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Legitimate Colorings of Projective Planes

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Abstract. For a projective plane $\mathbb{P}_n$ of order n, let $\chi(\mathbb{P}_n)$ denote the minimum number k, so that there is a coloring of the points of $\mathbb{P}_n$ in k colors such that no two distinct lines contain precisely the same number of points of each color. Answering a question of A. Rosa, we show that for all sufficiently large n, $5 \le \chi(\mathbb{P}_n) \le 8$ for every projective plane $\mathbb{P}_n$ of order n.

1. Introduction

Let $\mathbb{P} = \mathbb{P}_n = (P, \mathscr{L})$ be a projective plane of order n, with a set of points P and a set of lines $\mathscr{L}$. As is well known, $\mathbb{P}$ has $n^2 + n + 1$ points and $n^2 + n + 1$ lines with $n + 1$ points on every line. A *χ-coloring* of $\mathbb{P}$ is a function f from P to the set $\{1, 2, \dots, \chi\}$, which may also be viewed as the (ordered) χ-partition $(P_1, P_2, \dots, P_\chi)$ of P defined by $P_i = f^{-1}(i)$. Let C be a χ-coloring of $\mathbb{P}$, corresponding to the partition $(P_1, \dots, P_\chi)$. For a line $L \in \mathscr{L}$, we define the *type* $t_{L,C}$ of L (with respect to C) to be the following vector of length χ: $t_{L,C} = (|P_1 \cap L|, |P_2 \cap L|, \dots, |P_\chi \cap L|)$. Thus, $t_{L,C}$ is a vector with nonnegative integer coordinates whose sum in $|L| = n + 1$. The coloring C is called *legitimate* if no two distinct lines have the same type. Finally, let $\chi(\mathbb{P})$ denote the minimum integer χ, such that there exists a legitimate χ-coloring of $\mathbb{P}$. A. Rosa raised the problem of studying the numbers $\chi(\mathbb{P})$ and observed that $\chi(\mathbb{P}) \ge 4$ for every projective plane of order $n \ge 5$. Indeed, this follows from the fact that the number of vectors with χ nonnegative coordinates whose sum is $n + 1$ is $\binom{n+\chi}{\chi-1}$. Since $\binom{n+3}{2} < n^2 + n + 1$ for all $n \ge 5$, it follows that in any 3-coloring of a projective plane of order $n \ge 5$ there are two lines having the same type. Somewhat surprisingly, the set $\{\chi(\mathbb{P})\}$, as $\mathbb{P}$ ranges over all projective planes, is bounded. In fact, as shown in the next section, a rather straightforward application of the probabilistic method shows that for all sufficiently large n, $\chi(\mathbb{P}_n) \le 10$

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for every projective plane of order n. In the present paper we study the numbers $\chi(\mathbb{P}_n)$ for large n. We improve both the easy upper and lower bounds stated above and show that for all sufficiently large n

$$5 \leq \chi(\mathbb{P}_n) \leq 8$$

for every projective plane $\mathbb{P}_n$ of order n. The upper bound is proved in section 2, and the lower bound in section 3. The final section 4 contains several generalizations and open problems.

2. Eight Colors Suffice

In this section we prove the following theorem.

Theorem 2.1. *For all sufficiently large n,*

$$\chi(\mathbb{P}_n) \leq 8$$

for every projective plane $\mathbb{P}_n$ of order n.

Throughout the section we assume, whenever it is needed, that n is sufficiently large. Let $\mathbb{P} = \mathbb{P}_n = (P, \mathscr{L})$ be a projective plane of order n. We first show the easy proof that $\chi(\mathbb{P}) \leq 10$. A *random χ-coloring* C of $\mathbb{P}$ is a function f from P to $\{1, 2, \dots \chi\}$, where for each $p \in P$, $f(p) \in \{1, 2, \dots, \chi\}$ is chosen, independently, according to a uniform distribution. Let us call a pair $\{L, L'\}$ of two distinct lines of $\mathbb{P}$ *bad* (with respect to C) if $t_{L,C} = t_{L',C}$. One can easily check that for every fixed χ and every fixed pair of lines $\{L, L'\}$, the probability that $\{L, L'\}$ is bad (with respect to the random χ-coloring C) is $\Theta\left(\frac{1}{n^{(\chi-1)/2}}\right)$. Therefore, the expected number of bad pairs is $O\left(\binom{n^2+n+1}{2} \cdot \frac{1}{n^{(\chi-1)/2}}\right) = O(n^{4-(\chi-1)/2})$. In particular, for $\chi = 10$ the expected number of bad pairs, is smaller than 1 and hence there is a 10-coloring with no bad pairs which is, by definition, a legitimate coloring. Thus $\chi(\mathbb{P}) \leq 10$. Moreover, the proof actually shows that almost all 10-colorings of P are legitimate. Our objective is to improve the bound 10 to 8. As the details are somewhat complicated, let us first sketch the idea in the proof of this improvement. Our objective is to show that with positive probability a random 8-coloring of $\mathbb{P}$ is legitimate. However, unlike in the previous case, here the probability that it is indeed legitimate is extremely small. To obtain the required estimate for the probability that a random 8-coloring is legitimate, we apply the Lovász Local Lemma. This is a tool that enables one to conclude that with positive probability the complements of many events happen simultaneously, provided each of them is mutually independent of almost all the others. The events we would like to consider here are all the $\binom{n^2+n+1}{2}$ events that a fixed pair of lines is bad. However, here no reasonable condition on mutual independence is satisfied, and thus we have to be a little trickier. This is done by first considering a random coloring of most, but not all, the points, and then by applying the Local Lemma to the rest of the coloring.

We now present the proof in detail, starting with a few lemmas.

Lemma 2.2 (See also [5] for a similar statement) *There exists a subset $S \subset P$ of the set of points of $\mathbb{P}_n = (P, \mathscr{L})$, such that for every $L \in \mathscr{L}$*

$$\log n \leq |S \cap L| \leq 20 \log n. \tag{2.1}$$

Remark 2.3. All logarithms here and throughout the paper are in the natural base e. The constant 20 can be easily reduced. We make no attempts to optimize the constants here and in the following proof.

Proof of Lemma 2.2. Let us pick each point $p \in P$ independently, with probability $\frac{10 \log n}{n+1}$. Let S be the (random) set of all the points picked. For each line $L \in \mathscr{L}$, let A_L be the event that inequality (2.1) is violated for L. Clearly, $|S \cap L|$ is a Binomial random variable with expectation $10 \log n$ and standard deviation $\sqrt{10 \log\left(1 - \frac{10 \log n}{n+1}\right)} < \sqrt{10 \log n}$. Hence, by the standard estimates for Binomial distributions (see, e.g., [2], p. 11) for every $L \in \mathscr{L}$

$$\Pr(A_L) < e^{-(81/20) \log n} < 1/n^4.$$

Therefore, the expected number of lines L that violate (2.1) is smaller than $(n^2 + n + 1)/n^4 < 1$ and thus there is a set S for which (2.1) holds for every $L \in \mathscr{L}$. This completes the proof of the lemma.

Let $S \subset P$ satisfy the assertions of Lemma 2.2. Put $F = P \backslash S$ and let $f\colon F \to \{1, 2, \ldots, 8\}$ be a random coloring of F by 8 colors, where for each $p \in F$, $f(p) \in \{1, 2, \ldots, 8\}$ is chosen independently, according to a uniform distribution. Thus f is a partial coloring of P. For each line $L \in \mathscr{L}$, define the type $t_{L,f}$ of L (with respect to the partial coloring f) by $t_{L,f} = (|f^{-1}(1) \cap L|, \ldots, |f^{-1}(8) \cap L|)$. For two vectors $\underline{x} = (x_1, \ldots, x_8)$ and $\underline{y} = (y_1, \ldots, y_8)$ define the distance $d(\underline{x}, \underline{y})$ to be the l_1-distance between $\underline{x}$ and $\underline{y}$, i.e., $d(\underline{x}, \underline{y}) = \sum_{i=1}^{8} |x_i - y_i|$. Let us call a pair $\{L, L'\}$ of distinct lines *dangerous* if $d(t_{L,f}, t_{L',f}) \leq 40 \log n$. Notice that f assigns colors to all but at most $40 \log n$ points of $L \cup L'$. Thus, if $\{L, L'\}$ is not a dangerous pair, then in any extension of f to a coloring C of all points of $\mathbb{P}$ the types $t_{L,C}$ and $t_{L',C}$ will be different, i.e., $\{L, L'\}$ will not be a bad pair. Therefore, when trying to extend f to a legitimate coloring of $\mathbb{P}$, our only concern is to avoid making any dangerous pair into a bad one. In order to show that this can be done, we first study the structure of the dangerous pairs. We need the following simple, somewhat technical, lemma.

Lemma 2.4. *Let L be a line of $\mathbb{P}$, and let $T \subseteq L$ be a set of k points of L. Let $\underline{t} = (t_1, \ldots, t_8)$ be an arbitrary vector with nonnegative integer coordinates. Then for any given function $g\colon T \to \{1, 2, \ldots, 8\}$ and for the random coloring $f\colon F \to \{1, 2, \ldots, 8\}$:*

$$\Pr(t_{L,f} = \underline{t} \mid f(p) = g(p) \text{ for all } p \in T)$$
$$< \frac{\binom{m-k}{\left\lfloor \frac{m-k}{8} \right\rfloor, \left\lfloor \frac{m-k+1}{8} \right\rfloor, \ldots, \left\lfloor \frac{m-k+7}{8} \right\rfloor}}{8^{m-k}}, \tag{2.2}$$

where $m = |L \cap F|$. *In particular, if* $k \leq \sqrt{n}$, *the above conditional probability is smaller than* $100/n^{7/2}$.

Proof. For $1 \leq i \leq 8$, put $s_i = |g^{-1}(i)|$. Knowing that $f(p) = g(p)$ for all $p \in T$, the type $t_{L,f}$ is equal to $\underline{t}$ if and only if the number of points in $(L \cap F) \setminus T$ colored i is precisely $t_i - s_i$. There are 8^{m-k} equally likely possible colorings of $(L \cap F) \setminus T$ and the number of those making $t_{L,f} = \underline{t}$ is $\binom{m-k}{t_1 - s_1, t_2 - s_2, \ldots, t_8 - s_8}$. Therefore, the left hand side of (2.2) is equal to the ratio between the last multinomial coefficient and 8^{m-k}. Since this multinomial coefficient, for given m and k, attains its maximum when the numbers $t_i - s_i$ are as equal as possible, inequality (2.2) follows. The fact that by (2.1) $m \geq n + 1 - 20 \log n$, together with the standard estimates for multinomial coefficients obtained from Stirling's Formula (see, e.g., [2], p. 4), show that for all sufficiently large n the conditional probability considered is smaller than $100/n^{7/2}$, provided $k \leq \sqrt{n}$. □

Corollary 2.5. *Let* L_1 *and* L_2 *be two distinct lines of* $\mathbb{P} = (P, \mathscr{L})$ *and let* $T \subset P$ *be an* arbitrary set of points of $\mathbb{P}$ *satisfying* $|L_2 \cap T| \leq \sqrt{n}$. *Then, given any information on the coloring of the points in* T, *the conditional probability that* $\{L_1, L_2\}$ *is a dangerous pair (with respect to* f*) is smaller than* $\frac{(100 \log n)^8 \cdot 100}{n^{7/2}} < \frac{(\log n)^9}{n^{7/2}}$.

Proof. For every possible coloring of $L_1 \cup T$ (consistent with the given information on the coloring of T), and for every fixed type vector $\underline{t} = (t_1, \ldots, t_8)$ whose distance from $t_{L_1,f}$ is at most $40 \log n$, the conditional probability that $t_{L_2,f} = \underline{t}$ is, by Lemma 2.4, smaller than $100/n^{7/2}$. As there are less than $(100 \log n)^8 < \frac{(\log n)^9}{100}$ vectors $\underline{t}$ of distance at most $40 \log n$ from each such possible $t_{L_1,f}$, the desired result follows. □

Lemma 2.6. *The probability that there are three distinct lines* L, L' *and* L'' *such that both pairs* $\{L, L'\}$ *and* $\{L, L''\}$ *are dangerous (with respect to the partial coloring* f*) is smaller than* $(\log n)^{18}/n$.

Proof. Fix a line L and two other distinct lines L' and L''. By Corollary 2.5, the probability that $\{L, L'\}$ is a dangerous pair is smaller than $(\log n)^9/n^{7/2}$. By another application of Corollary 2.5 (with $L_1 = L$, $L_2 = L''$, $T = L \cup L'$) the conditional probability that $\{L, L''\}$ is a dangerous pair, given that $\{L, L'\}$ is a dangerous pair, is smaller than $(\log n)^9/n^{7/2}$. Thus, the probability that both pairs $\{L, L'\}$ and $\{L, L''\}$

are dangerous is smaller than $(\log n)^{18}/n^7$. There are $(n^2+n+1)\binom{n^2+n}{2} < n^6$ choices for the line L and the two other lines L' and L''. Thus, the expected number of pairs of the form $(L, \{L', L''\})$, where L, L' and L'' are three distinct lines and $\{L, L'\}$, $\{L, L''\}$ are dangerous pairs is smaller than $(\log n)^{18}/n$. Hence, the probability that this number is not zero (i.e., that it is at least 1) is smaller than $(\log n)^{18}/n$. □

Lemma 2.7. *The probability that there is a point $p \in P$ and ten distinct lines $\{L_1, L_2, \ldots, L_5, L'_1, L'_2, \ldots, L'_5\}$ such that $p \in L_1 \cap L_2 \cap \cdots \cap L_5$ and $\{L_i, L'_i\}$ is a dangerous pair for all $1 \le i \le 5$ is smaller than $(\log n)^{45}/\sqrt{n}$.*

Proof. Fix a point $p \in P$ and ten distinct lines $L_1, \ldots, L_5, L'_1, \ldots, L'_5$ such that $p \in L_1 \cap L_2 \cap \cdots \cap L_5$. By Corollary 2.5, the probability that $\{L_1, L'_1\}$ is a dangerous pair is smaller than $(\log n)^9/n^{7/2}$. Also, for every $1 \le i \le 4$, Corollary 2.5 implies that the conditional probability that $\{L_{i+1}, L'_{i+1}\}$ is a dangerous pair, given that $\{L_1, L'_1\}, \ldots, \{L_i, L'_i\}$ are all dangerous pairs, is smaller than $\frac{(\log n)^9}{n^{7/2}}$. It follows that the probability that all 5 pairs $\{L_i, L'_i\}$ are dangerous is smaller than $(\log n)^{45}/n^{35/2}$. The number of choices for p, $L_1, \ldots, L_5$ and $L'_1, \ldots, L'_5$ with $p \in L_1 \cap \cdots \cap L_5$ is smaller than $(n^2+n+1)\cdot\binom{n+1}{5}(n^2+n+1)^5 < n^{17}$. Thus, the probability that there are such p, $L_1, \ldots, L_5$ and $L'_1, \ldots, L'_5$ with all 5 pairs $\{L_i, L'_i\}$ dangerous is smaller than $n^{17}\cdot\frac{(\log n)^{45}}{n^{35/2}} = (\log n)^{45}/\sqrt{n}$. □

For a point p of $\mathbb{P}$ and a pair $\{L, L'\}$ of lines of $\mathbb{P}$, we say that p *lies* in $\{L, L'\}$ if $p \in L \cup L'$. An immediate consequence of Lemma 2.6 and Lemma 2.7 is the following.

Proposition 2.8. *The probability that no point of $\mathbb{P}_n$ lies in more than 4 dangerous pairs is at least $1 - \frac{(\log n)^{18}}{n} - \frac{(\log n)^{45}}{\sqrt{n}}$. In particular, there is an 8-coloring f of $F = P \backslash S$ in which no point belongs to more than 4 dangerous pairs. (In fact, almost all 8-colorings have this property, for sufficiently large n.)* □

Let $f\colon F \to \{1, 2, \ldots, 8\}$ be a partial 8-coloring of $\mathbb{P}$, satisfying the assertion of the last proposition. To complete the proof of Theorem 2.1 we show that f can be extended to a legitimate 8-coloring C of $\mathbb{P}$. Let C be a random extension of f, i.e., choose the color of each point $p \in S$ independently, in $\{1, 2, \ldots, 8\}$, according to a uniform distribution. Recall that by (2.1) S contains at least $\log n$ points of each line L, so there is still a considerable amount of freedom in determining the type $t_{L,C}$ of each line. By definition, C is legitimate if and only if there are no bad pairs of lines (with respect to the coloring C). Recall that the only pairs that may become bad (with respect to C) are those which are dangerous with respect to f. Our objective is to show that with positive (though exponentially small) probability, no dangerous pair becomes bad. To do so, we apply the Lovász Local Lemma proved in [4] (see also, e.g., [6]), which is the following:

Lemma 2.9 (Lovász Local Lemma: Symmetric case [4]). *Let $A_1, A_2, \ldots, A_n$ be events in a probability space. Suppose that, for all i, $\Pr(A_i) \le q$ and that each event A_i is mutually independent of all but at most b of the other events. If $eq(b+1) < 1$ then $Pr\left(\bigcap_{i=1}^{n} \bar{A}_i\right) > 0$; i.e., with positive probability no A_i occurs.*

For every dangerous pair $\{L_1, L_2\}$ (with respect to the fixed partial coloring f satisfying the assertion of Proposition 2.8 we chose), let A_{L_1,L_2} be the event that the pair $\{L_1, L_2\}$ is bad with respect to the random extension C of f. Let $S_1 = S \cap L_1$, $S_2 = S \cap L_2$ be the points of L_1 and L_2, respectively, that receive their new colors during the random choice of C. By our choice of S (see inequality (2.1)) both $|S_1|$ and $|S_2|$ are between $\log n$ and $20 \log n$. Therefore, one can easily check that

$$\Pr(A_{\{L_1,L_2\}}) \le \frac{100}{(\log n)^{7/2}}. \tag{2.3}$$

Indeed, for every given coloring of L_2, the conditional probability that L_1 will have the same type can be bounded, as in the proof of Lemma 2.4, by an expression of the form $\frac{\binom{m}{m_1, m_2, \ldots, m_8}}{8^m}$, where $m = |S_1 \backslash S_2| \ge \log n - 1$, and this expression is smaller than $100/(\log n)^{7/2}$.

We claim that the event $A_{\{L_1,L_2\}}$ is mutually independent of all the events $A_{\{L',L''\}}$ with

$$(S_1 \cup S_2) \cap (L' \cup L'') = \varnothing. \tag{2.4}$$

This is because the coloring f is already fixed and the only random process considered is its extension to C. Thus, the only colors that determine the event $A_{\{L_1,L_2\}}$ are those assigned to the points of $S_1 \cup S_2$, and no information on the coloring of L' and L'' is relevant to the the choice of these colors, provided (2.4) holds. Since by Proposition 2.8 no point belongs to more than 4 dangerous pairs and, since $|S_1 \cup S_2| \le 80 \log n$, it follows that the number of dangerous pairs $\{L', L''\}$ (besides $\{L_1, L_2\}$) that violate (2.4) does not exceed $|S_1 \cup S_2| \cdot 3 \le 240 \log n$. Combining this with (2.3) and Lemma 2.9 (with $q = 100/(\log n)^{7/2}$, $b = 240 \log n$) we conclude that with positive probability no event $A_{\{L,L'\}}$ occurs. In particular, there is at least one extension C of the partial coloring f which is an 8-coloring of $\mathbb{P}$ with no bad pairs. Thus $\chi(\mathbb{P}_n) \le 8$, completing the proof of Theorem 2.1. □

3. Four Colours do not Suffice

In this section we prove the following theorem.

Theorem 3.1. *For all sufficiently large n.*

$$\chi(\mathbb{P}_n) \geq 5$$

for every projective plane $\mathbb{P}_n$ of order n.

To prove this theorem we need the following simple but useful lemma. See also [1] and [3] for similar statements.

Lemma 3.2. *Let $\mathbb{P} = \mathbb{P}_n = (P, \mathscr{L})$ be a projective plane of order n and let $X \subseteq P$ be an arbitrary set of points of $\mathbb{P}$. Then*

$$\sum_{L \in \mathscr{L}} \left(|L \cap X| - \frac{(n+1)|X|}{n^2+n+1} \right)^2 = |X| n \left(1 - \frac{|X|}{n^2+n+1} \right). \tag{3.1}$$

Proof. Since every point of X belongs to precisely $n+1$ lines we have:

$$\sum_{L \in \mathscr{L}} |L \cap X| = (n+1)|X|.$$

Similarly, since every pair of points of X lie in a unique common line:

$$\sum_{L \in \mathscr{L}} \binom{|L \cap X|}{2} = \binom{|X|}{2}.$$

The above two inequalities enable us to compute any polynomial of the form $\sum_{L \in \mathscr{L}} (\alpha |L \cap X|^2 + \beta |L \cap X| + \gamma)$ in terms of n and $|X|$. In particular, an easy computation gives equality (3.1). □

Remark. In the next section we present another proof of Lemma 3.2, which uses the eigenvalues of the lines versus points incidence matrix of the projective plane $\mathbb{P}$. Although that proof is (a little) more complicated than the one above, it has the advantage that it can be generalized to other, more complicated structures provided some information on the eigenvalues of their corresponding incidence matrices is available.

In order to deduce Theorem 3.1 from Lemma 3.2, rather rough estimates suffice. We next present this proof. Afterwards, we describe briefly a more careful analysis which, although it does not enable us to improve the lower bound in Theorem 3.1 to $\chi(\mathbb{P}_n) \geq 6$, it provides some interesting properties of any legitimate 5-coloring of $\mathbb{P}_n$ for all sufficiently large n. We believe that in fact $\chi(\mathbb{P}_n) \geq 6$ for all sufficiently large n but at the moment we are unable to prove it.

Proof of Theorem 3.1. Let C be an arbitrary 4-coloring of $\mathbb{P} = \mathbb{P}_n = (P, \mathscr{L})$, corresponding to the partition $P = P_1 \cup P_2 \cup P_3 \cup P_4$ of the points of $\mathbb{P}$. For $1 \leq i \leq 4$, put $t_i = |P_i|(n+1)/(n^2+n+1)$. By Lemma 3.2, for each fixed i, $1 \leq i \leq 4$, we have

$$\begin{aligned}\sum_{L \in \mathscr{L}} (|L \cap P_i| - t_i)^2 &= |P_i| \cdot n \left(1 - \frac{|P_i|}{n^2+n+1} \right) \\ &= \frac{n}{n^2+n+1} \cdot |P_i| (n^2+n+1-|P_i|) \leq \frac{n(n^2+n+1)}{4} \leq n^3.\end{aligned}$$

Therefore, the number of lines $L \in \mathscr{L}$ that satisfy $||L \cap P_i| - t_i| > 3\sqrt{n}$ is smaller than $n^2/9$. It follows that there are at least $(n^2 + n + 1) - 3n^2/9 > n^2/2$ lines L for which

$$||L \cap P_i| - t_i| \le 3\sqrt{n} \text{ for all } 1 \le i \le 3. \tag{3.2}$$

We claim that there are at most $(6\sqrt{n} + 1)^3 < 250n^{3/2}$ possible type vectors $t_{L,C} = (|L \cap P_1|, \dots, |L \cap P_4|)$ for lines L that satisfy (3.2). Indeed, by (3.2) there are at most $6\sqrt{n} + 1$ possibilities for each of the three quantities $|L \cap P_1|$, $|L \cap P_2|$ and $|L \cap P_3|$, and as the sum of the 4 coordinates of $t_{L,C}$ is precisely $n + 1$ these three quantities determine the fourth. As there are at least $n^2/2$ lines L that satisfy (3.2), and the type of each of them belongs to a set of less than $250n^{3/2}$ possible type vectors it follows that for sufficiently large n there are two distinct lines having the same type. (In fact, there are at least $\sqrt{n}/500$ lines having the same type.) In particular, C is not legitimate and $\chi(\mathbb{P}_n) \ge 5$, as needed. □

In the rest of this section we briefly present a more careful analysis of colorings of projective planes using Lemma 3.2. Although this analysis does not suffice to improve the estimate in Theorem 3.1, it does supply some additional interesting information on colorings of projective planes. Let k be a fixed integer. Let $\mathbb{P} = \mathbb{P}_n = (P, \mathscr{L})$ be a projective plane of order n, where $n > n_0(k)$ is sufficiently large. Let C be an arbitrary k-coloring of $\mathbb{P}$, corresponding to the partition $(P_1, P_2, \dots, P_k)$ of P, and put $\underline{t} = (t_1, t_2, \dots, t_k)$, where $t_i = |P_i| \cdot (n + 1)/(n^2 + n + 1)$. For two k-dimensional vectors $\underline{x} = (x_1, \dots, x_k)$ and $\underline{y} = (y_1, \dots, y_k)$, put $\|\underline{x} - \underline{y}\|^2 = \sum_{i=1}^{k} |x_i - y_i|^2$. Combining Lemma 3.2 with the convexity of the function z^2 we obtain:

$$\begin{aligned}
\sum_{L \in \mathscr{L}} \|t_{L,C} - \underline{t}\|^2 &= \sum_{i=1}^{k} |P_i| \cdot n \left(1 - \frac{|P_i|}{n^2 + n + 1}\right) \\
&= (n^2 + n + 1)n - \frac{n}{n^2 + n + 1} \sum_{i=1}^{k} |P_i|^2 \\
&\le (n^2 + n + 1)n - \frac{n(n^2 + n + 1)}{k} = \frac{k-1}{k} n(n^2 + n + 1).
\end{aligned} \tag{3.3}$$

Suppose now, that $\mathscr{M} \subseteq \mathscr{L}$ is a set of lines, and $t_{L,C} \ne t_{L',C}$ for every two distinct lines $L, L' \in \mathscr{M}$. (In particular, if C is legitimate, this holds for $\mathscr{M} = \mathscr{L}$.) Inequality (3.3) provides an upper bound for $\sum_{L \in \mathscr{M}} \|t_{L,C} - \underline{t}\|^2 \le \sum_{L \in \mathscr{L}} \|t_{L,C} - \underline{t}\|^2$. On the other hand, it is obvious that the quantity $\sum_{L \in \mathscr{M}} \|t_{L,C} - \underline{t}\|^2$ is at least as big as the sum $\sum_{i=1}^{m} \|\underline{x}_i - \underline{t}\|^2$, where $m = |\mathscr{M}|$ and $\{\underline{x}_i\}_{i=1}^m$ are m distinct lattice points on the hyperplane $\langle \underline{x}, \underline{1} \rangle = n + 1$ in $\mathbb{R}^k$, chosen as close as possible to the point $\underline{t} \in \mathbb{R}^k$. (Here $\underline{1}$ is a k-dimensional vector of 1's.) This set $\{\underline{x}_i\}_{i=1}^m$ is simply the set of all lattice points inside a ball centered at $\underline{t}$ (in the hyperplane $\langle \underline{x}, 1 \rangle = n + 1$), with an appropriately chosen radius R, plus, if necessary, some of the points on the boundary of this ball. We thus need the following estimate.

Proposition 3.3. *Let k be a fixed integer and let $\underline{t} \in \mathbb{R}^k$ be a point on the hyperplane $H = \{\underline{x}: \langle \underline{x}, \underline{1} \rangle = n + 1\}$. Let $M^k(R, \underline{t})$ denote the sum $\sum \|\underline{x} - \underline{t}\|^2$, where $\underline{x}$ ranges over all lattice points on the hyperplane H inside the ball of radius R centered at $\underline{t}$. Then, as R tends to infinity*

$$M^k(R, \underline{t}) = (1 + o(1)) \cdot R^{k+1} \frac{\pi^{(k-1)/2}}{\Gamma\left(\frac{k+1}{2}\right)} \cdot \frac{k-1}{\sqrt{k}(k+1)}.$$

Proof. The volume B_r of a $k - 1$ dimensional ball of radius r is $\dfrac{\pi^{(k-1)/2}}{\Gamma\left(\frac{k+1}{2}\right)} r^{k-1}$. The volume of the basic parallelepipedon of the lattice of the hyperplane H is the following k by k determinant

$$\det \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & & & \vdots \\ 0 & 0 & 1 & -1 & & & \vdots \\ & & & & \ddots & & \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & -1 \\ \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k}} & \cdot & \cdot & \cdot & \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k}} \end{bmatrix} = \sqrt{k}.$$

Thus the number of lattice points inside a ball of radius r in H is $(1 + o(1))\dfrac{B_r}{\sqrt{k}}$.

Therefore

$$\begin{aligned} M^k(R, \underline{t}) &= (1 + o(1)) \int_{r=0}^{R} r^2 \frac{\partial}{\partial r}\left[\frac{B_r}{\sqrt{k}}\right] dr \\ &= (1 + o(1)) \int_{r=0}^{R} (k-1) r^{k+1} \frac{\pi^{(k-1)/2}}{\Gamma\left(\frac{k+1}{2}\right)} \cdot \frac{1}{\sqrt{k}} dr \\ &= (1 + o(1)) R^{k+1} \frac{\pi^{(k-1)/2}}{\Gamma\left(\frac{k+1}{2}\right)} \frac{k-1}{\sqrt{k}(k-1)}. \end{aligned}$$

This completes the proof of the proposition. □

Proposition 3.3 and inequality (3.3) easily imply Theorem 3.1 (which we have already proved). Indeed, suppose the theorem is false, let C be a legitimate 4-coloring of $\mathbb{P}_n$ and define $\underline{t}$ as above. Then the vectors $\{t_{L,C}: L \in \mathscr{L}\}$ are $n^2 + n + 1$ distinct lattice points on the hyperplane $H = \{\langle \underline{x}, \underline{1} \rangle = n + 1\}$ in $\mathbb{R}^4$. Consequently $\sum_{L \in \mathscr{L}} \|t_{L,C} - \underline{t}\|^2$ is at least $M^4(R, \underline{t})$, where R is the smallest radius of a 4 dimensional ball containing $n^2 + n + 1$ lattice points from H. Hence $\frac{4}{3}\pi R^3 \cdot \frac{1}{\sqrt{4}} \approx n^2 + n + 1$ and thus $R = \Omega(n^{2/3})$ and $M^4(R, \underline{t}) = \Omega(n^{10/3}) > \frac{4}{5}n(n^2 + n + 1)$, contradicting inequality (3.3).

A similar argument enables us to show that in any 5-coloring of $\mathbb{P}_n$ there are many pairs of distinct lines whose type vectors are very close to each other. Let us call two type vectors $t_{L,C}$ and $t_{L',C}$ *neighbours* if either $t_{L,C} = t_{L',C}$ or $t_{L,C}$ can be obtained from $t_{L',C}$ by changing the color of a single point of L' (i.e., by increasing one coordinate of $t_{L',C}$ by 1 and decreasing another coordinate by 1.)

Claim 3.4. *Let k be a fixed integer, and suppose $\underline{t} \in \mathbb{R}^k$. Let H be the hyperplane $\{\underline{x}: \langle \underline{x}, \underline{1} \rangle = n + 1\}$. Let Y be a set of lattice points on H and suppose no two distinct vectors in Y are neighbours, and each $y \in Y$ lies inside the ball of of radius R centered at $\underline{t}$. Then, as R tends to infinity*

$$|Y| \leq (1 + o(1)) \cdot \frac{1}{k\sqrt{k}} \frac{\pi^{(k-1)/2}}{\Gamma\left(\frac{k+1}{2}\right)} R^{k-1}.$$

I.e., Y does not contain more than a fraction of $\frac{(1 + o(1))}{k}$ of the lattice points of H inside this ball.

Proof. For each vector $\underline{y} = (y_1, \ldots, y_k) \in Y$, define k vectors $\underline{y}^1, \underline{y}^2, \ldots, \underline{y}^k$ by $y^i = (y_1, y_2, \ldots, y_{i-1}, y_i + 1, y_{i+1}, \ldots, y_k)$. Clearly all the $k \cdot |Y|$ vectors $\{\underline{y}^i: y \in Y, 1 \leq i \leq k\}$ lie inside the ball of radius $R + 1$ centered at $\underline{t}$, and they all belong to the hyperplane $\bar{H} = \{\underline{x}: \langle \underline{x}, 1 \rangle = n + 2\}$. Furthermore, as Y contains no neighbours, all these $k|Y|$ points are distinct. Therefore

$$k|Y| \leq (1 + o(1)) \frac{1}{\sqrt{k}} \frac{\pi^{(k-1)/2}}{\Gamma\left(\frac{k+1}{2}\right)} (R + 1)^{k-1},$$

and the assertion of the claim follows. □

We conclude this section with the following proposition.

Proposition 3.5. *For all sufficiently large n, for every projective plane $\mathbb{P} = \mathbb{P}_n = (P, \mathcal{L})$ of order n and for every 5-coloring C of $\mathbb{P}$ there are at least $n^2/100$ distinct pairs $\{L, L'\}$ of lines of $\mathbb{P}$ such that $t_{L,C}$ and $t_{L',C}$ are neighbours.*

Proof. Suppose this is false and let C be a 5-coloring of $\mathbb{P}$ with less than $n^2/100$ neighbouring pairs. Let $\mathcal{M} \subseteq \mathcal{L}$ be a set of lines obtained from $\mathcal{L}$ by omitting one line from each such pair. Clearly $|\mathcal{M}| > 0.99n^2$. By Claim 3.4, the number of vectors $\{t_{L,C}: L \in \mathcal{M}\}$ inside each ball of radius r does not exceed $(1 + o(1)) \frac{1}{5\sqrt{5}} \frac{\pi^2}{2} r^4$. Therefore, proceeding as in the proof of Proposition 3.3, we conclude that if R is defined by $\frac{1}{5\sqrt{5}}\frac{\pi^2}{2}R^4 = 0.99n^2$ then for any vector $\underline{t}$

$$\sum_{L\in\mathcal{M}}(t_{L,C}-\underline{t})^2 \geq (1+o(1))\frac{1}{5}R^6\frac{\pi^2}{2}\cdot\frac{4}{\sqrt{5}\cdot 6} = (1+o(1))R^6\frac{\pi^2}{15\sqrt{5}}.$$

These two equations give

$$\sum_{L\in\mathcal{M}}(t_{L,C}-\underline{t}^2)^2 \geq (1+o(1))\cdot(0.99)^{3/2}\frac{2\sqrt{10\sqrt{5}}}{3\pi}n^3 \approx (1+o(1))0.988n^3. \quad (3.4)$$

In particular, this holds for the vector $\underline{t}$ defined by the coloring C in the usual manner. However, by inequality (3.3)

$$\sum_{L\in\mathcal{M}}\|t_{L,C}-\underline{t}\|^2 \leq 0.8n(n^2+n+1) = (1+o(1))0.8\cdot n^3.$$

This contradicts inequality (3.4) and completes the proof of the proposition. □

4. Concluding Remarks

The main tool in the proof of Theorem 3.1 is Lemma 3.2. As mentioned in the remark following this lemma, the lemma and some more general statements can be proved using the eigenvalues of an appropriate incidence matrix. Let $H = (V, E)$ be a k-uniform l-regular hypergraph with a set V of p vertices and a set E of q edges $(p\cdot l = q\cdot k)$. The *incidence matrix* of H is the matrix $A = A_H = (a_{ev})_{e\in E, v\in V}$ defined by $a_{ev} = 1$ if $v\in e$ and $a_{ev} = 0$ if $v\notin e$. One can easily check that $k\cdot l$ is the maximum eigenvalue of the symmetric matrix A^TA, with a corresponding eigenvector $(1, 1, \ldots, 1)$. Let λ denote the second largest eigenvalue of A^TA. By Rayleigh's principle, for any vector $\underline{y} = (y_v)_{v\in V}$ that satisfies $\sum_{v\in V} y_v = 0$ the inequality

$$\langle A^TA\underline{y}, \underline{y}\rangle \leq \lambda \sum_{v\in V} y_v^2 \quad (4.1)$$

holds. Let $X\subseteq V$ be an arbitrary set of vertices of H. Define a vector $\underline{y} = \underline{y}_X = (y_v)_{v\in V}$ by $y_v = -\frac{|X|}{p}$ if $v\notin X$ and $y_v = 1-\frac{|X|}{p}$ if $v\in X$. Clearly $\sum_{v\in V} y_v = (p-|X|)\left(-\frac{|X|}{p}\right) + |X|\left(1-\frac{|X|}{p}\right) = 0$. Therefore, by (4.1)

$$\langle A^TA\underline{y}, y\rangle \leq \lambda\sum_{v\in V} y_v^2 = \lambda|X|\left(1-\frac{|X|}{p}\right). \quad (4.2)$$

However

$$\langle A^TA\underline{y}, \underline{y}\rangle = \langle A\underline{y}, A\underline{y}\rangle = \sum_{e\in E}\left(|e\cap X|\cdot\left(1-\frac{|X|}{p}\right) - |e\setminus X|\frac{|X|}{p}\right)^2$$
$$= \sum_{e\in E}\left(|e\cap X| - \frac{k}{p}|X|\right)^2.$$

Therefore, (4.2) implies

$$\sum_{e \in E}\left(|e \cap X| - \frac{k}{p}|X|\right)^2 \leq \lambda |X|\left(1 - \frac{|X|}{p}\right) \tag{4.3}$$

For the projective plane of order n, $p = q = n^2 + n + 1$, $k = n + 1$ and $\lambda = n$. Moreover, as for the projective plane all eigenvalues of $A^T A$ besides the first are equal to n, inequality (4.3) is an equality, which is equivalent to Lemma 3.2. For our purposes in this paper, only the inequality corresponding to (4.3) was used. Consequently, the proof in section 3 can be generalized to any analogous coloring problems of uniform regular hypergraphs, provided we have an estimate on the eigenvalues involved. The well known generalized n-gons supply one possible family of examples. Other possible examples arise from higher dimensional projective geometries $PG(d, q)$; one can consider the hypergraph whose vertices are the points of $PG(d, q)$ and whose edges are all subspaces of dimension r in $PG(d, q)$, where $1 \leq r < d$.

Returning to projective planes, we note that it seems that both our upper and lower bounds are not tight. We conclude the paper with the following conjecture.

Conjecture 4.1. For all sufficiently large n

$$6 \leq \chi(\mathbb{P}_n) \leq 7$$

for every projective plane of order n.

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