| \le k-2 \quad \textit{for all distinct} \quad F, F' \in \mathcal{F}.$$

Proof. Suppose that $F_1, F_2 \in \mathcal{F}$ and $|F_1 \cap F_2| = k-1$. Then $|F_1 \Delta F_2| = 2$. By 2-completeness $F_1 \Delta F_2 \subset F_3$ holds for some $F_3 \in \mathcal{F}$. Moreover, $\{F_1, F_2, F_3\} \in \mathcal{C}$, a contradiction.

(3.3) THEOREM. (*Sidorenko* [Si]) *If* $\emptyset \neq \mathcal{F} \subset \binom{X}{k}$ *is* $\mathcal{C}$*-free then*

$$\lambda(\mathcal{F}) = 1/k^k \quad \textit{for} \quad k = 2,\ 3 \textit{ and } 4.$$

Proof. We have to prove $\lambda(\mathcal{F}) \leq 1/k^k$ only. In view of (2.3) we may assume that $\mathcal{F}$ is 2-complete. Thus by (3.2) one has $|F \cap F'| \leq k-2$ for all distinct $F, F' \in \mathcal{F}$. Equivalently the links $\mathcal{F}(i)$ are pairwise disjoint, $i \in X$. Thus the polynomials $p(\mathcal{F}(i), \vec{x}_{-i})$ have no common term. Since $x_i \geq 0$, we infer using (2.5) that

$$\sum_i p(\mathcal{F}(i), \vec{x}_{-i}) \leq \sigma_{k-1}(X) \leq \binom{n}{k-1}/n^{k-1}.$$

Combining this with (2.3) we obtain

$$\lambda(\mathcal{F}) = p(\mathcal{F}, \vec{x}) \leq \binom{n}{k-1}/kn^k. \tag{3.4}$$

For $k = 2, 3$ the RHS is at most $1/k^k$ for all $n \geq k$. The same holds for $k = 4$ and $n \geq 4$, $n \neq 5$. However, $n = 5$ is impossible, because either $|\mathcal{F}| = 1$ and then $\mathcal{F}$ is not 2-complete, or $|\mathcal{F}| \geq 2$ and $|F \cap F'| = 3 = k-1$ holds for all $F, F' \in \mathcal{F}$-contradicting $|F \cap F'| \leq k-2$.

(3.5) COROLLARY. *For* $k = 2, 3, 4$ *and* $k|n$ *one has*

$$\text{ex}\,(n, \mathcal{C}) = (n/k)^k.$$

Proof. The upper bound follows from (3.3) and (1.2). To show the lower bound, consider the complete, equipartite k-graph $\mathcal{P}(n, k)$ which is $\mathcal{C}$-free.

(3.6) REMARK. In the case $k = 2$ (3.5) is an old result of Mantel [M] (see also [T]). For $k = 3$ it was proved by Bollobás [Bo] in a completely different way. Finally, for $k = 4$, (3.5) is due to Sidorenko [Si]. Let us mention also, that actually $\text{ex}(n, \mathcal{C}) = |\mathfrak{P}(n, k)|$ holds even if k does not divide n and $\mathfrak{P}(n, k)$ is the unique optimal family (see [Si]).

Recall that a $S(n, k, t)$ is a family $\mathscr{S} \subset \binom{X}{k}$ such that every t-set $T \subset X$ is contained in a *unique* member of $\mathscr{S}$. For $(n, k, t) = (11, 5, 4)$ and $(12, 6, 5)$ there exists a unique $S(n, k, t)$ W_5, W_6 called the Witt-design, cf. Beth *et al.* [BJL].

In general, call a family $\mathscr{D} \subset \binom{X}{k}$ *sparse* if $|D \cap D'| \leq k-2$ for all distinct distinct $D, D' \in \mathscr{D}$.

(3.7) PROPOSITION. *$\mathscr{D} \otimes \vec{m}$ is a $\mathscr{C}$-free family for any sparse k-graph $\mathscr{D}$ and all $\vec{m} = (m_1, \cdots, m_n)$.*

Proof. Clearly, $\mathscr{D}$ is $\mathscr{C}$-free. The rest follows from $\mathscr{C}$ being closed.

(3.8) THEOREM. *$\pi(\mathscr{C}) = k! \max \lambda(\mathscr{D})$ where the maximum is over all sparse k-graphs $\mathscr{D} \subset \binom{Y}{k}$ with $|Y| \leq k^k/k!$*

Proof. Set $\lambda = k! \max \lambda(\mathscr{D})$. In view of (1.1) it is sufficient to show that $k! \lambda(\mathscr{G}) \leq \lambda$ for all $\mathscr{C}$-free k-graphs $\mathscr{G} \subset \binom{X}{k}$. By (2.3) we may assume that $S(\vec{x}) = X$ (otherwise replace $\mathscr{G}$ by $\mathscr{G}_{S(\vec{x})}$) and therefore $\mathscr{G}$ is 2-complete. By (3.2) $\mathscr{G}$ is sparse. This leads to (3.4). For $n > k^k/k!$ the RHS of (3.4) is less than $1/k^k$, thus the maximum occurs for $n \leq k^k/k!$.

(3.9) REMARK. (3.8) shows that the determination of $\pi(\mathscr{C})$ is a finite problem, however, the number of cases to check increases very fast. To compute or bound the value of $\lambda(\mathscr{D})$ for a specific sparse k-graph is very difficult in general. Let us mention without proof the following result.

(3.10) THEOREM.

(i) $\pi(\mathscr{C}) = 720/11^4$ *for* $k = 5$

(ii) $\pi(\mathscr{C}) = 55/12^3$ *for* $k = 6$.

Moreover, for $n > n_0$ the only optimal $\mathscr{C}$-free k-family is obtained by blowing up the appropriate Witt design. (see (3.7)).

4. **A problem of Katona and Bollobás.** Let $s(n, k)$ denote $\max|\mathscr{F}|$, where $\mathscr{F} \subset \binom{X}{k}$ and $\mathscr{F}$ contains no three distinct sets F_1, F_2, F_3 with $F_1 \triangle F_2 \subset F_3$.

Clearly,

$$|\mathcal{P}(n, k)| \le s(n, k) \le \mathrm{ex}(n, \mathcal{C}).$$

(4.1) CONJECTURE. (Katona [K] and Bollobás [Bo])

$$|\mathcal{P}(n, k)| = s(n, k).$$

From (3.5) and (3.6) it follows that (4.1) is true for $k=2$, 3 and 4. It was proved in [FF1] that (4.1) holds for $n \le 2k$.

Since the class of k-graphs excluded by (4.1) is closed, the analogue of (3.8) holds for this problem. One has only to add that $\mathcal{D}$ contains no D_1, D_2, D_3 with $D_1 \triangle D_2 \subset D_3$.

Katona [K] asked what happens in the non-uniform case. Determine $s(n) = \max|\mathcal{F}|$, $\mathcal{F} \subset 2^X$ and $\mathcal{F}$ contains no F_1, F_2, F_3 with $F_1 \triangle F_2 \subset F_3$.

(4.2) CONJECTURE. (Erdös-Katona [K])

$$s(n) = s(n, [n/3]) = |\mathcal{P}(n, [n/3])|.$$

Let us note that $|\mathcal{P}(n, [n/3])| \le 3^{n/3} = 1.44^n_{\dots}$. In [FF1] the upper bound $s(n) \le n\, 1.5^n$ is proved.

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