

# Covering Pairs by $q^2 + q + 1$ Sets

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For given  $k$  and  $s$  let  $n(k, s)$  be the largest cardinality of a set whose pairs can be covered by  $s$   $k$ -sets. We determine  $n(k, q^2 + q + 1)$  if a  $PG(2, q)$  exists,  $k > q(q + 1)^2$ , and the remainder of  $k$  divided by  $(q + 1)$  is at least  $\sqrt{q}$ . Asymptotic results are also given for  $n(k, s)$  whenever  $s$  is fixed and  $k \rightarrow \infty$ . Our main tool is the theory of fractional matchings of hypergraphs. © 1990 Academic Press, Inc.

## 1. DEFINITIONS

This paper is organized as follows. In this section we recall some definitions. The first part of the paper is devoted to the fractional matchings of intersecting hypergraphs. (The proofs can be found in Section 7 and 8). In the second part we apply the results to the following problem: How large a set can be if its pairs can be covered by  $s$   $k$ -sets.

A *hypergraph*  $\mathbf{H}$  is a pair  $(V(\mathbf{H}), E(\mathbf{H}))$ , where  $V(\mathbf{H})$  is a (finite) set, the vertices or points, and  $E(\mathbf{H})$ , the edge-set, a collection of subsets of  $V(\mathbf{H})$ . If we want to emphasize that  $\mathbf{H}$  contains multiple edges, then we call  $\mathbf{H}$  a *multihypergraph*.  $\mathbf{G}$  is a subhypergraph of  $\mathbf{H}$  if  $V(\mathbf{G}) \subset V(\mathbf{H})$  and  $E(\mathbf{G}) \subset E(\mathbf{H})$ . The *dual* of  $\mathbf{H}$ ,  $\mathbf{H}^T$  is obtained by interchanging the role of vertices and edges, i.e.,  $V(\mathbf{H}^T) = E(\mathbf{H})$ , and  $E(\mathbf{H}^T) = \{E(p) : p \in V(\mathbf{H})\}$ , where  $E(p) = \{E \in E(\mathbf{H}) : p \in E\}$ . A hypergraph is an *r-graph*, or *r-uniform*, if all edges have  $r$  elements. The *rank* of  $\mathbf{H}$  is  $\max\{|E| : E \in E(\mathbf{H})\}$ . The *degree* of a point  $p$  is  $\deg_{\mathbf{H}}(p) = |\{E : E \in E(\mathbf{H}), p \in E\}|$ . The maximum degree,  $\max \deg_{\mathbf{H}}(p)$ , is denoted by  $D(\mathbf{H})$ . A hypergraph is *regular* if for all  $p \in V(\mathbf{H})$  we have  $\deg(p) = D$ . A *matching*  $\mathcal{M}$  is a subfamily of pairwise disjoint edges; the *matching number*  $\nu(\mathbf{H})$ , is the maximum number of edges in

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a matching in  $\mathbf{H}$ . If  $v(\mathbf{H}) = 1$ , i.e., if  $E \cap E' \neq \emptyset$  holds for all  $E, E' \in E(\mathbf{H})$ , then  $\mathbf{H}$  is called *intersecting*. A *cover*  $T$  is a subset of  $V(\mathbf{H})$  which meets all edges of  $\mathbf{H}$ , and the covering number  $\tau(\mathbf{H})$  is the minimum size of  $T$ . An  $r$ -uniform hypergraph  $\mathbf{H}$  is called a *projective plane* of order  $r-1$  if  $|V(\mathbf{H})| = |E(\mathbf{H})| = r^2 - r + 1$  and every two edges intersect in exactly one element. Briefly,  $\mathbf{H}$  is a  $PG(2, r-1)$ . Projective planes are known to exist whenever  $r-1$  is a power of a prime. An  $r$ -graph is a *truncated projective plane* (of order  $r-1$ ) if it is obtained from a  $PG(2, r-1)$  by deleting a vertex  $p$  and all the lines through  $p$ . A  $TPG(2, r-1)$  is the dual of the affine plane  $AG(2, r-1)$ . The notations  $\lfloor x \rfloor$  and  $\lceil x \rceil$  stand for the lower and upper integer parts of the real  $x$ , respectively.

## 2. FRACTIONAL MATCHINGS OF INTERSECTING HYPERGRAPHS

A *fractional matching* of the (multi)hypergraph  $\mathbf{H}$  is a non-negative function on the edges  $w: E(\mathbf{H}) \rightarrow \mathbf{R}^+$ , such that

$$\sum_{p \in E} w(E) \leq 1$$

holds for every vertex  $p \in V(\mathbf{H})$ . The *value* of  $w$ ,  $\|w\|$ , is the total sum  $\sum w(E)$ . The supremum of  $\|w\|$ , denoted by  $v^*(\mathbf{H})$ , is the *fractional matching number* of  $\mathbf{H}$ . A *fractional cover* of  $\mathbf{H}$  is a function on vertices,  $t: V(\mathbf{H}) \rightarrow \mathbf{R}^+$ , such that

$$\sum_{x \in E} t(x) \geq 1$$

holds for every edge  $E \in E(\mathbf{H})$ . The value of  $t$  is  $\|t\| = \sum_{x \in V} t(x)$ . The *fractional covering number*,  $\tau^*(\mathbf{H})$ , of  $\mathbf{H}$  is the infimum of  $\|t\|$ . The calculation of  $\tau^*$  and  $v^*$  are dual linear programming problems, so their optima coincide, i.e., for all  $\mathbf{H}$  we have

$$v \leq v^* = \tau^* \leq \tau \leq rv.$$

The value of  $\tau^*$  is always a rational, and there are *optimal* fractional matching  $w: E(\mathbf{H}) \rightarrow \mathbf{Q}^+$  and cover  $t: V(\mathbf{H}) \rightarrow \mathbf{Q}^+$  with  $\|w\| = \|t\| = \tau^*(\mathbf{H})$ . In [Fü] the following theorem is proved.

(2.1) Suppose that  $\mathbf{H}$  is an intersecting hypergraph of rank  $r$ . Then either  $\tau^*(\mathbf{H}) \leq r-1$ , or  $\mathbf{H}$  is a finite projective plane of order  $r-1$ .

In the latter case  $\tau^*(\mathbf{H}) = r-1 + (1/r)$ . In general, one cannot improve (2.1), because if  $\mathbf{H}$  is a hypergraph obtained from  $PG(2, r-1)$  by deleting

a line, then  $\tau^*(\mathbf{H}) = r - 1$ . There is another intersecting hypergraph  $\mathbf{G}$  with  $\tau^*(\mathbf{G}) = r - 1$ , the so-called *twisted projective plane*. Then  $|V(\mathbf{G})| = |E(\mathbf{G})| = r^2 - r$ , it is  $r$ -uniform, every degree is  $r$ , and the edges cover all pairs. Such a hypergraph is known to exist only for  $r \leq 4$  (see, e.g., in [F]), and it is proved that its existence implies that  $r$  or  $r - 2$  is a square. (Further constraints about the existence of twisted planes can be found in [LMV].)

(2.2) THEOREM. Suppose that  $\mathbf{H}$  is an intersecting hypergraph of rank  $q + 1$ . Then either

- (i)  $\mathbf{H}$  is a  $PG(2, q)$ , and then  $\tau^*(\mathbf{H}) = q + 1/(q + 1)$ , or
- (ii)  $\mathbf{H}$  contains a truncated projective plane  $TPG(2, q)$ , i.e.,  $TPG(2, q) \subseteq E(\mathbf{H})$ , and then  $\tau^*(\mathbf{H}) = q$ , or
- (iii)  $\mathbf{H}$  is a twisted projective plane, and then  $\tau^*(\mathbf{H}) = q$ , or
- (iii/a)  $\mathbf{H}$  contains a twisted projective plane, and then  $\tau^*(\mathbf{H}) = q$  (in this case  $q = 2$ ), or
- (iv)  $\tau^*(\mathbf{H}) \leq q - 1/(q^2 + q - 1)$ .

The proof of this theorem is postponed to Section 7. Let  $\varepsilon = \varepsilon(q + 1)$  denote the largest real such that in (iv) one can write  $\tau^*(\mathbf{H}) \leq q - \varepsilon$ . Delete three nonconcurrent lines of a  $PG(2, q)$ . The obtained hypergraph  $\mathbf{F}$  has fractional matching number  $\tau^*(\mathbf{F}) = q - 1/(2q - 1)$ . Hence in this case  $\varepsilon \leq 1/(2q - 1)$ . It is known that  $\varepsilon(3) = \frac{1}{5}$  (see [CFGG]).

(2.3) Conjecture. For  $r \geq 4$ ,  $\varepsilon(r) \geq 1/(2r - 3)$ .

Later we will see some partial evidence that  $\varepsilon(r) = O(1/r)$ .

We determine the maximum of  $\tau^*$  for another class of hypergraphs. Define

$$\tau_i^*(s) = \max\{\tau^*(\mathbf{H}) : \mathbf{H} \text{ is intersecting, } |V(\mathbf{H})| \leq s\}. \quad (2.4)$$

It is not difficult to see that

$$\tau_i^*(q^2 + q + 1) \leq q + 1/(q + 1),$$

and here equality holds iff a  $PG(2, q)$  exists. (This was proved, e.g., in [AKL, PS]). We will use the following improvement of this statement.

(2.5) THEOREM. Let  $\mathbf{H}$  be an intersecting hypergraph over  $q^2 + q + 1$  elements ( $q$  an integer). If  $\mathbf{H}$  does not contain a  $PG(2, q)$  as a subhypergraph then  $\tau^*(\mathbf{H}) \leq q + (q - 1)/(q^2 + q - 1)$ .

If we replace a line  $L$  of a  $PG(2, q)$  by a superset  $L \cup \{x\}$ , where  $x \in V(PG(2, q)) - L$ , then for the obtained intersecting hypergraph  $\mathbf{F}$  we have equality in (2.5). So the upper bound in Theorem (2.5) could not be improved in general. (To see that  $\tau^*(\mathbf{F}) = (q^3 + q^2 - 1)/(q^2 + q - 1)$  one can consider the fractional matching  $w$ ,

$$w(E) = \begin{cases} (q-1)/(q^2+q-1) & \text{if } x \in E, E \neq L \cup \{x\} \\ q/(q^2+q-1) & \text{otherwise,} \end{cases}$$

and fractional cover  $t$ ,

$$t(p) = \begin{cases} (q-1)/(q^2+q-1) & \text{if } p \in L \\ q/(q^2+q-1) & \text{if } p \in V(\mathbf{F}) - L, \end{cases}$$

with values  $\|w\| = \|t\|$ .)

### 3. COVERING OF PAIRS BY A SMALL NUMBER OF SUBSETS

$C(n, k, t)$  denotes the minimal number of  $k$ -sets required to cover all pairs of an  $n$ -set. For fixed  $k$ , and for  $n \rightarrow \infty$ , Erdős and Hanani [EH] proved that

$$\binom{n}{2} / \binom{k}{2} \leq C(n, k, 2) \leq (1 + o(1)) \binom{n}{2} / \binom{k}{2}. \quad (3.0)$$

This limit theorem easily follows from the following theorem of Wilson [W]. For all  $n > n_0(k)$  if  $\binom{n}{2}/\binom{k}{2}$  and  $(n-1)/(k-1)$  are integers then a Steiner system  $S(n, k, 2)$  exists. But the lower bound in (3.0) is very poor if  $n$  is not much bigger than  $k$ . Mills [M79] determined the solution of  $C(n, k, 2) = s$  for all  $s$  up to 12. For  $s = 13$  he [M84] and Todorov [T] determine all  $(n, k)$  pairs with  $C(n, k, 2) = 13$ , except the pairs  $(28, 9)$  and  $(41, 13)$  which are undecided.

Suppose  $s$  is given, and let  $n(k, s) = \max\{n : C(n, k, 2) \leq s\}$ ; i.e.,  $n(k, s)$  is the largest size of a set whose pairs can be covered by  $s$   $k$ -sets. In the following theorem  $\tau_i^*(s)$  was defined in (2.4).

(3.1) THEOREM. For all  $s$  and  $k$  one has

$$\tau_i^*(s)k - s < n(k, s) \leq \tau_i^*(s)k.$$

For any given  $s$  equality holds for infinitely many  $k$ .

Mills [M79] also proved that  $\lim_{k \rightarrow \infty} n(k, s)/k$  exists and equals to its maximum. He has determined this limit for  $s \leq 13$ . With our notations his results is the following:

$s$	1	2	3	4	5	6	7	8	9	10	11	12	13
$\tau_i^*(s)$	1	1	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{9}{5}$	2	$\frac{7}{3}$	$\frac{12}{5}$	$\frac{5}{2}$	$\frac{8}{3}$	$\frac{14}{5}$	3	$\frac{13}{4}$

Some of his result ( $s \leq 6$ ) was rediscovered in [SVZ].

*Proof of (3.1).* Considering the dual problem we obtain

(3.2) PROPOSITION.  $n(k, s) = \max\{|E(\mathbf{H})|: \mathbf{H} \text{ is an intersecting (multi)-hypergraph over } s \text{ elements with maximum degree at most } k\}$ .

Then the upper bound follows from the fact that for all hypergraphs

$$|E(\mathbf{H})| \leq \tau^*(\mathbf{H}) D(\mathbf{H}). \quad (3.3)$$

(Indeed,  $w(E) = 1/D$  is a fractional matching with value  $|E(\mathbf{H})|/D$ .)

To prove the lower bound let  $\mathbf{G}$  be an intersecting hypergraph with  $|V(\mathbf{G})| \leq s$ ,  $\tau^*(\mathbf{G}) = \tau_i^*(s)$ . We may suppose that  $|E(\mathbf{G})| \leq |V(\mathbf{G})| \leq s$  (see [Fü], or later (7.4)) Let  $w: E(\mathbf{G}) \rightarrow \mathbf{R}^+$  be an optimal fractional matching of  $\mathbf{G}$ . Replace every edge  $E$  of  $\mathbf{G}$  by  $\lfloor w(E)k \rfloor$  copies. We obtain a multi-hypergraph  $\mathbf{H}$ :

$$\begin{aligned} n(k, s) &\geq |E(\mathbf{H})| \\ &= \sum_{E \in E(\mathbf{G})} \lfloor w(E)k \rfloor > \sum (w(E)k - 1) \geq \tau_i^*(s)k - s. \quad \blacksquare \end{aligned}$$

In the case  $s = q^2 + q + 1$  if a  $PG(2, q)$  exists, (3.3) and (2.5) imply much more. We will state our main result in Section 5 after some preparations.

#### 4. GENERALIZED $r$ -COVERS

Let  $\mathbf{H}$  be an intersecting hypergraph,  $r$  a non-negative integer,  $0 \leq r \leq q$ . The pair of (multi)hypergraphs  $(\mathbf{B}, \mathbf{L})$  over  $V(\mathbf{H})$  is a *generalized  $r$ -cover* of  $\mathbf{H}$  if the following holds

- (i)  $\mathbf{L}$  is a subset of edges of  $\mathbf{H}$  (with multiplicities)
- (ii)  $\mathbf{B} \cup \mathbf{H}$  is intersecting (i.e.,  $\mathbf{B} \cup \mathbf{H} \cup \mathbf{L}$  is intersecting),
- (iii)  $|E(\mathbf{B})| \geq |E(\mathbf{L})|$ ,
- (iv)  $E(\mathbf{B}) \cap E(\mathbf{L}) = \emptyset$  (i.e., an edge of  $\mathbf{H}$  cannot appear in both  $\mathbf{B}$  and  $\mathbf{L}$ ),
- (v)  $\deg_{\mathbf{B}}(x) \leq \deg_{\mathbf{L}}(x) + r$ .

The value of the  $r$ -cover is  $v_r(\mathbf{B}, \mathbf{L}) = |E(\mathbf{B})| - |E(\mathbf{L})|$ . Finally,  $v_r(\mathbf{H}) = \max\{v_r(\mathbf{B}, \mathbf{L}): (\mathbf{B}, \mathbf{L}) \text{ is a generalized } r\text{-cover of } \mathbf{H}\}$ .

In this section  $\mathbf{H}$  will be a finite projective plane  $\mathbf{P}$  of order  $q$ . Let  $v_r(q) = \max\{v_r(\mathbf{P}): \mathbf{P} \text{ is a projective plane of order } q\}$ .

(4.1) We have  $v_r(q) \geq rq - q + r$ .

*Proof.* We give a construction. Let  $L_0$  be a line,  $\{p_1, \dots, p_r\} \subset L_0$  an  $r$ -element set. Then define  $\mathbf{L}$  as  $(q-r)$  copies of  $L_0$ , and let  $E(\mathbf{B}) = \{L \in E(\mathbf{P}): L \neq L_0, L \cap \{p_1, \dots, p_r\} \neq \emptyset\}$ . ■

(4.2) We have  $v_r(q) \leq rq$  for all  $0 < r \leq q$ .

*Proof.* Every set  $B \in E(\mathbf{B})$  has at least  $q+1$  elements, so we have

$$r(q^2 + q + 1) \geq \sum (\deg_{\mathbf{B}}(x) - \deg_{\mathbf{L}}(x))$$

$$= \sum_{B \in E(\mathbf{B})} |B| - \sum_{L \in E(\mathbf{L})} |L| \geq (q+1)(|E(\mathbf{B})| - |E(\mathbf{L})|). \quad \blacksquare \quad (4.3)$$

(4.4) THEOREM. If  $r \geq \sqrt{q}$ , then  $v_r(q) = rq - q + r$ .

For the proof we need a new definition and a lemma. A set  $B$  is a *blocking set* of the hypergraph  $\mathbf{H}$  if it intersects all edges but does not contain any. The investigation of the blocking sets of block designs was initiated by Pelikán [P]. He observed that there is no blocking set  $T$  of the projective plane  $\mathbf{P}$  of order  $q \leq 2$ , and for  $q \geq 3$  one has  $|T| \geq q+1 + \sqrt{q/2}$ .

(4.5) (Pelikán [Pe], Bruen [B]) Suppose  $q \geq 3$  and  $1 \leq |T \cap L| < q+1$  holds for all line of a  $PG(2, q)$ . Then  $|T| \geq q+1 + \sqrt{q}$ . Moreover if equality holds then  $T$  induces a Bear subplane.

That is, the system  $\{L \cap T: L \text{ is a line, } |L \cap T| > 1\}$  is a projective plane of order  $\sqrt{q}$ .

*Proof of (4.4).* Let  $(\mathbf{B}, \mathbf{L})$  be an  $r$ -cover of  $\mathbf{P}$ . If  $\mathbf{B}$  contains a line,  $L_0$ , of  $\mathbf{P}$  then by definition we have

$$\begin{aligned} v_r(\mathbf{B}, \mathbf{L}) &= |E(\mathbf{B})| - |E(\mathbf{L})| \\ &\leq 1 + \sum_{x \in L_0} \max\{0, \deg_{\mathbf{B}}(x) - \deg_{\mathbf{L}}(x)\} \\ &\leq 1 + (r-1)(q+1). \end{aligned} \quad (4.6)$$

If  $\mathbf{B}$  does not contain any line, then  $|B| \geq 1 + q + \sqrt{q}$  holds for all  $B \in E(\mathbf{B})$ , by (4.5). Hence we have from (4.3) that

$$\begin{aligned} r(q^2 + q + 1) &\geq (q+1) \cdot v_r(\mathbf{B}, \mathbf{L}) + \sum_{B \in E(\mathbf{B})} (|B| - q - 1) \\ &\geq (q+1 + \sqrt{q}) v_r(\mathbf{B}, \mathbf{L}). \end{aligned} \quad (4.7)$$

This implies

$$rq + r - r\sqrt{q} \geq v_r(\mathbf{B}, \mathbf{L}).$$

The left hand side is less than  $rq + r - q$  for  $r > \sqrt{q}$ . ■

The determination of  $v_r(q)$  for  $1 \leq r < \sqrt{q}$  seems to be very difficult. The following example shows that  $v_r(q)$  can be much larger than the lower bound in (4.1).

(4.8) EXAMPLE. Let  $\mathbf{P}$  be a Desarguesian projective plane of order  $q$ , where  $\sqrt{q}$  is an integer. Let  $B_1, \dots, B_{q-\sqrt{q}+1}$  be a decomposition of  $V(\mathbf{P})$  into Baer subplanes. (Such a decomposition exists, see [Br, Y].) Let  $L_0$  be a line and let  $\mathbf{A} = \{A_1, \dots, A_{q-\sqrt{q}+1}\}$  be an intersecting family on  $L_0$  such that the maximum degree  $D(\mathbf{A}) \leq \sqrt{q}(1 + o(1))$ . (It is easy to prove that such a family exists.) Define

$$E(\mathbf{B}) = \{A_i \cup B_i : 1 \leq i \leq q - \sqrt{q} + 1\}$$

$$E(\mathbf{L}) = D(\mathbf{A}) \text{ copies of } L_0.$$

Then  $(\mathbf{B}, \mathbf{L})$  is a generalized 1-cover of  $\mathbf{P}$  with value  $q - 2\sqrt{q} + o(\sqrt{q})$ . Hence

$$v_r(q) \geq rq - 2r(\sqrt{q} + o(\sqrt{q})).$$

(4.9) COROLLARY.  $v_1(3) = 1$ , and  $v_1(4) = 2$ .

*Proof.*  $v_1(3) \geq 1$  follows from (4.1).  $v_1(4) \geq 2$  is given by (4.8) with the following modifications. Let  $B_1, B_2, B_3$  be three disjoint Baer subplanes of  $GF(2, 4)$  ( $|B_i| = 7$ ) and let  $L_0 = \{x_1, \dots, x_5\}$  be an arbitrary line. Suppose that  $B_i \cap L_0 = \{x_i\}$  for  $i = 1, 2$  and  $B_3 \cap L_0 = \{x_3, x_4, x_5\}$ . Then the following family is a generalized 1-cover with value 2.  $\mathbf{B} = \{B_1 \cup \{x_2, x_3\}, B_2 \cup \{x_4, x_5\}, B_3\}$ ,  $\mathbf{L} = \{L_0\}$ .

To obtain an upper bound for  $v_1(q)$ , the proof of (4.4) gives that if  $(\mathbf{B}, \mathbf{L})$  is a generalized 1-cover with  $\mathbf{B}$  not containing any line, then  $v_1(\mathbf{B}, \mathbf{L}) \leq 1$  by (4.6).  $v_1 > 1$  implies  $|\mathbf{B}| > v_1$ , i.e.,  $|\mathbf{B}| \geq v_1 + 1$ . Hence (4.7) implies

$$q^2 + q + 1 \geq (q + 1 + \lceil \sqrt{q} \rceil) v_1 + \lceil \sqrt{q} \rceil$$

i.e.,  $v_1(3) \leq 1$  and  $v_1(4) \leq 2$ . ■

Call a generalized  $r$ -cover  $(\mathbf{B}, \mathbf{L})$  of  $\mathbf{P}$  *optimal* if  $v_r(\mathbf{B}, \mathbf{L}) = v_r(q)$ .

(4.10) PROPOSITION. There exists an optimal  $r$ -cover  $(\mathbf{B}, \mathbf{L})$  of  $\mathbf{P}$  such that every line  $L \in \mathbf{L}$  has multiplicity  $m(L)$  at most  $\lfloor \sqrt{q} \rfloor (q - r + 1) - 1$ .

*Proof.* If  $v_r(q) = rq + r - q$ , then Example (4.1) is optimal, and then  $\max m(L) = q - r$ . Consider the case  $v_r(q) > rq + r - q$ . Then  $r < \sqrt{q}$ , by Theorem (4.4). Moreover, (4.6) implies that  $\mathbf{B}$  does not contain any line. Hence (4.7) gives that

$$\begin{aligned} r(q^2 + q + 1) &\leq (q + 1) v_r(q) + \sum_{B \in \mathbf{B}} (|B| - q - 1) \\ &> (q + 1)(qr + r - q) + \lceil \sqrt{q} \rceil |\mathbf{B}|, \end{aligned}$$

implying  $\lfloor \sqrt{q} \rfloor (q - r + 1) \geq |\mathbf{B}|$ . Finally, clearly,  $m(L) \leq |\mathbf{L}| < |\mathbf{B}|$ . ■

## 5. THE CASE OF $s = q^2 + q + 1$

(5.1) THEOREM. Suppose that a  $PG(2, q)$  exists and  $s = q^2 + q + 1$ . Let  $k = (q + 1)a + r$ , where  $0 \leq r \leq q$ . Then if  $a$  is large enough ( $a \geq q^2 + q - rq - 1$ ) we have

$$n(k, s) = (q^2 + q + 1)a + v_r(q).$$

This is a large improvement on a result from [To] if  $k$  tends to infinity. Theorems (3.1) and (2.5) imply

(5.2) COROLLARY. If a  $PG(2, q)$  exists then

$$n((q + 1)a, q^2 + q + 1) = a(q^2 + q + 1)$$

and

$$n((q + 1)a + q, q^2 + q + 1) = a(q^2 + q + 1) + q^2$$

hold for all integers  $a \geq 0$ .

(5.3) COROLLARY. We have

$$\begin{array}{ll} n(4a + 1, 13) = 13a + 1 & \text{if } a \geq 8, \\ n(4a + 2, 13) = 13a + 5 & \text{if } a \geq 5, \\ n(5a + 1, 21) = 21a + 2 & \text{if } a \geq 15, \\ n(5a + 2, 21) = 21a + 6 & \text{if } a \geq 11, \\ n(5a + 3, 21) = 21a + 11 & \text{if } a \geq 7. \end{array}$$



This corollary follows from (5.1) and (4.4) for  $r \geq \sqrt{q}$ . In the remaining cases  $r = 1$  and  $v_1(3) = 1$ ,  $v_1(4) = 2$  follow from Corollary (4.9).

*Proof of Theorem (5.1).* First we prove the lower bound. Suppose that  $(\mathbf{B}, \mathbf{L})$  is an optimal  $r$ -cover of  $\mathbf{P}$ , and let  $m(L)$  be the multiplicity of the line  $L \in \mathbf{L}$ ,  $M = \max\{m(L) : L \in \mathbf{L}\}$ . Suppose that  $k = a(q+1) + r$ , ( $0 \leq r \leq q$ ), where  $a \geq M$ . We may suppose that  $M < \lfloor \sqrt{q} \rfloor (q-r+1)$  by Proposition (4.10). Define the following hypergraph  $\mathbf{H}$  with the vertex set  $V(\mathbf{P})$ :  $E(\mathbf{H}) = E(\mathbf{B}) \cup \{(a - m(L)) L : L \in E(\mathbf{P})\}$ , i.e., the multiplicity of a line from  $\mathbf{P}$  is  $(a - m(L))$ . Obviously,  $D(\mathbf{H}) \leq k$ ,  $\mathbf{H}$  is intersecting, and

$$|E(\mathbf{H})| = (q^2 + q + 1)a + v_r(\mathbf{B}, \mathbf{L}) = (q^2 + q + 1)a + v_r(q). \quad (5.4)$$

(5.4) and (4.1) imply that for  $a \geq q - r$  one has

$$n((q+1)a + r, q^2 + q + 1) \geq (q^2 + q + 1)a + rq + r - q. \quad (5.5)$$

Proof of the upper bound: Let  $\mathbf{H}$  be an intersecting multihypergraph over  $q^2 + q + 1$  elements. According to Theorem (2.5) we distinguish two cases.

(i) If  $\tau^*(\mathbf{H}) \leq q + (q-1)/(q^2 + q - 1)$ , then (3.3) and (5.5) imply that

$$\begin{aligned} |E(\mathbf{H})| &\leq ((q+1)a + r) \tau^*(\mathbf{H}) < (q^2 + q + 1)a \\ &\quad + rq + r - q \leq n((q+1)a + r, q^2 + q + 1) \end{aligned}$$

for  $a \geq q(q+1-r) - 1$ .

(ii) If  $\tau^*(\mathbf{H}) > q + (q-1)/(q^2 + q - 1)$  then a finite plane  $\mathbf{P}$  is a subhypergraph of  $\mathbf{H}$ . Define the (multi)hypergraphs  $\mathbf{B}$  and  $\mathbf{L}$  as follows: Denote the multiplicity of a line  $L \in E(\mathbf{P})$  by  $m(L)$ . Then  $E(\mathbf{L})$  consists of the lines of  $\mathbf{P}$  with multiplicities  $\max\{0; a - m(L)\}$ .  $E(\mathbf{B})$  consists of the edges of  $\mathbf{H}$  different from the lines and with the lines  $L$  of  $\mathbf{P}$  with multiplicities  $\max\{0; m(L) - a\}$ . Then  $(\mathbf{B}, \mathbf{L})$  is a generalized  $r$ -cover of  $\mathbf{P}$ , hence

$$|E(\mathbf{H})| = a(q^2 + q + 1) + |E(\mathbf{B})| - |E(\mathbf{L})| \leq a(q^2 + q + 1) + v_r(q). \quad \blacksquare$$

## 6. DIRECTION OF FURTHER RESEARCH

The method of the previous chapters is a powerful tool to determine  $n(k, s)$  asymptotically, whenever we are able to calculate  $\tau_i^*(s)$ . E.g., Theorem (2.2) easily implies the case  $s = q^2 + q$ .

(6.1) THEOREM. Let  $\mathbf{H}$  be an intersecting hypergraph over  $q^2 + q$  elements. Then either

- (i)  $\mathbf{H}$  contains a  $TPG(2, q)$ , or
- (ii)  $\mathbf{H}$  contains a twisted plane, or
- (iii)  $\tau^*(\mathbf{H}) \leq q - 1/3(q + 1)^3$ .

The proof of (6.1) is analogous to the proof of (2.5) (see Section 9). Suppose that a  $TPG(2, q)$  or a twisted plane on  $q^2 + q$  vertices exist, let  $k \geq 10q^4$ , and write  $k$  in the forms  $k = a_1q + r_1 = a_2(q + 1) + r_2$ , where  $0 \leq r_1 < q$ ,  $0 \leq r_2 < q + 1$ . Then (6.1) and (3.3) imply that

$$n(k, q^2 + q) = \max\{a_1q^2 + v_{r_1}^1(q), a_2(q^2 + q) + v_{r_2}^2(q)\}, \quad (6.2)$$

where  $v_r^\alpha(q)$  is the maximum value of a generalized  $r$ -cover in a  $TPG(2, q)$  (in the case  $\alpha = 1$ ) or in a twisted plane (in the case  $\alpha = 2$ ).

It seems to be hopeful to determine  $\tau_i^*(q^2 + q + 1 + a)$  if  $|a|$  is small and a  $PG(2, q)$  exists.

(6.3) Conjecture. We have

$$\tau_i^*(q^2 + q + 2) \leq q + 2/(2q + 1),$$

and here equality holds if a  $PG(2, q)$  exists.

Our method also can be extended to the following generalization of  $n(k, s)$  (also investigated in [M79]). Recall that  $C(n, k, t)$  denotes the minimal number of  $k$ -sets required to cover all  $t$ -sets of an  $n$ -set. For large  $n$  Rödl [R] proved that

$$\binom{n}{t} / \binom{k}{t} \leq C(n, k, t) \leq (1 + o(1)) \binom{n}{t} / \binom{k}{t}.$$

We are interested in the case when  $k$  is large. Let  $n_t(k, s) = \max\{n : C(n, k, t) \leq s\}$ , i.e., the largest size of a set whose  $t$ -sets can be covered by  $s$   $k$ -sets. The followings are simple generalization of (3.2), (3.1).  $\mathbf{H}$  is  $t$ -wise intersecting if  $E_1 \cap \dots \cap E_t \neq \emptyset$  for all  $t$  edges of  $\mathbf{H}$ .

(6.4) PROPOSITION.  $n_t(k, s) = \max\{|E(\mathbf{H})| : \mathbf{H} \text{ is a } t\text{-wise intersecting multihypergraph over } s \text{ elements with maximum degree at most } k\}$ .

Let  $\tau^*(s, t) = \max\{\tau^*(\mathbf{H}) : \mathbf{H} \text{ is } t\text{-wise intersecting over } s \text{ elements}\}$ .

(6.5) PROPOSITION.  $\tau^*(s, t)k - s < n_t(k, s) \leq \tau^*(s, t)k$ , and here equality holds for infinitely many values of  $n$ .

## 7. PROOF OF THEOREM (2.2)

A hypergraph  $\mathbf{H}$  is called  $\tau^*$ -critical if  $\tau^*(\mathbf{H}') < \tau^*(\mathbf{H})$  holds for all sub-hypergraphs  $\mathbf{H}'$ ; i.e., we cannot delete an edge without decreasing the value of  $\tau^*$ . Let  $w$  be an optimal fractional matching of the hypergraph  $\mathbf{H}$ . The support  $T$  of  $w$  is the set of vertices  $p$  for which  $\sum_{p \in E} w(E) = 1$ , i.e., the set of saturated vertices. The following lemma easily follows from the basic properties of linear programming.

(7.1) LEMMA. (see in [F88]) Let  $\mathbf{H}$  be  $\tau^*$ -critical and  $T$  be a maximal support. Then  $|E(\mathbf{H})| \leq |T|$ .

This lemma implies, e.g., that if  $\mathbf{H}$  is  $\tau^*$ -critical of rank  $r$  then

$$|E(\mathbf{H})| \leq |V(\mathbf{H})|, \quad (7.2)$$

$$|E(\mathbf{H})| \leq r\tau^*, \quad (7.3)$$

and if equality holds, then  $V(\mathbf{H}) = T$ . We will need the following sharpening of (7.1).

(7.4) LEMMA. Let  $\mathbf{H}$  be  $\tau^*$ -critical and  $T$  a maximal support. Then the characteristic vectors of  $E \cap T$  ( $E \in E(\mathbf{H})$ ) are linearly independent in  $\mathbf{R}^T$ .

*Proof.* Let  $\mathbf{v}(E)$  denote the characteristic vector of  $E \cap T$ , i.e.,

$$\mathbf{v}(E)(x) = \begin{cases} 1 & \text{if } x \in T \cap E \\ 0 & \text{if } x \in T - E \end{cases}$$

Suppose on the contrary that one has reals  $\alpha(E)$  such that  $\sum_E \alpha(E) \mathbf{v}(E) = 0$ . Suppose that  $\sum_E \alpha(E) \geq 0$ . Let  $z$  be a real, and  $w$  be an optimal fractional matching with the maximal support  $T$ . Then  $w(E) > 0$  for all edges. Define

$$w(E, z) = w(E) + z\alpha(E).$$

This is a fractional matching of  $\mathbf{H}$  if  $|z|$  is sufficiently small with value  $\|w\| + z \sum \alpha(E)$ . Start with  $z = 0$  and increase it until we hit a constraint either of the type

$$w(E, z) \geq 0$$

or of the type

$$\sum_{x \in E} w(E, z) \leq 1,$$

where now  $x \notin T$ . In both cases we get a contradiction to one of the earlier constraints (i.e., that  $\mathbf{H}$  is  $\tau^*$ -critical, and  $T$  is maximal). ■

(7.5) LEMMA. *Let  $\mathbf{H}$  be an arbitrary hypergraph and  $w: E(\mathbf{H}) \rightarrow \mathbf{R}^+$  an optimal fractional matching,  $t: V(\mathbf{H}) \rightarrow \mathbf{R}^+$  an optimal fractional cover. Suppose that for some  $p \in V(\mathbf{H})$  we have  $t(p) > 0$ . Then  $\sum_{p \in E} w(E) = 1$ , i.e.,  $p$  is saturated by  $w$ .*

This is a well-known lemma in linear programming.

*Proof.* Let  $s(x) = \sum_{x \in E} w(E)$ . Then we have

$$\begin{aligned} 0 &\leq \sum t(x)(1 - s(x)) = \sum t(x) - \sum_{x \in E} \sum t(x) w(E) \\ &= \tau^* - \sum_E w(E) \left( \sum_{x \in E} t(x) \right) \leq \tau^* - \sum_E w(E) = 0, \end{aligned}$$

i.e.,  $1 - s(x) = 0$  whenever  $t(x) > 0$ . ■

We will prove Theorem (2.2) in the following form.

(7.6) THEOREM. *If  $\mathbf{H}$  is  $\tau^*$ -critical, intersecting, and  $(q+1)$ -uniform and  $\tau^*(\mathbf{H}) = q - \varepsilon$  where  $0 \leq \varepsilon < 1/(q^2 + q - 1)$ , then  $\mathbf{H}$  is either a truncated or a twisted projective plane.*

The next step of the proof requires the following

LEMMA. *If  $T$  is a  $(q+1)$ -element set and it intersects all edges of a twisted plane  $\mathbf{H}$  of order  $q$ ,  $q \geq 3$ , then  $T \in E(\mathbf{H})$ .*

*Proof.* Suppose that  $T$  does not contain any edge of  $\mathbf{H}$ . Let  $m = \max\{|T \cap E| : E \in E(\mathbf{H})\}$ ,  $|E_0 \cap T| = m$ . Let  $x \in E_0 \setminus T$ . There are  $q$  disjoint edges of  $\mathbf{H}$  through  $x$  which are pairwise disjoint outside of  $x$ . This implies that

$$(q-1) + |E_0 \cap T| \leq |T|,$$

i.e.,  $m \leq 2$ . For  $q \geq 3$  there exists an  $x \in E_0 \setminus T$  such that all the  $q+1$  edges through  $x$  are disjoint outside of  $E_0 \setminus T$ . Hence  $|E_0 \cap T| = 1$ . This is a contradiction, because there is no  $(q+1)$ -element set intersecting every edge in a singleton. ■

*Proof of (2.2) from (7.6).* Let now  $\mathbf{H}$  be an arbitrary intersecting hypergraph of rank  $q+1$  with  $\tau^* = q - \varepsilon$ . Deleting edges we obtain a  $\tau^*$ -critical subhypergraph  $\mathbf{H}^1$  of  $\mathbf{H}$  with  $\tau^*(\mathbf{H}) = \tau^*(\mathbf{H}^1)$ . If  $\mathbf{H}^1$  is not  $(q+1)$ -uniform we can add extra points of degree 1, obtaining  $\mathbf{H}^2$ . Then Theorem (7.6)

implies that  $\mathbf{H}^2$  is one of two extreme cases. Neither of them has vertex of degree 1, hence  $\mathbf{H}^1 = \mathbf{H}^2$ . In the case  $\mathbf{H}^1$  is a twisted plane and  $q \geq 3$ , we obtain that  $\mathbf{H}^1 = \mathbf{H}$ , because if a  $(q+1)$ -element set  $T$  intersects all the edges of  $\mathbf{H}^1$  then  $T \in E(\mathbf{H}^1)$  by the above lemma. ■

The rest of this section is devoted to the proof of (7.6). Let  $w$  be an optimal fractional matching of  $\mathbf{H}$  with a support  $T$  of maximal size. As  $\mathbf{H}$  is critical we have

$$w(E) > 0 \quad \text{for all edges } E \in E(\mathbf{H}). \quad (7.7)$$

For a vertex  $x \in V(\mathbf{H})$  define  $s(x) = \sum_{x \in E} w(E)$ . Then  $0 < s(x) \leq 1$ . Let  $E_0$  be an arbitrary edge,  $x \in E_0$ . We have

$$\begin{aligned} s(x) + q &\geq \sum_{y \in E_0} s(y) = \sum_E w(E) |E \cap E_0| \\ &= \tau^*(\mathbf{H}) + qw(E_0) + \sum_{E \neq E_0} w(E) (|E \cap E_0| - 1). \end{aligned} \quad (7.8)$$

This implies that

$$\frac{s(x) + \varepsilon}{q} \geq w(E_0) \quad (7.9)$$

holds for all  $x \in E_0 \in E(\mathbf{H})$ . We can improve on (7.9) if  $\deg(x) < q$ . Fix  $x$  and add up (7.9) for all  $x \in E_0 \in E(\mathbf{H})$ . We obtain

$$\frac{d}{q} (s(x) + \varepsilon) \geq s(x),$$

i.e.,  $s(x) \leq d\varepsilon/(q-d)$ . Then (7.9) implies that

$$\varepsilon \geq \frac{\varepsilon}{q-d(x)} \geq w(E_0) \quad (7.10)$$

holds for  $x \in E_0$  if  $\deg(x) = d(x) < q$ . For this  $x$

$$(q-1)\varepsilon \geq d(x)\varepsilon \geq s(x). \quad (7.11)$$

Divide  $V(\mathbf{H})$  into three parts:  $T$  is the set of saturated vertices,  $A = \{x \in V(\mathbf{H}) - T: \deg(x) < q\}$ ,  $B = \{x \in V(\mathbf{H}) - T: \deg(x) \geq q\}$ . The inequality (7.10) implies that

$$\deg(x) \geq q \quad (7.12)$$

holds for each point  $x \in T \cup B$ . If  $E \cap A \neq \emptyset$  for an edge  $E \in E(\mathbf{H})$  then we have

$$|E \cap A| = 1. \quad (7.13)$$

Indeed, as we used in (7.8) we have

$$\tau^* = q - \varepsilon < \sum_{x \in E} s(x) \leq |E \cap A| + \sum_{x \in E \cap A} s(x).$$

However, by (7.11), for  $x \in A$  one has  $s(x) \leq \varepsilon(q - 1)$ , so the right-hand side is not larger than

$$(q + 1) - |E \cap A|(1 - (q - 1)\varepsilon),$$

i.e.,

$$|E \cap A| \leq \lfloor (1 + \varepsilon)/(1 - (q - 1)\varepsilon) \rfloor = 1.$$

Let  $\mathcal{A} = \{E \in E(\mathbf{H}) : A \cap E \neq \emptyset\}$ ,  $|\mathcal{A}| = a$ . Now (7.13) implies that

$$\sum_{x \in A} s(x) \leq \varepsilon a. \quad (7.14)$$

Our next claim is

$$|\mathcal{A}| < 2q. \quad (7.15)$$

Indeed, (7.9) and (7.10) imply that

$$q - \varepsilon = \tau^* = \sum w(E) \leq a\varepsilon + (|E(\mathbf{H})| - a) \frac{1 + \varepsilon}{q}.$$

Using  $|E(\mathbf{H})| \leq (q + 1)(q - \varepsilon)$  (by (7.3)) and the fact that  $\varepsilon < 1/(3q - 1)$  we obtain (7.15).

(7.16) PROPOSITION.  $|T \cup B| = q^2 + q$ .

*Proof.* The lower bound for  $|T \cup B|$  follows from (7.14) and from (7.15).

$$\begin{aligned} |T \cup B| &\geq \sum_{x \in T \cup B} s(x) = (q + 1)\tau^* - \sum_{x \in A} s(x) \\ &\geq (q + 1)(q - \varepsilon) - \varepsilon a = q^2 + q - \varepsilon(q + 1 + a) > q^2 + q - 1. \end{aligned}$$

To prove an upper bound for  $|T \cup B|$  start again with the inequality  $\sum_{x \in E} s(x) \geq \tau^* + qw(E)$  and add it up for all edge  $E$ . We obtain

$$\sum s(x) \deg(x) \geq |E(\mathbf{H})| \tau^* + q\tau^*.$$

Subtract  $\sum \deg(x) = (q+1)|E(\mathbf{H})|$  from both sides, we have after rearranging that

$$|E(\mathbf{H})|(1+\varepsilon) - q\tau^* \geq \sum_{x \in V(\mathbf{H})} \deg(x)(1-s(x)). \quad (7.17)$$

Continue it, using (7.12); we have

$$\begin{aligned} &\geq \sum_{x \in T \cup B} \deg(x)(1-s(x)) \geq q \sum_{x \in T \cup B} (1-s(x)) \\ &= q \left( |T \cup B| - \sum_{x \in T \cup B} s(x) \right) \geq q|T \cup B| - q(q+1)\tau^*. \end{aligned} \quad (7.18)$$

Rearranging between the extreme sides of (7.17) and (7.18) we have

$$\frac{1+\varepsilon}{q} |E(\mathbf{H})| + q\tau^* \geq |T \cup B|. \quad (7.19)$$

Using (7.3) we have

$$q^2 + q + 1 - \varepsilon^2 - \frac{\varepsilon + \varepsilon^2}{q} \geq |T \cup B|,$$

implying  $|T \cup B| \leq q^2 + q$  in the case  $\varepsilon > 0$ . If  $\varepsilon = 0$  and  $|E(\mathbf{H})| \leq q^2 + q - 1$  then again (7.19) implies that  $|T \cup B| \leq q^2 + q$ . Finally, if  $\tau^* = q$  and  $|E(\mathbf{H})| = q^2 + q = (q+1)\tau^*$ , then by (7.2) we have that  $V(\mathbf{H}) = T$ , i.e.,  $A = B = \emptyset$ , so the obvious  $|T| \leq (q+1)\tau^*$  inequality implies the proposition. ■

Using the above Proposition (7.16) and (7.17) we improve on (7.15) by

(7.20) PROPOSITION.  $|\mathcal{A}| \leq q - 1$ .

*Proof.* If  $\varepsilon = 0$  then  $T = V(\mathbf{H})$  so  $\mathcal{A} = \emptyset$ . We may suppose that  $\varepsilon > 0$  and so  $|E(\mathbf{H})| < q^2 + q$ . From (7.17) we have

$$\begin{aligned} |E(\mathbf{H})|(1+\varepsilon) - q\tau^* &\geq \sum_{x \in T \cup B} \deg(x)(1-s(x)) \\ &\quad + \sum_{x \in A} \deg(x)(1-s(x)). \end{aligned} \quad (7.21)$$

Here

$$\begin{aligned}
 \sum_{x \in T \cup B} \deg(x) (1 - s(x)) &\geq q \sum_{x \in T \cup B} (1 - s(x)) = q \left( q^2 + q - \sum_{x \in T \cup B} s(x) \right) \\
 &= q \left( q^2 + q - (q + 1) \tau^* + \sum_{x \in A} s(x) \right) \\
 &= (q^2 + q) \varepsilon + q \sum_{x \in A} s(x).
 \end{aligned}$$

Then (7.21) implies

$$\begin{aligned}
 |E(\mathbf{H})|(1 + \varepsilon) - q^2 + q\varepsilon &\geq q^2\varepsilon + q\varepsilon + \sum_{x \in A} \deg(x) + \sum_{x \in A} (q - d(x)) s(x) \\
 &\geq q^2\varepsilon + q\varepsilon + a.
 \end{aligned}$$

Then we have

$$(|E(\mathbf{H})| - q^2)(1 + \varepsilon) \geq a. \quad \blacksquare$$

A corollary of (7.20) is

$$\sum_{x \in T \cup B} (1 - s(x)) \leq 2q\varepsilon. \quad (7.22)$$

Indeed,

$$\sum_{x \in T \cup B} (1 - s(x)) = q^2 + q - \left( (q + 1) \tau^* - \sum_{x \in A} s(x) \right) \leq (q + 1) \varepsilon + a\varepsilon.$$

Another corollary of (7.16) and (7.22) is

$$(7.23) \text{ COROLLARY. } \text{ If } \tau^* = q \text{ then } |T| = |V(\mathbf{H})| = q^2 + q.$$

Indeed by (7.7) and (7.10) we have  $\mathcal{A} = \emptyset$ , implying  $T \cup B = V(\mathbf{H})$ . Then (7.22) gives that  $B = \emptyset$ .

(7.24) CLAIM. *If  $\deg_{\mathbf{H}}(x) \geq q + 1$  for all  $x \in T \cup B$  then  $\mathbf{H}$  is a twisted plane.*

*Proof.* We have

$$(q^2 + q)(q + 1) \leq \sum_{x \in T \cup B} \deg(x) = \sum_E |E \cap (T \cup B)| \leq (q + 1)|E(\mathbf{H})|,$$



which implies that  $|V(\mathbf{H})| = |E(\mathbf{H})| = q^2 + q$ , and  $\mathbf{H}$  is  $(q+1)$ -regular. Let  $\mathcal{G}$  be the set of pairs covered at least twice by  $E(\mathbf{H})$ . Then every edge  $E$  contains exactly one member of  $\mathcal{G}$ , because

$$\begin{aligned} \sum_{F \in E(\mathbf{H}) - \{E\}} |E \cap F| - 1 &= \left( \sum_{F \neq E} |E \cap F| \right) - (|E(\mathbf{H})| - 1) \\ &= \left( \sum_{x \in E} \deg(x) - 1 \right) - (q^2 + q - 1) = 1. \end{aligned}$$

This implies  $|\mathcal{G}| \leq \frac{1}{2}|E(\mathbf{H})| = \frac{1}{2}(q^2 + q)$ . On the other hand every point is covered by  $\bigcup \mathcal{G}$ , hence  $|\mathcal{G}| = \frac{1}{2}(q^2 + q)$ , and it is a matching. Then a simple counting shows that  $E(\mathbf{H})$  covers every pair exactly once except the pairs in  $\mathcal{G}$  are covered twice. Shortly,  $\mathbf{H}$  is a twisted plane. ■

(7.25) PROPOSITION. If  $x_i \in E_i$ ,  $E_i \in E(\mathbf{H})$  for  $i = 1, 2$  and  $|E_1 \cap E_2| > 1$ , then

$$w(E_1) + w(E_2) \leq \frac{2\varepsilon + s(x_1) + s(x_2)}{q+1}.$$

Here  $E_1 \neq E_2$  but  $x_1 = x_2$  is allowed.

*Proof.* (7.8) implies that

$$s(x_1) + q - \tau^* \geq qw(E_1) + w(E_2),$$

and here the roles of  $E_1$  and  $E_2$  can be exchanged. Adding up these two inequalities we obtain (7.25). ■

From now on we suppose that  $q \geq 3$ . (In the case  $q = 2$  we can use the fact that  $\varepsilon(3) = \frac{1}{5}$ ; see (2.3).)

Let  $Q = \{x \in V(\mathbf{H}) : \deg_{\mathbf{H}}(x) = q\}$  and define  $\mathcal{Q} = \{E \in E(\mathbf{H}) : E \cap Q \neq \emptyset\}$ . By definition  $Q \subset T \cup B$ . By (7.24) we may suppose that

$$|Q| \geq 1. \quad (7.26)$$

(7.27) PROPOSITION.  $\bigcup \mathcal{Q} \subset T \cup B$ .

*Proof.* Indeed, if  $x \in E \cap Q$ ,  $E \in E(\mathbf{H})$ , then  $s(x) \geq 1 - 2q\varepsilon$  by (7.22). Then by (7.9) we have

$$w(E) = s(x) - \sum_{x \in F, E \neq F} w(F) \geq s(x) - (q-1) \frac{s(x) + \varepsilon}{q}.$$

These imply that

$$w(E) \geq \frac{1}{q} - \frac{3q-1}{q} \varepsilon \quad (7.28)$$

On the other hand if  $E \cap A \neq \emptyset$  then by (7.10) we have  $w(E) \leq \varepsilon$ , a contradiction to (7.28). ■

Until this point we used only that  $0 \leq \varepsilon < 1/(4q-1)$ . However, in the next steps we really need that  $\varepsilon < 1/(q^2 + q - 1)$ .

(7.29) LEMMA. *Every two edges of  $\mathcal{E}$  intersect in exactly one element.*

*Proof.* Suppose on the contrary that  $E_i \in \mathcal{E}$  with  $x_i \in E_i \cap Q$  for  $i = 1, 2$  such that  $|E_1 \cap E_2| > 1$ . We will get a contradiction in three steps. First we suppose that  $x_1 = x_2 = x$ . By (7.22) we have

$$s(x) \geq 1 - 2q\varepsilon. \quad (7.30)$$

On the other hand (7.9) and (7.25) give

$$s(x) \leq (q-2) \frac{s(x) + \varepsilon}{q} + \frac{2\varepsilon + 2s(x)}{q+1},$$

which implies that

$$s(x) \leq \frac{\varepsilon}{2} (q^2 + q - 2). \quad (7.31)$$

Now (7.30) and (7.31) imply that  $\varepsilon \geq 2/(q^2 + 5q - 2)$ , a contradiction.

As a second step we have

(7.32) CLAIM. *If  $\deg_{\mathbf{H}}(p) = q$  then  $s(p) = 1$ .*

*Proof.* If  $\varepsilon = 0$  then  $T = V(\mathbf{H})$  by (7.23), so there is nothing to prove. Suppose that  $\varepsilon > 0$ . Denote the edges through  $p$  by  $E_1, \dots, E_q$ . By the above part of the proof of (7.29) these edges are disjoint outside  $p$ . Let  $t: V(\mathbf{H}) \rightarrow \mathbf{R}^+$  be an optimal fractional cover. Then

$$q - \varepsilon \geq \sum_{x \in \bigcup E_i} t(x) = \sum_{i=1}^q \left( \sum_{x \in E_i} t(x) \right) - (q-1) t(p) \geq q - (q-1) t(p).$$

We obtained that  $t(p) \geq \varepsilon/(q-1) > 0$ . Then Lemma 7.5 implies  $s(p) = 1$ . ■

*Proof of (7.29) (Conclusion).* Suppose now that  $x_1 \neq x_2$ . Instead of (7.30) we have  $s(x_i) = 1$  by (7.32). Again (7.9) gives

$$1 = s(x_i) \leq (q-1) \frac{1+\varepsilon}{q} + w(E_i),$$

yielding

$$\frac{1 - (q-1)\varepsilon}{q} \leq w(E_i).$$

Apply (7.25), we obtain

$$2 \frac{1 - (q-1)\varepsilon}{q} \leq w(E_1) + w(E_2) \leq \frac{2+2\varepsilon}{q+1},$$

a contradiction if  $\varepsilon < 1/(q^2 + q - 1)$ .

For a point  $x \in Q$  define  $C(x) = T \cup B - \cup \{E - \{x\} : x \in E \in E(\mathbf{H})\}$ . That is  $x \in C(x)$ ,  $C(x) \subset T \cup B$  and it consists of those points which cannot be reached from  $x$  by one step. Clearly,  $|C(x)| = q$  and

$$|C(x) \cap E| \leq 1 \quad \text{for all } E \in E(\mathbf{H}). \quad (7.33)$$

Indeed, (7.33) holds for the edges  $E$  if  $x \in E$ , by definition. If  $x \notin E$  then let  $E_1, \dots, E_q$  denote the edges through  $x$ . As  $E_i - \{x\}$  are pairwise disjoint and  $E$  meets each of them, only at most one point of  $E$  can lie outside  $\cup E_i$ .

This proof also gave that

$$|C(x) \cap E| = 1 \quad \text{if } E \in \mathcal{E}. \quad (7.34)$$

(7.35) **LEMMA** Suppose  $x, y \in Q$ . Then either  $C(x) = C(y)$  or  $C(x) \cap C(y) = \emptyset$ .

*Proof.* If  $y \in C(x)$  then all edges  $F$  through  $y$  intersect  $C(x)$  only in  $y$  by (7.34). Hence

$$C(x) \cap (\cup \{F - \{y\} : y \in F \in E(\mathbf{H})\}) = \emptyset,$$

i.e.,  $C(x) \subset C(y)$ . If  $y \notin C(x)$  then the  $q$  edges through  $y$  cover the points of  $C(x)$ , so  $C(y) \cap C(x) = \emptyset$ . ■

Suppose that the collection  $\{C(x) : x \in Q\}$  consists of  $s$  sets  $C_1, \dots, C_s$ . These are disjoint  $q$ -sets and we have  $Q \subset \cup C_i$ . Hence

$$|Q| \leq qs. \quad (7.36)$$

Let  $|E(\mathbf{H})| = q^2 + q - m$ . We have

$$(q^2 + q - m)(q + 1) = \sum |E| \geq \sum_{x \in T \cup B} \deg(x) \geq (q + 1)(q^2 + q) - |Q|.$$

This implies

$$|Q| \geq m(q + 1). \quad (7.37)$$

Now (7.36), (7.37), and (7.26) give

$$s \geq m + 1. \quad (7.38)$$

(7.39) CLAIM.  $\varepsilon = 0$ .

*Proof.* Suppose  $\varepsilon > 0$ . Then by (7.33) we have

$$\sum_{x \in C_i} s(x) = \sum_E w(E) |E \cap C_i| \leq \tau^* = |C_i| - \varepsilon. \quad (7.40)$$

So  $C_i \cap B \neq \emptyset$ . As the  $C_i$ 's are pairwise disjoint we get that  $|B| \geq s$ . Applying (7.1) to  $\mathbf{H}$  we have

$$q^2 + q - m = |E(\mathbf{H})| \leq |T| \leq q^2 + q - s.$$

This contradicts (7.38). ■

From now on we suppose that  $\varepsilon = 0$ . As in this case every point is saturated, (7.40) implies that

$$\text{for all } E \in E(\mathbf{H}) \quad \text{one has } |E \cap C_i| = 1. \quad (7.41)$$

(7.42) CLAIM.  $m = q$ .

*Proof.* Suppose first that  $m < q$ . Then (7.41) implies that the characteristic vectors  $\mathbf{v}_i$  of  $C_i$  ( $1 \leq i \leq m + 1 \leq s \leq q + 1$ ) are linearly independent together with the characteristic vectors  $\mathbf{v}(E)$  ( $E \in E(\mathbf{H})$ ). Indeed, suppose on the contrary that for some  $\alpha_i$  and  $\alpha(E)$  reals we have

$$\sum_{i=1}^{m+1} \alpha_i \mathbf{v}_i + \sum_E \alpha(E) \mathbf{v}(E) = 0. \quad (7.43)$$

Consider the scalar product of (7.43) with  $\mathbf{v}_j$ . We obtain that

$$q\alpha_j + \sum \alpha(E) = 0 \quad (7.44)$$

holds for all  $1 \leq j \leq m+1$ . Now multiply (7.43) by the characteristic vector of  $V(\mathbf{H})$ . We obtain

$$q \sum_{j=1}^{m+1} \alpha_j + (q+1) \sum \alpha(E) = 0. \quad (7.45)$$

Now (7.44) and (7.45) imply that  $\alpha_1 = \dots = \alpha_{m+1} = 0$ , and

$$\sum_E \alpha(E) \mathbf{v}(E) = 0.$$

But the vectors  $\mathbf{v}(E)$  are linearly independent by Lemma 7.4. So we have proved that all of these vectors are linearly independent, hence  $|E(\mathbf{H})| + m + 1 \leq q^2 + q$ , a contradiction. ■

If  $E(\mathbf{H}) = q^2$  then every degree is exactly  $q$ . So we have obtained that  $\mathbf{H}$  is a  $q$ -regular intersecting hypergraph over  $q^2 + q$  elements, any two edges and  $C_1, \dots, C_{q+1}$  intersect in exactly one point. So it is a truncated projective plane. ■

## 8. PROOF OF THEOREM (2.5)

Suppose that  $\tau^*(\mathbf{H}) > q + (q-1)/(q^2 + q - 1)$ . We will prove that  $\mathbf{H}$  contains a  $PG(2, q)$  and hence  $\tau^*(\mathbf{H}) = q + 1/(q+1)$ . Every edge of  $\mathbf{H}$  has at least  $\lceil \tau^*(\mathbf{H}) \rceil = q+1$  elements. Let  $\mathbf{G}$  consist of the  $q+1$  element edges of  $\mathbf{H}$ . Put a weight  $1/(q+2)$  into every vertex of  $\mathbf{H}$ . In this way we have covered all the large (i.e.,  $\geq q+2$  elements) edges and  $(q+1)/(q+2)$  part of the edges of  $\mathbf{G}$ . Hence

$$q + \frac{q-1}{q^2 + q - 1} < \tau^*(\mathbf{H}) \leq \frac{q^2 + q + 1}{q+2} + \frac{\tau^*(\mathbf{G})}{q+2}, \quad (8.1)$$

implying that  $\tau^*(\mathbf{G}) > q - 1/(q^2 + q - 1)$ . Then Theorem (2.2) implies that  $\tau^*(\mathbf{G}) \geq q$ , and one of the cases (2.2)(i), (ii), or (iii) holds. In the case of (2.2) (iii)  $\mathbf{G}$  is a twisted plane on  $q^2 + q$  points only. Then

$$t(x) = \begin{cases} 1/(q+1) & \text{if } x \in V(\mathbf{G}), \\ 0 & \text{if } x \in V(\mathbf{H}) - V(\mathbf{G}), \end{cases} \quad (8.2)$$

is a fractional covering of  $\mathbf{H}$  with value  $q$ , contradicting (8.1).

In the case (2.2) (ii) let  $\mathbf{A}$  be a truncated projective plane of order  $q$ ,  $E(\mathbf{A}) \subset E(\mathbf{G}) \subset E(\mathbf{H})$ . Let  $p = V(\mathbf{H}) - V(\mathbf{A})$ , and  $\mathcal{B} = \{E \in E(\mathbf{G}) : p \in E\}$ . If

$\mathcal{B} = \emptyset$  then the cover (8.2) shows that  $\tau^*(\mathbf{H}) \leq q$ , a contradiction. If  $|\mathcal{B}| = 1$ , i.e.,  $\mathcal{B} = \{B\}$ , then let

$$t_2(x) = \begin{cases} (q-1)/q^2 & \text{if } x \notin B, \\ 1/q & \text{if } x \in B - \{p\}, \\ 1/q^2 & \text{if } x = p. \end{cases}$$

This is a fractional cover of  $\mathbf{H}$  with value  $q + 1/q^2$ , which is less than the left hand side of (8.1). If  $|\mathcal{B}| \geq 2$ , then  $E(\mathbf{G}) = E(\mathbf{A}) \cup \mathcal{B}$  is a subhypergraph of a  $PG(2, q)$ . If any line  $L$  of this plane is missing from  $\mathbf{G}$  (this line  $L$  contains  $p$ ) then

$$t_3(x) = \begin{cases} (q-1)/(q^2 + q - 1) & \text{if } x \in L, \\ q/(q^2 + q - 1) & \text{if } x \notin L \end{cases}$$

is a fractional cover of  $\mathbf{H}$  with value  $q + (q-1)/(q^2 + q - 1)$ , again contradicting (8.1).

The only remaining case is when  $\mathbf{G}$  contains a  $PG(2, q)$ , so  $\tau^*(\mathbf{H}) \geq \tau^*(\mathbf{G}) = q + 1/(q+1)$ , as desired.

## 9. PROOF OF THEOREM (6.1)

Let  $\mathbf{H}_0$  be the set of  $(q+1)$ -element edges of  $\mathbf{H}$ . Suppose that (i) and (ii) do not hold; then, by Theorem (2.2), we have

$$\tau^*(\mathbf{H}_0) \leq q - \frac{1}{q^2 + q - 1}. \quad (9.1)$$

Suppose that  $\mathbf{H}$  is  $\tau^*$ -critical, and let  $w: E(\mathbf{H}) \rightarrow \mathbf{R}^+$  be an optimal fractional matching. Obviously  $\tau^*(\mathbf{H}) \leq q$ . (If every edge has at least  $q+1$  elements then  $t(x) \equiv 1/(q+1)$  is a fractional cover with  $|t| = q$ , and if there is an edge with at most  $q$  elements then  $\tau^*(\mathbf{H}) \leq \tau(\mathbf{H}) \leq q$ .)

Let  $\tau^*(\mathbf{H}) = q - \varepsilon$ , where  $\varepsilon \geq 0$ ; suppose  $\varepsilon < 1$ . Then every edge has at least  $q$  elements. Let  $E_0$  be an edge of  $q$  elements. Then

$$q \geq \sum_{x \in E_0} \left( \sum_{x \in E} w(E) \right) \geq \tau^* + (q-1) w(E_0)$$

implies

$$w(E_0) \leq \varepsilon/(q-1).$$

Thus

$$\sum_{|E|=q} w(E) \leq \varepsilon \frac{q^2 + q}{q - 1}. \quad (9.2)$$

Then we have

$$\begin{aligned} (q^2 + q) - (q + 1) \tau^* &\geq \sum w(E) |E| - \sum (q + 1) w(E) \\ &\geq \sum_{|E| > q+1} w(E) - \sum_{|E|=q} w(E). \end{aligned}$$

This and (9.2) imply that

$$\sum_{|E| > q+1} w(E) \leq (q + 1) \varepsilon + \frac{q^2 + q}{q - 1} \varepsilon. \quad (9.3)$$

Finally (9.2) and (9.3) imply that

$$q - \frac{3q^2 + 2q - 1}{q - 1} \varepsilon \leq \sum_{|E|=q+1} w(E) \leq \tau^*(\mathbf{H}_0). \quad (9.4)$$

Then (9.1) and (9.4) yield that  $\varepsilon > 1/(3(q + 1)^3)$ .

*Note added in proof.* Our main result (Theorem (5.1)) settles a conjecture of Todorov [T89] for almost all  $n$ . However, as Proposition (4.10) shows, this conjecture does not hold for every  $n$ .

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