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COVERING THE COMPLETE GRAPH BY PARTITIONS*

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A (D, c)-coloring of the complete graph K^n is a coloring of the edges with c colors such that all monochromatic connected subgraphs have at most D vertices. Resolvable block designs with c parallel classes and with block size D are natural examples of (D, c)-colorings. However, (D, c)-colorings are more relaxed structures. We investigate the largest n such that K^n has a (D, c)-coloring. Our main tool is the fractional matching theory of hypergraphs.

1. Definitions

This paper is organized as follows. In this section we recall some definitions and introduce notations. The first part of the paper is devoted to the fractional matchings of r-partite hypergraphs. In the second part we apply the results to the (D, c)-colorings of the complete graphs.

A *hypergraph* $\mathbf{H}$ is a pair $(V(\mathbf{H}), E(\mathbf{H}))$, where $V(\mathbf{H})$ is a (finite) set, the set of vertices or points, and $E(\mathbf{H})$, the edge set, is a collection of subsets of $V(\mathbf{H})$. If we want to emphasize that $\mathbf{H}$ contains (or might contain) multiple edges, then we call it a *multihypergraph*. If $\mathbf{H}$ does not contain multiple edges then it is called a *simple* hypergraph. G is a *subhypergraph* of $\mathbf{H}$ if $V(G) \subset V(\mathbf{H})$ and $E(G) \subset E(\mathbf{H})$. The *dual* $\mathbf{H}^*$ of $\mathbf{H}$ is obtained by interchanging the role of vertices and edges and keeping the incidences, i.e. $V(\mathbf{H}^*) = E(\mathbf{H})$ and $E(\mathbf{H}^*) = \{E(p) : p \in V(\mathbf{H})\}$, where $E(p) =: \{E \in E(\mathbf{H}) : p \in E\}$. A hypergraph is an *r-graph*, or *r-uniform* hypergraph, if all edges have r elements. The *rank* of $\mathbf{H}$ is r if $\max\{|E| : E \in E(\mathbf{H})\} = r$. An r-graph $\mathbf{H}$ is *r-partite* if the vertex-set has a partition $V(\mathbf{H}) = X_1 \cup \cdots \cup X_r$ such that $|X_i \cap E| = 1$ holds for all $E \in E(\mathbf{H})$, $1 \leq i \leq r$. The *degree* of a vertex p is $\deg_{\mathbf{H}}(p) = |\{E : p \in E \in E(\mathbf{H})\}|$. The maximum degree, $\max \deg(p)$, is denoted by $D(\mathbf{H})$. A *matching* $\mathcal{M}$ is a subset of $E(\mathbf{H})$ consisting of pairwise disjoint edges. The matching number, $\nu(\mathbf{H})$, is the maximum number of edges in a matching in $\mathbf{H}$. If $\nu(\mathbf{H}) = 1$, i.e. $E \cap E' \neq \emptyset$ for all $E, E' \in E(\mathbf{H})$, then $\mathbf{H}$ is called *intersecting*. A *cover* T of $\mathbf{H}$ is a subset which meets all the edges of $\mathbf{H}$, and the covering number, $\tau(\mathbf{H})$, is the minimum size of a cover. An i-cover, where i is a positive integer, is a function $t : V(\mathbf{H}) \to \{0, 1, \ldots, i\}$ such that

$$\sum_{x \in E} t(x) \geq i$$

holds for all $E \in E(\mathbf{H})$. The complete graph on n points is denoted by K^n.

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An r-uniform hypergraph $\mathbf{H}$ is called a *finite projective plane* of order $r-1$ if $|V(\mathbf{H})| = |E(\mathbf{H})| = r^2 - r + 1$ and every two distinct edges intersect in exactly one element. Briefly, $\mathbf{H}$ is a PG$(2, r-1)$. Projective planes are known to exist whenever $r-1$ is a prime or prime power. An *affine plane*, AG$(2, r-1)$, is obtained from a PG$(2, r-1)$ by deleting an edge E_0 from $V(\text{PG}(2, r-1))$ and setting the edge set $E(\text{AG}(2, r-1)$ equal to $\{E \setminus E_0 : E_0 \neq E \in E(\text{PG}(2, r-1))\}$. An r-graph is called a *truncated projective plane* of order $r-1$ (or briefly a TPG$(2, r-1)$) if it is obtained from a PG$(2, r-1)$ by deleting a vertex p and the r edges through p. It is the dual of an AG$(2, r-1)$. Let $\mathbf{A}$ be an $AG(2, q)$, $p \in V(A)$, and let $E_1, \ldots, E_{i+1}$ be edges through p $(i \leq q)$. Then the following function t is an i-cover:

$$t(x) = \begin{cases} i & \text{if } x = p, \\ 1 & \text{if } x \in (\bigcup E_j) - \{p\}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for $1 \leq i < q$ we have

$$\tau_i(\mathbf{A}) \leq (i+1)q - 1. \tag{1.1}$$

On the other hand let t be a minimal i-cover $(1 \leq i \leq q-1)$. Then there exists a vertex $x \in V(\mathbf{A})$ with $t(x) = 0$. Considering the $q+1$ lines through x we obtain

$$i(q+1) \leq \tau_i(\mathbf{A}). \tag{1.2}$$

Hence equality holds in (1.1) for $i = q-1$, i.e. for every affine plane $\mathbf{A}$ of order q we have

$$\tau_{q-1}(\mathbf{A}) = q^2 - 1. \tag{1.3}$$

There are other cases when (1.2) gives the optimal bound. If q is a power of 2, and $\mathbf{A}$ is a galois plane, then there exists a *hyperoval* $C \subset V(\mathbf{A})$, i.e. $|C| = q+2$ and $|C \cap E| \leq 2$ for all $E \in E(\mathbf{A})$. Then $V(\mathbf{A}) - C$ is a $(q-2)$-cover with cardinality $(q+1)(q-2)$, i.e. in this case

$$\tau_{q-2}(\mathbf{A}) = q^2 - q - 2. \tag{1.4}$$

We use the notations $\lfloor x \rfloor$ and $\lceil x \rceil$ for the lower and upper integer part of x, respectively.

2. Fractional matchings of r-partite hypergraphs

A *fractional matching* w of the hypergraph $\mathbf{H}$ is a non-negative function on the edges, $w : E(\mathbf{H}) \to \mathbf{R}^+$, such that

$$\sum_{p \in E} w(E) \leq 1$$

holds for all vertices $p \in V(\mathbf{H})$. The *value* of w, $\|w\|$, is the total sum $\sum w(E)$. The supremum of $\|w\|$, denoted by $\nu^*(\mathbf{H})$, is the *fractional matching number of* $\mathbf{H}$. A *fractional cover* of $\mathbf{H}$ is a function on the vertices, $t: V(\mathbf{H}) \to \mathbf{R}^+$, such that

$$\sum_{p \in E} t(p) \geq 1$$

holds for all edges $E \in E(\mathbf{H})$. The *value* of t is $\|t\| =: \sum_{x \in V} t(x)$. The *fractional covering number*, $\tau^*(\mathbf{H})$, of $\mathbf{H}$ is the infimum of $\|t\|$. As the calculation of τ^* and ν^* are dual linear programming problems their optima coincide, i.e. for all $\mathbf{H}$ we have

$$\nu \leq \nu^* = \tau^* \leq \tau \leq r\nu.$$

Hence the value of τ^* is always rational, and there is an optimal fractional matching w and a cover t with $\|w\| = \|t\| = \tau^*(\mathbf{H})$. In [8] the following theorem is proved.

(2.1). *If* $\mathbf{H}$ *is an intersecting hypergraph of rank* r, *then either* $\tau^*(\mathbf{H}) \leq r-1$, *or* $\mathbf{H}$ *is a finite projective plane of order* $r-1$.

In the latter case $\tau^*(\mathbf{H}) = r - 1 + (1/r)$. One cannot improve (2.1) in general, because if $\mathbf{H}$ is a truncated projective plane of order $r-1$, then $\tau^*(\mathbf{H}) = r-1$. However, we show the following sharpening of (2.1).

Theorem 2.1. *Suppose that* $\mathbf{H}$ *is an* r*-partite, intersecting hypergraph. Then either* $\tau^*(\mathbf{H}) \leq r - 1 - (1/(r-1))$, *or* $\mathbf{H}$ *is a truncated projective plane of order* $r-1$ (*and then* $\tau^*(\mathbf{H}) = r-1$).

We remark that if we delete a line of a truncated projective plane, then we obtain an *r-partite hypergraph with* $\tau^* = r - 1 - 1/(r-1)$.

For the proof we split into two parts the statement of the theorem.

(2.2). *Suppose that* $\mathbf{F}$ *is intersecting, r-partite and* $\tau^* = r-1$. *Then* $\mathbf{F}$ *is a truncated projective plane.*

(2.3) *Suppose that* $\mathbf{H}$ *is intersecting, r-partite and* $\tau^* < r-1$. *Then* $\tau^* \leq r - 1 - 1/(r-1)$.

Remark 2.2. It is easy to prove a weaker version of (2.3) using the following fact from [5]. Let $\mathbf{G}$ be an arbitrary hypergraph. Write $\tau^*(\mathbf{G})$ in the form u/v, where u, v are positive integers and $(u, v) = 1$. Then $v \leq r^{\frac{1}{2}r\tau^*}$. To finish the proof of the weaker form of (2.3) write $\tau^*(\mathbf{H})$ in the form u/v. Then $v \leq r^{-\binom{r}{2}}$, hence $\tau^*(\mathbf{H}) < r - 1 - r^{-\binom{r}{2}}$.

The proof of Theorem 2.1 combines the methods of [7] and [8]. We are going to use the following lemma. Let $\mathbf{H}$ be an arbitrary hypergraph, and w be an optimal fractional matching. The support S of w is the set of vertices p for which $\sum_{p \in E} w(E) = 1$, i.e. the set of saturated points. The hypergraph $\mathbf{H}$ is *τ^*-critical* if $\tau^*(\mathbf{H}') < \tau^*(\mathbf{H})$ holds for every subhypergraph $\mathbf{H}'$, i.e. we cannot delete an edge without decreasing the value of τ^*.

Lemma 2.3. *Let* $\mathbf{H}$ *be τ^*-critical and S be a maximal support. Then* $|E(\mathbf{H})| \leq |S|$.

Proof. Let $w \in \mathbf{R}^{E(\mathbf{H})}$ be an optimal fractional matching of $\mathbf{H}$ with support S. Then w lies on the boundary of the polytope P defined by the inequalities

$$w(E) \geq 0 \quad \text{for all} \quad E \in E(\mathbf{H}),$$

$$\sum_{p \in E} w(E) \leq 1 \quad \text{for all} \quad p \in V(\mathbf{H}).$$

There is vertex w_0 of P such that w_0 is also an optimal fractional matching, and w_0 lies on all the facets of P which contain w. This means that the support of w_0 contains the support of w, i.e. it is also S. Moreover $\mathbf{H}$ is τ^*-critical, so we have that $w_0(E) > 0$ holds for every edge $E \in E(\mathbf{H})$. Thus P is full dimensional. Then the number of facets of P through w_0 is at least $|E(\mathbf{H})|$. □

As a corollary we have (see [5]): *if* $\mathbf{H}$ *is τ^*-critical, then*

$$|E(\mathbf{H})| \leq \tau^* r. \tag{2.4}$$

Proof. Let S be a maximal support, then Lemma 2.3 implies that

$$|E(\mathbf{H})| \leq |S| \leq \sum_{p \in V(\mathbf{H})} \Big(\sum_{p \in E} w(E) \Big) = \sum |E|\, w(E) \leq \tau^* r.$$

Furthermore, equality holds in (2.4) if and only if $E \subset S$ for all edge $E \in E(\mathbf{H})$, i.e. every non-isolated point is saturated. □

As $w(E) \equiv 1/D$ is always a fractional matching with value $|E(\mathbf{H})|/D$, we have

$$|E(\mathbf{H})| \leq \tau^* D, \tag{2.5}$$

for all hypergraphs $\mathbf{H}$.

Proof of (2.2). Let $\mathbf{H}$ be a τ^*-critical subgraph of $\mathbf{F}$ with $\tau^*(\mathbf{F}) = \tau^*(\mathbf{H}) = r - 1$. Without loss of generality we may suppose that $V(\mathbf{H}) = \cup \{E : E \in E(\mathbf{H})\}$ with parts $X_1, \ldots, X_r$ (i.e. $|X_i \cap E| = 1$ for all $E \in E(\mathbf{H})$, $1 \leq i \leq r$). Then (2.4) implies that

$$|E(\mathbf{H})| \leq (r-1)r.$$

Claim 2.4. For all i one has $|X_i| = r - 1$.

Proof. Every X_i is a cover, hence $|X_i| \geq \tau \geq \tau^* = r - 1$. To prove an upper bound for $|X_i|$ we distinguish two cases. If $|E(\mathbf{H})| = r(r-1)$, i.e. equality holds in (2.4), then every point is saturated. So $|V(\mathbf{H})| = r\tau^* = r(r-1)$, and we are done. So we can suppose that

$$|E(\mathbf{H})| \leq r(r-1) - 1. \tag{2.6}$$

Let w be an optimal fractional matching of $\mathbf{H}$. We write $s(w, p)$ for $\sum_{p \in E} w(E)$, and if it does not cause confusion we write $s(p)$, briefly. Let p be a vertex and $p \in E_0 \in E(\mathbf{H})$. Then

$$s(p) + (r-1) \geq \sum_{q \in E_0} s(q) = \sum_{E \in E(\mathbf{H})} |E \cap E_0|\, w(E) \geq \tau^* + (r-1)w(E_0). \tag{2.7}$$

Hence

$$s(p) \geq (r-1)w(E_0). \tag{2.8}$$

If we add up (2.8) for every edge E_0 which contains p, then we have

$$s(p)\deg_{\mathbf{H}}(p) \geq (r-1)s(p). \tag{2.9}$$

As $w(E) > 0$ for all $E \in E(\mathbf{H})$, (2.9) implies that

$$\deg(p) \geq r - 1 \tag{2.10}$$

holds for all $p \in V(\mathbf{H})$. Finally, (2.10) and (2.6) imply that $|X_i| \leq (r(r-1) - 1)/(r-1) < r$, proving Claim 2.4. □

Now we return to the proof of (2.2). Joint a new element x to $V(\mathbf{H})$, and define the hypergraph $\mathbf{G}$ by the vertex set $V(\mathbf{G}) = V(\mathbf{H}) \cup \{x\}$ and the edge set $E(\mathbf{H}) \cup \{X_i \cup \{x\} : 1 \leq i \leq r\}$. Define $w' : E(\mathbf{G}) \to \mathbf{R}^+$ as follows:

$$w'(E) = \begin{cases} \dfrac{r-1}{r} w(E) & \text{if } E \in E(\mathbf{H}), \\ 1/r & \text{if } E \in E(\mathbf{G}) \setminus E(\mathbf{H}). \end{cases}$$

Then w' is a fractional matching of $\mathbf{G}$ with value $\|w\|\,(r-1)/r + 1$. Thus $\mathbf{G}$ is an intersecting r-graph with $\tau^* \geq r - 1 + 1/r$. Hence $\mathbf{G}$ is a finite projective plane, by (2.1), and $\mathbf{H}$ is a truncated projective plane.

It is easy to see, that if $\mathbf{H}$ is a truncated projective plane, and C is an r-element cover which intersects every part X_i as well, then $C \in E(\mathbf{H})$. This implies that $\mathbf{H} = \mathbf{F}$. □

Proof of (2.3). We may suppose that $\mathbf{H}$ is τ^*-critical. Let w be an optimal fractional matching of $\mathbf{H}$ with maximal support S, that is $S = \{p \in V(\mathbf{H}) : s(p) = 1\}$. Denote the parts of $\mathbf{H}$ by $X_1, \ldots, X_r$. As X_i intersects every edge in exactly one element we obtain

$$|S \cap X_i| \leq \sum_{p \in X_i} s(p) = \nu^*(\mathbf{H}) < r - 1. \tag{2.11}$$

Hence $|S| \leqslant r(r-2)$. Then Lemma 2.3 implies that

$$|E(\mathbf{H})| \leqslant r(r-2). \tag{2.12}$$

Let $A = \{p \in X_i : \deg p \geqslant r-1\}$. By (2.12) we have

$$|A| \leqslant r(r-2)/(r-1) < r-1. \tag{2.13}$$

If $|X_i| \leqslant r-2$, then $\tau^* \leqslant \tau \leqslant |X_i| \leqslant r-2$, and we are done. From now on we suppose that $|X_i| \geqslant r-1$. Then (2.13) implies that there exists a vertex $p \in X_i \backslash A$. The inequality (2.7) holds for all intersecting r-graphs. So let $p \in E_0 \in E(\mathbf{H})$, then

$$r-1-\tau^* \geqslant (r-1)w(E_0) - s(p). \tag{2.14}$$

Adding up (2.14) for all E_0 with $p \in E_0$, we have

$$\deg(p)(r-1-\tau^*) \geqslant (r-1-\deg(p))s(p) \geqslant s(p), \tag{2.15}$$

since $\deg(p) \leqslant r-2$. We now add up (2.15) for all $p \in X_i \backslash A$, and obtain

$$(r(r-2) - |A|\,(r-1))(r-1-\tau^*)$$
$$\geqslant \Big(|E(\mathbf{H})| - \sum_{p\in A} \deg(p)\Big)(r-1-\tau^*) \geqslant \sum_{p\in X_i - A} s(p) \geqslant \tau^* - |A|.$$

Rearranging the extremes of this inequality, we obtain that $\tau^* \leqslant r(r-2)/(r-1)$, as stated. □

3. (D, c)-colorings of complete graphs

In this section we deal with the following Ramsey type problem. Color the edges of a complete graph by c colors. How large is the largest monochromatic *connected* component? A (D, c)-coloring of the complete graph $\mathbf{K}$ is a coloring of the edges with c colors so that all monochromatic connected subgraphs have at most D vertices. A (D, c)-coloring can be viewed as c partitions of a ground set into sets of cardinality at most D such that all pairs of the elements appear together in some of the sets. Resolvable block designs with c parallel classes and with blocks of size D are natural examples of (D, c)-colorings. However, (D, c)-colorings are more relaxed structures since the blocks may have any sizes up to D, and the pairs of the ground set may appear together in many blocks. Let $f(D, c)$ denote the largest integer m such that $\mathbf{K}^m$ has a (D, c)-coloring. Obviously,

$$f(D, c) \leqslant +c(D-1). \tag{3.1}$$

The function $f(D, c)$ was introduced by Gerencsér and Gyárfás [9] in 1967. The value of $f(D, 2) = D$ and $f(D, 3)$ were determined in [1] and [9]. In [10] there are further results on $f(D, c)$. The problem of determining $f(D, c)$ was rediscovered by Bierbrauer and Brandis [3]. In [4] the value of $f(D, c)$ was given for all $c \leqslant 5$ or $D \leqslant 3$.

Theorem 3.2 [4].

$$f(D,3)=\begin{cases} 4p & \text{if } D=2p \\ 4p+1 & \text{if } D=2p+1 \end{cases}$$

$$f(D,4)=\begin{cases} 9p & \text{if } D=3p \\ 9p+1 & \text{if } D=3p+1 \\ 9p+4 & \text{if } D=3p+2 \end{cases}$$

$$f(D,5)=\begin{cases} 16p & \text{if } D=4p \\ 16p+1 & \text{if } D=4p+1 \\ 16p+6 & \text{if } D=4p+2 \\ 16p+9 & \text{if } D=4p+3 \end{cases}$$

$$f(2,c)=\begin{cases} c+1 & \text{if } c \text{ is odd} \\ c & \text{if } c \text{ is even} \end{cases}$$

$$f(3,c)=\begin{cases} 5 & \text{if } c=3 \\ 2c & \text{if } c\equiv 0 \pmod 3,\ c\geqslant 6 \\ 2c+1 & \text{if } c\equiv 1 \pmod 3 \\ 2c-1 & \text{if } c\equiv 2 \pmod 3. \end{cases}$$

In [2] and [3] there are further results for the case $D \leqslant c$. They use strong results from the theory of resolvable block designs. In this paper we give a theorem which asymptotically determines $f(D, c)$ whenever D is large, c is fixed, and $c-1=q$ is a prime power. Further interpretation of $f(D, c)$ from the point of view of Ramsey theory can be found in [6].

With a (D, c)-coloring of $\mathbf{K}^n$ we can associate a hypergraph $\mathbf{H}$ with $V(\mathbf{H}) = V(\mathbf{K}^n)$ and the edges of $\mathbf{H}$ as the vertex sets of the connected monochromatic components. The dual hypergraph $\mathbf{H}^*$ of $\mathbf{H}$ is a c-partite, intersecting hypergraph (where multiple edges are allowed). So we have

Proposition 3.2. *$f(D, c) = \max_G |E(G)|$, where* **G** *runs through all c-partite, intersecting multihypergraphs with maximum degree at most D.*

Recall the definition of the i-cover, $\tau_i(\mathbf{H})$ is the maximum of $\sum t(x)$ *where* $t: V(\mathbf{H}) \to \{0, 1, \ldots, i\}$ such that $\sum_{x \in E} t(x) \geqslant i$ holds for all $E \in E(\mathbf{H})$. For an integer i, whenever a projective plane of order q exists, define

$$\tau_i(q) = \min\{\tau_i(\mathbf{A}) : \mathbf{A} \text{ affine plane of order } q\}.$$

Let $\tau_0(q)=0$. We need one more definition.

$$\tau_c^* = \max\{\tau^*(\mathbf{H}) : \mathbf{H} \text{ is } c\text{-partite and intersecting}\}.$$

By Theorem 2.1 we have that $\tau_q^* = q$ if a PG(2, q) exists, and $\tau_q^* \leqslant q-(1/q)$ otherwise.

Theorem 3.3.

$$D\tau_c^* - c\tau_c^* < f(D, c) \leqslant D\tau_c^*,$$

and for any fixed c there are infinitely many D for which equality holds.

By Theorem 2.1 this implies that (see [10])

$$f(D, c) \leqslant D(c-1). \tag{3.2}$$

Theorem 3.4. *Suppose that there exists an affine plane of order q, and let $D = q\lceil D/q\rceil - i$ where $0 \leqslant i < q$. Then for $D \geqslant q^2 - q$ we have*

$$f(D, q+1) = \lceil D/q\rceil q^2 - \tau_i(q).$$

For $D > q^2 - q$ an extremal multihypergraph is obtained only from a truncated projective plane by multiplying its edges.

The case $D \equiv 0 \pmod q$ was proved in [4]. Their lower bound for $f(D, q+1)$ for general i is probably slightly smaller than the one given in Theorem 3.4.

Proof of 3.3. Let **H** be a c-partite, intersecting multihypergraph with maximum degree D. Then by (2.5) we have

$$|\mathrm{E}(\mathbf{H})| \leqslant \tau_c^* D,$$

which implies the upper bound.

To prove the lower bound, consider a τ^*-critical, c-partite, intersecting hypergraph **G** with $\tau^*(\mathbf{G}) = \tau_c^*$. (Such a **G** exists.) Let $w : E(\mathbf{G}) \to \mathbf{R}^+$ be an optimal fractional matching. Define the multihypergraph **H** on the edge set $E(\mathbf{G})$ such that the multiplicity of an edge E is $\lfloor w(E)D\rfloor$. Then $D(\mathbf{H}) \leqslant D$, and

$$|E(\mathbf{H})| > \sum_{E\in E(\mathbf{G})} (w(E)D - 1) = \tau_c^* D - |E(\mathbf{G})|. \tag{3.3}$$

Here $|E(\mathbf{G})| \leqslant c\tau_c^*$, so (3.3) implies the lower bound. □

Proof of Theorem 3.4. Let $D = q\lceil D/q\rceil - i$ and $n = q^2\lceil D/q\rceil - \tau_i(q)$. Using the affine planes we construct a (D, c)-coloring of $\mathbf{K}^n$, which implies the lower bound. Let **A** be an AG$(2, q)$ with an i-cover $t : V(\mathbf{A}) \to \{0, 1, \ldots, i\}$ such that $\sum t(x) = \tau_i(q)$. Let $\mathscr{L}_1, \ldots, \mathscr{L}_{q+1}$ be the parallel classes of **A**, that is $\mathscr{L}_u = \{L_{u,v} : 1 \leqslant v \leqslant q\}$ such that $\bigcup \mathscr{L}_u = E(\mathbf{A})$ and $L_{u,v} \cap L_{u,w} = \emptyset$ for $1 \leqslant v < w \leqslant q$.

Replace each point p of $V(\mathbf{A})$ by a $\lceil D/q \rceil - t(p)$ element set $Z(p)$, and define $Z(E) = \bigcup \{Z(p): p \in E\}$. Then $Z(\mathscr{L}_u)$ $(1 \leqslant u \leqslant q+1)$ is a $(D, q+1)$-coloring of $Z(V(\mathbf{A}))$.

To prove the upper bound for $f(D, q+1)$ we are going to use Proposition 3.2. Suppose that $\mathbf{H}$ is a $(q+1)$-partite, intersecting hypergraph with $D(\mathbf{H}) \leqslant D$, and $|E(\mathbf{H})| = f(D, q+1)$. Then the above construction and (1.1) imply that

$$|E(\mathbf{H})| \geqslant q^2 \lceil D/q \rceil - qi - q + 1 = qD - q + 1. \tag{3.4}$$

By (2.5) we have that $|E(\mathbf{H})| \leqslant D\tau^*(\mathbf{H})$, so (3.4) implies that

$$\tau^*(\mathbf{H}) \geqslant q - \frac{q-1}{D}.$$

Hence for $D > q^2 - q$ we obtain that $\tau^*(\mathbf{H}) > q - (1/q)$. Apply Theorem 2.1. Hence $\mathbf{H}$ is a multihypergraph obtained from the truncated projective plane $\mathbf{P}$. Denote the multiplicities of the edges $E \in E(\mathbf{P})$ by $m(E)$. We claim that

$$m(E) \geqslant \lceil D/q \rceil \tag{3.5}$$

holds for every edge E. Indeed, if $m(E_0) > \lceil D/q \rceil$, then

$$|E(\mathbf{H})| = \sum m(E) = -qm(E_0) + \sum_{p \in E_0} \Big(\sum_{p \in E} m(E) \Big) \leqslant -q \lceil D/q \rceil - q + (q+1)D.$$

This is less than the right hand side of (3.4), thus (3.5) follows. Let $t(E) = \lceil D/q \rceil - m(E)$. Then t is an i-cover of the dual of $\mathbf{P}$, that is $\sum t(E) \geqslant \tau_i(\mathbf{A}) \geqslant \tau_i(q)$. Finally,

$$|E(\mathbf{H})| = q^2 \lceil D/q \rceil - \sum_{E \in \mathbf{P}} t(E) \leqslant q^2 \lceil D/q \rceil - \tau_i(q). \quad \square$$

The case $D = q^2 - q$ also follows from the above argument.

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Note added in proof

The main result of the second part (Theorem 3.4) verifies a conjecture of Bierbrauer [11]. He also conjectures that Gyárfás' lower bound [4] for $f(D, q+1)$ coincides with the value given in Theorem 3.4. Moreover, he determines $f(D, 6) = 5D - 3$ for $D \geqslant 89$, $D \not\equiv (\text{mod } 5)$.

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