A Turanlike Neighborhood Condition and Cliques in Graphs

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INTRODUCTION

There have been many conditions placed on graphs to ensure the existence of certain kinds of subgraphs, in particular, conditions on the degrees of vertices have been useful. The following result of Ore is an example of the use of such a degree condition.

THEOREM A [5]: If G is a graph of order $n \geq 3$ such that the sum of degrees of any pair of nonadjacent vertices is at least n, then G is Hamiltonian.

Gould and Jacobson introduced a neighborhood condition that was patterned after the Ore type of degree condition, and that also implies the existence of certain subgraphs. An example of a result using this condition is the following, which parallels the previously cited result of Ore.

THEOREM B [3]: If G is a graph of order $n \geq 3$ such that the union of the neighborhoods of each pair of nonadjacent vertices is of cardinality at least $(2n + 1)/3$, then G is Hamiltonian.

Our purpose is to investigate the neighborhood condition of the preceding type needed to ensure a clique of a fixed order. If $n = km$, then the Turan graph [6], which is the complete k-partite graph, $K_{m,m,\ldots,m}$, does not contain a complete K_{k+1} as a subgraph. However, for $m \geq t \geq 1$, the union of the neighborhoods of any set of t independent vertices has precisely $(k-1)m = (k-1)n/k$ vertices. Therefore, the following theorem, which is the main result to be proved, is the best possible of this type.

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THEOREM 1: Let k and t be fixed integers greater than or equal to 2. If any set of t independent vertices of a graph of order $n > n_0(k, t)$ has more than $(k-1)n/k$ vertices in the union of the neighborhoods of the vertices, then G has a clique of order at least $k+1$.

NOTATION

All graphs will be finite and without loops or multiple edges. Notation will generally follow that of [4] unless otherwise stated. Some special notation and terminology will be introduced, and standard notation that is used extensively will be briefly described. For example, the complete multipartite graph with k parts each with m vertices will be expressed as $K(k; m)$, and the special case when $m = 1$, which is the complete graph on k vertices, will be expressed simply as K_k.

Let v be a vertex of a graph H. The neighborhood of v (the vertices that are adjacent in H to v) will be denoted by $N_H(v)$, or simply $N(v)$ when the identity of H is clear. If t is a positive integer, then $H_v(t)$ will denote the graph obtained from H by replacing v with t independent vertices, each with the same neighborhood as v. We will say that $H_v(t)$ is obtained from H by expanding the vertex v into t vertices. The graph obtained when each vertex of H is expanded into t vertices will be denoted by $H(t)$. Therefore, if H has order h (the number of vertices in H), then $H_v(t)$ and $H(t)$ have orders $h + t - 1$ and ht, respectively. Also, with this notation, $K_k(t) = K(k; t)$.

The maximum number of edges a graph G of order n can have without having a copy of a graph H is the extremal number $\text{ex}(n, H)$. Additional edges in G will ensure at least one copy, but possibly many copies of H. By $n_G(H)$ we will denote the number of copies of H in G, where H is considered as a labeled graph. If the order of H is p, then $n_G(H) \leq cn^p$ for some $c = c(H)$, because there are at most that many subsets of p vertices of G. If, on the other hand, $n_G(H) \geq c'n^p$ for some $c' = c'(H)$, we will say that H *saturates* G.

We next carefully define the neighborhood condition that appears in the statement of Theorem 1, and is the basis of this investigation.

DEFINITION: For fixed positive integers k and t, a graph G of order n satisfies the *neighborhood condition* $N(k, t)$ if for each set $\{x_1, x_2, \ldots, x_t\}$ of t independent vertices,

$$|\{\cup N_G(x_i): 1 \leq i \leq t\}| > (k-1)n/k.$$

PROOFS

We begin with a restatement of the result to be proved in this section.

THEOREM 1: Let $k, t \geq 2$ be integers. If a graph G of order $n > n_0(k, t)$ satisfies the neighborhood condition $N(k, t)$, then G contains a K_{k+1}.

It should be noted that a graph G of order n that satisfies the neighborhood condition $N(k, t)$ does not necessarily have more than $\text{ex}(n, K_{k+1})$ edges. Thus, Theorem 1 is not a consequence of the extremal result of Turan [6].

The following lemma reduces the proof of Theorem 1 to proving the existence of an expansion of K_k, namely a $K(k; t)$, instead of a K_{k+1}.

LEMMA 1: Let $k, t \geq 2$ be integers. If a graph G of order n satisfies the neighborhood condition $N(k, t)$ and contains a $K(k; t)$, then G contains a K_{k+1}.

Proof: Let $A_1, A_2, \ldots, A_k$ be the vertices in the k parts of the complete multipartite graph $K(k; t)$, and let A be the remaining $n - kt$ vertices of G. We will assume that G does not contain a K_{k+1}, and show that this leads to a contradiction.

The vertices in each A_i are independent, and no vertex of A is adjacent to at least one vertex in each A_i $(1 \leq i \leq t)$, since there is no K_{k+1} in G. There is no loss of generality in assuming that there are $|A|/k = (n - kt)/k = (n/k) - t$ vertices of A with no adjacencies in A_1. Therefore, the t independent vertices of A_1 have a combined neighborhood of at most $n - (n/k)$ vertices, which implies that G does not satisfy the neighborhood condition $N(k, t)$. This contradiction completes the proof of Lemma 1. □

Our next objective is to show that a graph G that satisfies the neighborhood condition $N(k, t)$ contains a $K(k; t)$. We will show something stronger, namely that $K(i; t)$ saturates G for $(1 \leq i \leq k)$. The following lemma will be used in an inductive proof of the preceding statement. Lemma 2, and its proof, are patterned after a result of Erdös and Simonovits in [2].

LEMMA 2: Let t be a fixed positive integer and H a fixed graph of order p. If G is any graph of order n with

$$n_G(H) = m,$$

then there is a constant $c = c(p, t)$ such that

$$n_G(H_v(t)) \geq [(cm^t)/(n^{(p-1)(t-1)})]$$

for any vertex v of H.

Proof: Let $H' = H - v$, and $\{H'_r\colon r \in R\}$ be the copies of H' contained in G. For each copy H'_r, let L_r be the vertices of $G - H'_r$ that are adjacent in G to the neighborhood $N_H(v)$ of v in H'_r. If $l_r = |L_r|$, then $\sum_{r \in R} l_r = m$. Each subset of L_r with t vertices will give a copy of $H_v(t)$ in G. Therefore,

$$n_G(H_v(t)) = \sum_{r \in R} \binom{l_r}{t} \geq |R| \binom{m/|R|}{t}$$

$$\geq [c'(p, t)(m^t/|R|^{t-1})].$$

Since H' has order $p - 1$, $|R| \leq c''n^{p-1}$ and

$$n_G(H_v(t)) \geq [c(p, t)m^t/(n^{(p-1)(t-1)})].$$

This completes the proof of Lemma 2. □

The special case of Lemma 2 when H saturates G gives the following two corollaries, which are expressed in the form that we will apply them in the proof of Proposition 1.

COROLLARY 1: If $m = c'n^p$, then $n_G(H_v(t)) \geq [cn^{p+t-1}]$.

COROLLARY 2: If $m = c'n^p$, then $n_G(H(t)) \geq [cn^{pt}]$.

The proof of Theorem 1 will be complete with the proof of the following result, which states that $K(k; t)$ saturates any graph that satisfies the neighborhood condition $N(k, t)$.

PROPOSITION 1: Let $t \geq 2$, $k \geq 1$ be integers and let G be a graph of order n, which satisfies $N(k, t)$. Then, there exist positive constants $c = c_{k,t}$ and $c' = c'_{k,t}$ such that

$$n_G(K_k) \geq [cn^k] \tag{1}$$

and

$$n_G(K(k; t)) \geq [c'n^{kt}]. \tag{2}$$

Proof: The proof is by induction on k with t fixed throughout the proof. For $k = 1$, both (1) and (2) are trivially true. We assume that (1) and (2) are true for $k = r \geq 1$ and verify them for $k = r + 1$. Thus, we assume G satisfies the neighborhood condition $N(r + 1, t)$. We can also assume that n is large, because appropriate choice of constants c and c' make the result trivial for small values of n.

Since property $N(r + 1, t)$ implies $N(r, t)$, we have that both (1) and (2) are true for $k = r$, so G contains at least $[c'n^{rt}]$ copies of $K(r; t)$. There are two types of copies of $K(r; t)$: there are those with no edges in each of their parts and those with at least one edge in some part.

First consider the case of a copy of $K(r; t)$ with parts $A_1, A_2, \ldots, A_r$, each of which is independent. Let A be the remaining vertices of G. For each i $(1 \leq i \leq r)$, let B_i be the vertices of A that have no adjacencies in A_i. Let B be the remaining vertices of A. The neighborhood condition $N(r + 1, t)$ implies $|B_i| < |A|/(r + 1)$, and hence

$$|B| \geq |A|/(r + 1) \geq c''n$$

for some positive constant c''. Note that each vertex in B will give at least one copy of a K_{r+1} in G using precisely one vertex from each A_i.

If at least one half of the copies of $K(r; t)$ in G are of the first type, then there will be at least $[(c''n)(c'n^{rt})/2]$ copies of a K_{r+1}, counting multiplicities. However, any such K_{r+1} can come from at most n^{rt-r} different copies of a $K(r; t)$. Thus G would contain at least $[(c''c'n^{r+1})/2]$ copies of a K_{r+1} in this case.

We can now assume that at least one half of the copies of $K(r; t)$ in G are of the second type and have at least one edge in one of their parts. Associated with each of the $(c'n^{rt})/2$ copies of $K(r; t)$ of this type there is a copy of K_{r+1} in G. Also, any such K_{r+1} will arise from at most n^{rt-r-1} different copies of a $K(r; t)$. Hence there are at least

$$[(c'n^{rt})/(2n^{rt-r-1})] = [(c'n^{r+1})/2]$$

copies of a K_{r+1} in G. This verifies (1) for $k = r + 1$.

Since $K(r + 1, t) = K_{r+1}(t)$, Corollary 2 and (1) verify that (2) is true when $k = r + 1$. This completes the proof of Proposition 1. □

The proof of Theorem 1 is an immediate consequence of Proportion 1 and Lemma 1.

PROBLEMS

There are numerous unsolved problems related to neighborhood conditions like the one just considered. In [3] and [4] neighborhood conditions for nonadjacent pairs of vertices are used to ensure the existence of certain types of subgraphs. Theorem B is an example of one of these results. It would be nice to replace each of these conditions by a neighborhood condition involving t independent vertices where $t \geq 3$. Also, one can be concerned not with just the existence of a certain subgraph, but with how many subgraphs of this type there are. Proposition 1 is an example of a result of this type.

Bondy and Chvátal considered a "degree" closure that generalized results of the type given in Theorem A. Does there exist a "neighborhood" closure analogous to the "degree" closure that would generalize the results using neighborhood conditions?

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