

Maximal Intersecting Families and Affine Regular Polygons in $PG(2, q)$

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A family of mutually intersecting k -sets is called a k -clique. A k -clique is maximal if it is not contained in any larger k -clique. Using a classification result of Wettl we give a new upper bound for $m(k)$, the minimum number of members of a maximal k -clique, proving $m(k) \leq k^2/2 + 5k + o(k)$ whenever $k-1$ is a prime power. The proof is based on finite geometric results which are thought to be of independent interest. © 1989 Academic Press, Inc.

1. PRELIMINARIES ON MAXIMAL CLIQUES

Let k be a positive integer. A k -clique (or intersecting family of rank k) is a collection of pairwise nondisjoint k -sets. A k -clique is *maximal* if it cannot be extended to another k -clique by adding a new k -set (and possible new elements).

A set B is called a *blocking set* of the hypergraph \mathcal{F} if $B \cap F \neq \emptyset$ for

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every $F \in \mathcal{F}$. Clearly if \mathcal{F} is a k -clique then every superset of any $F \in \mathcal{F}$ is a blocking set, so we say a blocking set is *nontrivial* if it contains no member of \mathcal{F} . Thus a k -clique is *maximal* if and only if it contains no nontrivial blocking set of k or fewer elements. For example, the following hypergraphs are maximal k -cliques:

all the k -subsets of a given $(2k - 1)$ -element set, (1.1)

the system of lines of a finite projective plane of order $k - 1$. (1.2)

Denote by $m(k)$ the minimum size of a maximal k -clique. Erdős and Lovász [10] have given bounds for $m(k)$, proving in particular that $m(k) \geq \frac{8}{3}k - 3$. This was improved by Dow, Drake, Füredi, and Larson [4] to

$$m(k) \geq 3k \quad \text{for } k \geq 4. \quad (1.3)$$

J.-C. Meyer [16] observed that $m(1) = 1$, $m(2) = 3$, and $m(3) = 7$. That $m(4) = 12$ follows from (1.3) and the fact, proved in [11], that

$$m(k) \leq \frac{3}{4}k^2 \text{ whenever } k \text{ is even and a projective plane of order } k/2 \text{ exists.} \quad (1.4)$$

For $k > 4$ the value of $m(k)$ remains open.

The determination of $m(k)$ is one of a number of questions concerning minimum cardinality families which are maximal with respect to various restrictions. This type of problem was raised by Erdős and Kleitman [9], and there has been little progress in these investigations. It was conjectured by Meyer [17] and Erdős [7] that $m(k) \geq k^2 - k + 1$ (the bound being attained by (1.2) where $k - 1$ is a prime power). This was disproved in [11] (see (1.4) above), but it is still true that known constructions depend heavily on projective and affine geometries. There are at present just two other known classes of k -cliques of size less than k^2 . These give the following upper bounds:

$$m(q^n + q^{n-1}) \leq q^{2n} + q^{2n-1} + q^{2n-2} \quad (1.5)$$

obtained from any n -uniform projective Hjelmslev plane of order q (Babai and Füredi [11] for $n = 2$, Drake and Sane [6] for all n):

$$m(k) \leq \frac{3}{4}k^2 + \frac{3}{2}k - 1 \text{ if } k - 1 \text{ is an odd prime power, } k \geq 8 \quad (\text{Blokhuis [1]}). \quad (1.6)$$

The construction giving (1.6) was the first counterexample to the conjecture of [11] that any maximal k -clique \mathcal{F} satisfies $|\mathcal{F}| \geq |\bigcup \mathcal{F}|$. Using an

idea of Drake [5], Blokhuis [1] recently gave an ingenious inductive proof of

$$m(k) < k^5 \quad \text{for every } k. \quad (1.7)$$

It seems reasonable to conjecture that at least $m(k)/k \rightarrow \infty$ ($k \rightarrow \infty$). (Erdős [8, problem I.3] offers \$500 for resolution of a somewhat stronger conjecture.)

In this paper we show

THEOREM 1.8. *If $k - 1$ is a prime power then $m(k) \leq k^2/2 + 5k + o(k)$.*

(Actually we give the proof only for $q \equiv -1 \pmod{6}$, but the other cases are similar.) The proof, given in Section 4, is based on a result (Theorem 2.5) which is thought to be of independent interest. In particular it is shown (Corollary 2.6) that if B is a set of $q + 1$ points meeting all lines meeting a conic C of the Galois plane $\text{PG}(2, q)$, then the points $B \setminus C$ are collinear.

2. MINIMAL COVERINGS OF THE SET OF LINES MEETING A CONIC

Let $(\mathcal{P}, \mathcal{L})$ be the hypergraph formed by the points and lines of the Galois plane $\text{PG}(2, q)$. We write $L(x, y)$ for the line spanned by (distinct) points x, y , and $P(L, M)$ for the point of intersection of the lines L, M .

Let $C \subset \mathcal{P}$ be a proper conic, and define

$$\mathcal{L}(C) = \{L \in \mathcal{L} : L \cap C \neq \emptyset\},$$

the set of secants and tangents of C . Thus

$$|\mathcal{L}(C)| = \binom{q+1}{2} + q + 1. \quad (2.1)$$

The following result was proved by Bruen and Thas [3] for q even and by Korchmaros and Segre [15] for all q :

If B is a subset of $\mathcal{P} \setminus C$ of size at most $q + 1$ meeting all lines of $\mathcal{L}(C)$, then B is an exterior line of C . (2.2)

(The hypotheses immediately give $|B| = q + 1$ and $|B \cap L| = 1$ for each $L \in \mathcal{L}(C)$.) In this section we generalize (2.2), describing all the minimal blocking sets of $\mathcal{L}(C)$. For another nice generalization of (2.2) see [2].

We are interested in blocking sets of size $q + 1$ (the obvious lower bound) of the hypergraph $\mathcal{L}(C)$. Let us denote by $\mathcal{B}(C)$ the collection of

such blocking sets which are not equal to C and are not lines of \mathcal{L} . For example, it is an easy exercise to verify

PROPOSITION 2.3. *If $B \in \mathcal{B}(C)$ satisfies $|B \setminus C| \leq 3$, then one of the following holds:*

(a) $B = C \setminus \{x\} \cup \{y\}$ with $x \in C$ and $y \in L_x \setminus \{x\}$, where L_x denotes the tangent at x ;

(b) $B = C \setminus \{u, v\} \cup \{w, x\}$ with $u, v \in C$, $w \in L(u, v) \setminus \{u, v\}$ and $x = P(L_u, L_v)$, where L_u and L_v are the tangents at u and v .

(c) $B = C \setminus \{u_1, u_2, u_3\} \cup \{x_1, x_2, x_3\}$, where $u_i \in C$ and x_i is the intersection of the tangent at u_i and the secant spanned by u_{i+1}, u_{i+2} (subscripts mod 3).

Before giving a complete description of $\mathcal{B}(C)$ we recall a few geometric facts. Let $L \in \mathcal{L}$ and let c_1, \dots, c_m be the points of $C \setminus L$, l_1, \dots, l_m the points of $L \setminus C$ ($m \in \{q-1, q, q+1\}$) indexed so that l_i lies on $L(c_1, c_i)$. (As usual $L(c, c)$ is the tangent at $c \in C$.) If we multiply subscripts by the rule

$$xy = z \quad \text{if } l_z \text{ lies on } L(c_x, c_y),$$

then the set $\{1, \dots, m\}$ becomes an Abelian group (a consequence of Pascal's theorem), which we denote $G(L, c_1)$. It was shown by Korchmáros [13, 14] that $G(L, c_1)$ is cyclic if $m \in \{q-1, q+1\}$ and elementary Abelian if $m = q$.

EXAMPLE 2.4. Let N be a coset of a subgroup of $G(L, c_1)$, and let B be obtained by deleting from C the points of $C \setminus L$ corresponding to N and adding the points of $L \setminus C$ corresponding to $N \cdot N$. Then $|B| = q+1$ (since $|N \cdot N| = |N|$) and so $B \in \mathcal{B}(C)$.

Notice that Example 2.4 includes the examples in Proposition 2.3. It is also easy to see that replacing c_1 by c_i gives the same examples since the system of cosets does not change. Our main geometric result is

THEOREM 2.5. *The only sets in $\mathcal{B}(C)$ are those given by Example 2.4.*

The proof of Theorem 2.5 depends mainly on establishing

LEMMA 2.6. *If $B \in \mathcal{B}(C)$ then the points of $B \setminus C$ are collinear.*

We remark that Theorem 2.5 is closely related to results of Wettl [19] and Szönyi and Wettl [18]. They described the $(q+1)$ -element sets Q with the property that for some line L , $Q \setminus L$ is an arc (i.e., contains no three collinear points) and every line containing 2 points from $Q \setminus L$ avoids $L \setminus Q$.

3. PROOF OF THEOREM 2.5

We begin with a lemma of Korchmáros and Segre [15].

LEMMA 3.1. *Suppose a, b, c are noncollinear points of $\text{PG}(2, q)$ and D is a set of $q + 1$ points disjoint from $\{a, b, c\}$ but meeting every line which meets $\{a, b, c\}$. Then the points $D \cap L(a, b)$, $D \cap L(a, c)$, and $D \cap L(b, c)$ are collinear.*

Let a, b, c be collinear points disjoint from C and suppose that p_a, p_b, p_c are points of the conic C such that each of $\{a, p_b, p_c\}$, $\{p_a, b, p_c\}$, $\{p_a, p_b, c\}$ is a collinear triple. Then the triangle $\{p_a, p_b, p_c\}$ is said to be associated with the triple $\{a, b, c\}$.

LEMMA 3.2 (folklore; see [15]). *The number of triangles associated with a collinear triple spanning a line l is at most*

1. *if q is even and l is not a tangent or q is odd and l is a tangent*
2. *otherwise.*

Let $B \in \mathcal{B}(C)$ and denote $C \setminus B$ by K , $B \setminus C$ by W , and $|K| = |W|$ by k .

We first prove Lemma 2.6, after which Theorem 2.5 follows easily. Of course we may suppose $k \geq 3$. Consider the following 2-design \mathcal{A} on W :

$$\mathcal{A} = \{L \cap W : L \cap W \geq 2, L \in \mathcal{L}\}.$$

Our aim is to prove that \mathcal{A} is a trivial design, i.e., $\mathcal{A} = \{W\}$. Clearly,

$$\sum_{A \in \mathcal{A}} \binom{|A|}{2} = \binom{k}{2}. \quad (3.3)$$

CLAIM 3.4. $\sum_{A \in \mathcal{A}} \binom{|A|}{3} \geq \frac{1}{2} \binom{k}{3}$.

Proof. Indeed by Lemma 3.1 every triangle $\{a, b, c\} \subseteq K$ is associated with a collinear triple of W . So by Lemma 3.2 the number of collinear triples in W is at least $\frac{1}{2} \binom{k}{3}$. On the other hand, the number of collinear triples in W is exactly $\sum \binom{|A|}{3}$. ■

This immediately gives Lemma 2.6 if $k = 3$ or 4 , so from now on we suppose that $k \geq 5$. Suppose for Claims 3.5 and 3.7 that q is odd.

CLAIM 3.5. *For all $A \in \mathcal{A}$ we have $|A| \geq 3$.*

Our main tool in the proof of Claim 3.5 is the fact:

$$\text{If } L \in \mathcal{L}, \quad L \cap C = \{a, b\}, \quad L \cap K \neq \emptyset, \quad \text{and} \quad L \cap W \neq \emptyset, \quad \text{then} \\ \{a, b\} \subseteq K. \quad (3.6)$$

For if $a \in K$, e.g., then each line through a carries just one point of B , and since $B \cap L$ contains a point of W it cannot also contain b .

Proof of 3.5. Let $p, q \in W$. Since $k \geq 5$ there is some $a \in K$ such that both $L = L(p, a)$ and $L' = L(q, a)$ are secants of C . Set $L \cap C = \{a, b\}$, $L' \cap C = \{a, c\}$. By (3.6), $b, c \in K$ and applying Lemma 3.1 (with $D = B$) we find that $L(b, c) \cap W$ is a third point of the block A of \mathcal{A} containing p, q . ■

CLAIM 3.7. *There is an $A_0 \in \mathcal{A}$ with $|A_0| \geq \frac{1}{2}k + 1$.*

Proof. Indeed with $d = \max\{|A| : A \in \mathcal{A}\}$ we have

$$\frac{1}{2} \binom{k}{3} \leq \sum_{A \in \mathcal{A}} \binom{|A|}{3} \leq \frac{d-2}{3} \sum_{A \in \mathcal{A}} \binom{|A|}{2} = \frac{d-2}{3} \binom{k}{2}$$

(the last equality by (3.3)), and the claim follows. ■

Claims 3.5 and 3.7 quickly imply 2.6. For if A_0 is as in Claim 3.7 and $p \in W \setminus A_0$, then the lines joining p to A_0 contain at least

$$1 + 2|A_0| > k$$

points of W , a contradiction. ■

For q even a similar argument shows that the points of W are collinear with the possible exception of the core (say r) of C . But in this exceptional case Lemma 3.2 demands at least $\binom{k}{3}$ triples from the $k-1$ points of $W \setminus \{r\}$, which is impossible. So again W is collinear.

Proof of Theorem 2.5. We may suppose $k \geq 2$. Denote the line containing W by L . We work in the group $G = G(L, c_1)$, where c_1 is some point of $C \setminus L$. Denote by \hat{K} and \hat{W} the subsets of G corresponding to K and W . By the definitions of B, K, W we have $\hat{K} \cdot \hat{K} \subseteq \hat{W}$, whence $|\hat{K} \cdot \hat{K}| = |\hat{K}|$ implying that \hat{K} is a coset of some subgroup of G . ■

4. A MAXIMAL k -CLIQUE FROM $\mathcal{B}(C)$

For q a prime power, $q+1 \equiv 0 \pmod{6}$, let C be a proper conic of $PG(2, q)$, L an exterior line, and c_0 an arbitrary point of C . Let H be the 6-element subgroup of the (cyclic) group $G(L, c_0)$ and denote by K and W

the subsets of C and L corresponding to H . Then $B = (C \setminus K) \cup W$ is in $\mathcal{B}(C)$. Set

$$\mathcal{F} = \mathcal{L}(C) \cup \{L' \in \mathcal{L} : L' \cap W \neq \emptyset\} \cup \{B' \in \mathcal{B}(C) : W \subseteq B'\}.$$

THEOREM 4.1. *\mathcal{F} is a maximal $(q+1)$ -clique with*

$$|\mathcal{F}| < \frac{1}{2}(q+1)^2 + 4(q+1).$$

Proof. Clearly \mathcal{F} is intersecting. It is also easy to see that no line not in \mathcal{F} is a blocking set of \mathcal{F} , so the maximality of \mathcal{F} follows from

PROPOSITION 4.2. *If B' is a nontrivial $(q+1)$ -point blocking set of*

$$\mathcal{F}_0 := \mathcal{L}(C) \cup \{L \in \mathcal{L} : L \cap W \neq \emptyset\} \cup \{B\},$$

then $W \subseteq B' \in \mathcal{B}(C)$.

Proof. That $B' \in \mathcal{B}(C)$ follows from $\mathcal{L}(C) \subseteq \mathcal{F}_0$. Set $B' = (C \setminus K') \cup W'$ with $|K'| = |W'| = k$ and suppose by way of contradiction that there exists $x \in W \setminus W'$.

Set $L' = L(W')$. We must have $L' \neq L$, since otherwise any exterior line through x avoids B' . Thus

$$|W \cap L'| \leq 1. \quad (4.3)$$

We will show

$$|K \setminus K'| \leq 2. \quad (4.4)$$

For suppose $\{a, b, c\} \subseteq K \cap B'$. Since $L' \neq L$, one of the lines $L(a, b)$, $L(a, c)$, $L(b, c)$ (say $L(a, b)$) does not contain $p(L, L')$. Let $L(a, b) \cap W = \{y\}$. Then $y \in W \setminus B'$ implies that every line through y contains exactly one point of B' , a contradiction since $L(a, b)$ contains two such points. This proves (4.4).

By (4.3) some tangent to C at a point of K meets W in a point $w \notin L'$. Then w is on 4 lines which meet C only in K , at least 2 of which avoid $K \setminus K'$ (by (4.4)). These two lines, together with the $(q-1)/2$ exterior lines through w , must be met by W' , whence $|W'| > (q+1)/2$. It follows that $W' = L' = B'$, contradicting the assumption that B' is nontrivial. This proves Proposition 4.2. ■

It remains to prove the upper bound on $|\mathcal{F}|$. As noted in (2.1) we have

$$|\mathcal{L}(C)| = \binom{q+2}{2}. \quad (4.5)$$

Since at least 3 of the 6 points of W are exterior points (all 6 if $4 \mid q+1$),

$$|\{L \in \mathcal{L} \setminus \mathcal{L}(C) : L \cap W \neq \emptyset\}| \leq 3q - 5. \quad (4.6)$$

To bound the size of $\{B' \in \mathcal{B}(C) : W \subset B'\}$ note that for any such B' , $B' \setminus C = W'$ corresponds to a (proper) subgroup of $G(L, c_0)$ containing H . The number of such subgroups is the number of proper divisors of $(q+1)/6$ and so is less than $2\sqrt{(q+1)/6} - 1$. The assertion

$$|\{B' \in \mathcal{B}(C) : W \subset B'\}| < 4\sqrt{(q+1)/6} - 2 \quad (4.7)$$

thus follows from

PROPOSITION 4.8. *For any $B' \in \mathcal{B}(C)$ there is at most one $B'' \in \mathcal{B}(C) \setminus \{B'\}$ for which $B'' \setminus C = B' \setminus C$.*

Proof. Let $B' \setminus C = W'$. It is clear that the complements of any two B'' satisfying $B'' \setminus C = W'$ are disjoint inside C , since for such a B'' and any $c \in C \setminus B''$ we have

$$C \setminus B'' = \left(\bigcup \{L(c, d) : d \in W'\} \right) \cap C.$$

But since all tangents to C at points outside such a B'' meet W' , at most $2|W'|$ points of C lie in $C \setminus B''$ for some B'' , so the proposition is proved. ■

Finally, the desired bound on $|\mathcal{F}|$ follows from (4.5)–(4.7) and the proof of Theorem 4.1 is complete. ■

Similar constructions can be given for the remaining values of q . (Briefly: for $q \equiv 1 \pmod{6}$ we may take L a secant and $|H| = 6$; for $q = 2^k$ we may take L a tangent and $|H| = 4$; the linear term in the bound of Theorem 1.8 derives from the cases $q = 3^k$, where for some values of k we must take $|H|$ as large as 9.)

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