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A PROJECTIVE PLANE IS AN OUTSTANDING 2-COVER

Zoltán FÜREDI

Mathematical Institute of the Hungarian Academy of Sciences, 1364, Budapest, P.O.B. 127, Hungary

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$C(v, k, 2)$ denotes the minimum number of k-subsets required to cover all pairs of a v-set. Obviously, $C(n^2+n+1, n+1, 2) \geqslant n^2+n+1$ where equality holds if and only if a finite projective plane exists. In this note the following conjecture of Mendelsohn is proved. If a PG(2, n) does not exist, then $C(n^2+n+1) \geqslant n^2+n+3$.

1. Definitions

A *hypergraph* $\mathscr{H}$ is a pair $(V, \boldsymbol{E})$ where V, the set of *vertices*, is a finite set, and $\boldsymbol{E}$, the set of *edges*, is a collection of subsets of V. Let $\boldsymbol{E}(x)$ denote the set of edges containing $x \in V$. $\deg(\mathscr{H}, x)$ stands for $|\boldsymbol{E}(x)|$ (i.e. the *degree* of x). If all the degrees are d, then $\mathscr{H}$ is called *d-regular*. If all the edges have k elements, then $\mathscr{H}$ is *k-uniform*. The hypergraph $\mathscr{H}$ is called *intersecting* if $E \cap E' \neq \emptyset$ for all edges $E, E' \in \boldsymbol{E}$. Moreover it is called *1-intersecting* if $|E \cap E'| = 1$ holds for all distinct edges. The *restriction* $\mathscr{H} \mid X$ stands for the hypergraph $(V \cap X, \{E \cap X : E \in \boldsymbol{E}\})$. The *dual* hypergraph $\mathscr{H}^*$ is obtained by interchanging the roles of vertices and edges of $\mathscr{H}$ keeping the incidencies, i.e. $V(\mathscr{H}^*) = \boldsymbol{E}$ and $\boldsymbol{E}(\mathscr{H}^*) = \{\boldsymbol{E}(x) : x \in V\}$. Now we are going to define two classes of hypergraphs, the linear spaces and the 2-covers.

A *linear space* $\mathscr{L}$ is a pair $(P, \boldsymbol{L})$ consisting of a set P of *points* and a set of $\boldsymbol{L}$ of subsets of P called lines with the properties that

(1) any two distinct points p and q are contained in a unique line, and
(2) every line contains at least two points.

The linear space is called *trivial* if it has only one line, $\boldsymbol{L} = \{P\}$. The linear space is called a *near pencil* if it has a line which contains all but one of the points of P. In 1948 deBruijn and Erdös [3] proved that for every nontrivial linear space one has

$$|\boldsymbol{L}| \geqslant |P|. \tag{1.1}$$

Moreover, here equality holds if and only if $\mathscr{L}$ is either a near pencil or a finite projective plane PG(n, 2).

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A pair (V, E) is called a $(v, k, 2)$-*cover*, (or briefly, a 2-*cover*), iff
(1) $|V| = n$,
(2) E is a collection of k-element subsets of V, called edges,
(3) every pair of elements of V is contained in at least one edge.
Denote by $C(v, k, 2)$ the minimum number of edges in a $(v, k, 2)$-cover. Then

$$C(v, k, 2) \geq \binom{v}{2} \Big/ \binom{k}{2}. \tag{1.2}$$

Note that the dual of a 2-cover is an intersecting hypergraph, and the dual of a 1-intersecting family is a linear space. A finite projective plane, PG$(n, 2)$, of order n is a $(n^2+n+1, n+1, 2)$-cover with n^2+n+1 edges. A hypergraph $\mathscr{H}$ is said to be *embedded* in the linear space $\mathscr{L}$, if $V(\mathscr{H}) \subset P$ and $E(\mathscr{H})L$. Vanstone [5] pointed out that if $\mathscr{H}$ is an $(n+1)$-uniform, 1-intersecting hypergraph with at most n^2+n+1 vertices, moreover

$$|E| \geq n^2, \tag{1.3}$$

then $\mathscr{H}$ can be embedded into a projective plane of order n. (This result was recently improved by Metsch [4], who replaced (1.3) by $|E| > n^2 - (n/6)$.)

2. Results

Theorem 2.1. *Suppose that V is a set of n^2+n+1 elements and E is a family of $(n+1)$-elements subsets covering all pairs of V, such that $|E| = n^2+n+2$. Then E contains a finite projective plane of order n.*

Corollary 2.2.

$$C(n^2+n+1, n+1, 2)\begin{cases} = n^2+n+1 & \text{if a PG}(n, 2) \text{ exists,} \\ \geq n^2+n+3 & \text{otherwise.}\end{cases}$$

This was a conjecture of Assaf and Mendelsohn [1]. They investigated the minimal 2-designs (what they call "imbrical" designs and "failed goemetries"). They have an analogous conjecture for affine geometries, which seems to me much more difficult.

Conjecture 2.3 [1].

$$C(n^2, n, 2)\begin{cases} = n^2+n & \text{if a PG}(n, 2) \text{ exists,} \\ \geq n^2+n+2 & \text{otherwise.}\end{cases}$$

As Baker [2] showed, the direct analog of Theorem 2.1 is not true. Using Baer subplanes she constructed minimal $(n^2, n, 2)$-covers of size n^2+n+1 for infinitely many values of n.

3. Proof of the theorem

If $\boldsymbol{E}$ is not a minimal 2-cover, i.e. $\boldsymbol{E}\backslash\{E\}$ is still a 2-cover, then $\boldsymbol{E}\backslash\{E\}$ is necessarily a projective plane and the theorem follows. So from now on we suppose on the contrary that every edge $E \in \boldsymbol{E}$ there exists a pair $\{x, y\}$ such that

$$\{x, y\} \text{ is covered only by } E. \tag{3.1}$$

As $\boldsymbol{E}(x)$ covers all vertices of V

$$\deg(x) \geqslant n+1 \tag{3.2}$$

holds for all $x \in V$. Moreover if equality holds in (3.2) then for all $y \in V\backslash\{x\}$

$$\{x, y\} \text{ is covered by } \boldsymbol{E} \text{ exactly once.} \tag{3.3}$$

Denote W the set of those vertices whose degree exceeds $n+1$. (3.3) implies that if $\{x, y\}$ is contained in more than one edge from $\boldsymbol{E}$, then $\{x, y\} \subset W$. We claim thtat

$$|W| \leqslant n+1. \tag{3.4}$$

Indeed, we obtain an upper bound as follows.

$$(n^2+n+2)(n+1) = \sum_{E \in \boldsymbol{E}} |E| = \sum_{x} \deg(x) \geqslant |V|\,(n+1) + |W|.$$

We distinguish two cases.

(i) If W intersects all the edges of $\boldsymbol{E}$.

Let p be any vertex from $V\backslash W$. Then all the sets $E\backslash\{p\}$ intersect W for $E \in \boldsymbol{E}(p)$. But these sets are pairwise disjoint by (3.3), so we have

$$|W| \geqslant \sum_{p \in E} |W \cap E| \geqslant \deg(p) = n+1. \tag{3.5}$$

Hence equality holds in (3.4). Now, (3.4) and (3.5) implies that for every edge E with $E \not\subset W$ one has $|E \cap W| = 1$. However

$$\sum_{E} |E \cap W| = \sum_{x \in W} \deg(x) \geqslant |W|\,(n+2) = (n+1)(n+2) > |\boldsymbol{E}|.$$

So there exist at least 2 edges E_1, E_2 with $|E_i \cap W| \geqslant 2$, and then $E_i \subset W$ $(i=1,2)$. Therefore by (3.4), actually $E_i = W$. Hence $E_1 = E_2$, so $\boldsymbol{E}\backslash\{E_1\}$ also forms a 2-cover, contradicting to (3.1). From now one we may suppose that

(ii) $W \cap E_0 = \emptyset$ for some $E_0 \in \boldsymbol{E}$.

Let $\boldsymbol{E}_0 = \{E \in \boldsymbol{E} : E \cap E_0 \neq \emptyset\}$. By (3.3) we have that $\boldsymbol{E}_0(x)$ covers all vertices of $V\backslash\{x\}$ exactly once. Hence the hypergraph $(V, \boldsymbol{E}_0)$ is $n+1$-regular and $|\boldsymbol{E}_0| = n^2+n+1$. So there exists an edge $E_1 \in \boldsymbol{E}$ disjoint from $\boldsymbol{E}_0$. Then for all $x \in E_1$ one has $\deg(\boldsymbol{E}, x) = \deg(\boldsymbol{E}_0, x) + 1$, i.e. $E_1 \cap W$. Therefore by (3.4) we have

$$E_1 = W \in \boldsymbol{E}. \tag{3.6}$$

Moreover $\deg(x) = n+2$ for all $x \in W$. As we have supposed in (3.1) there is a pair $\{x, y\} \subset W$ which is not covered by $\boldsymbol{E}\setminus\{W\}$. Let $\mathcal{H}_0$ be the restriction of $\boldsymbol{E}\setminus\{W\}$ to $(V\setminus W) \cup \{x, y\}$. Consider the dual of $\mathcal{H}_0$. The edges of $\mathcal{H}_0^*$ corresponding to the vertices from $V-W$ are denoted by $\boldsymbol{A}$, the duals of x and y are denoted by B_1, B_2, resp. Then $\mathcal{H}_0^*$ is an $(n+1)$-uniform hypergraph over n^2+n+1 elements. Moreover any two of its edges intersect in exactly one element, except $B_1 \cap B_2 = \emptyset$. As $\boldsymbol{A} \cup \{B_1\}$ has more than n^2 members (1.3) implies that there is a family $\boldsymbol{B}$ such that $\boldsymbol{A} \cup \{B_1\} \cup \boldsymbol{B}$ forms a projective plane. Then the restriction $\boldsymbol{B} \mid B_2$ is a linear space. If it is trivial linear space, then we obtain the contradiction that B_2 belongs to the line set of the projective plane, so $B_1 \cap B_2 \neq \emptyset$. Finally, if it is a nontrivial linear space, then (1.1) leads to the contradiction.

$$n = |\boldsymbol{B}| \geqslant |B_2| = n+1.$$

References

[1] A. Assaf and E. Mendelsohn, On the spectrum of imbrical designs, in Combinatorial Design Theory, Annals of Discrete Mathematics 34 (1987)
[2] Catherine Baker, Failed geometries, Lecture on Eleventh British Combinatorial Conference, London (July 1987).
[3] N.G. De Bruijn and P. Erdös, On a combinatorial problem, Indag. Math. 10 (1948) 421–423.
[4] K. Metsch, An improved bound for the embedding of linear spaces into projective planes, Geom. Dedicata 26 (1988) 333–340.
[5] S.A. Vanstone, The extendability of $(r, 1)$-designs, Proc. Third Manitoba Conf. on Numerical Math. (1973) 409–418.