Extremal Problems Whose Solutions Are the Blowups of the Small Witt-Designs

P. FRANKL

CNRS, Paris, France

AND

Z. FÜREDI

MIT, Cambridge, Massachusetts, 02139*

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Let $f(k, n, \Sigma)$ denote the maximum of $|\mathscr{F}|$, where $\mathscr{F} \subset (\begin{smallmatrix}1,\dots,n\\k\end{smallmatrix})$ and there are no $F_1, F_2, F_3 \in \mathscr{F}$ with $|F_1 \cap F_2| = k - 1$, $F_1 \, \Delta F_2 \subset F_3$. The function $f(k, n, \Sigma)$ was determined for $k = 2$ by Mantel, for $k = 3$ by Bollobás and for $k = 4$ by Sidorenko. Here we determine it for $k = 5, 6$ and $n > n_0$. Moreover, we show that the only optimal families, i.e., $|\mathscr{F}| = f(k, n, \Sigma)$ arise from the unique (11, 5, 4) or (12, 6, 5) Steiner-systems by a simple operation, called blowup. © 1989 Academic Press, Inc.

1. INTRODUCTION

Let $X = \{1, ..., n\}$ be an n-element set and let $\mathscr{F} \subset (\begin{smallmatrix}X\\k\end{smallmatrix})$ be a k-graph, that is, a collection of distinct k-subsets of X, $2 \leqslant k \leqslant n$. Following de Caen [C] we consider the property:

(Σ)   there are no three edges $F_1, F_1, F_3 \in \mathscr{F}$ with $|F_1 \cap F_2| = k - 1$ and $F_3 \supset F_1 \, \Delta F_2$.

Note that $B \Delta C = (B - C) \cup (C - B)$ and that for $k = 2$ (Σ) means that the graph $\mathscr{F}$ is triangle-free. There is a related, stronger property, which was introduced by Katona [K]:

(Δ)   there are no three edges $F_1, F_2, F_3 \in \mathscr{F}$ with $F_1 \, \Delta F_2 \subset F_3$. Note that for $k = 2, 3$ (Σ) and (Δ) coincide.

The simplest way to obtain a relatively large k-graph $\mathscr{F}$ with property (Δ) (and thus with (Σ)) is to consider the complete equipartite k-graph.

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Let $X = X_1 \cup \cdots \cup X_k$ be a partition with $\lfloor \frac{n}{k} \rfloor \leqslant |X_i| \leqslant \lceil \frac{n}{k} \rceil$ for $1 \leqslant i \leqslant k$ and set $\mathscr{P} = \mathscr{P}(k, n) = \{F \in \binom{X}{k}: |F \cap X_i| = 1, i = 1, ..., k\}$. Then

$$\left(1 - O\left(\frac{1}{n}\right)\right)\left(\frac{n}{k}\right)^k \leqslant |\mathscr{P}| \leqslant \left(\frac{n}{k}\right)^k, \tag{1.1}$$

with equality on the right-hand side if k divides n.

Define $f(k, n, \Sigma)$ $(f(k, n, \Delta))$ as max $|\mathscr{F}|$ where the maximum is over all $\mathscr{F} \subset \binom{X}{k}$ having property $\Sigma(\Delta)$, respectively.

By an old result of Mantel [M],

$$f(2, n, \Sigma) = f(2, n, \Delta) = \left\lfloor \frac{n^2}{4} \right\rfloor = |\mathscr{P}(2, n)|. \tag{1.2}$$

Katona [K] conjectured and Bollobás [B1] proved, that for $k = 3$,

$$f(3, n, \Sigma) = f(3, n, \Delta) = |\mathscr{P}(3, n)|. \tag{1.3}$$

Bollobás [B] conjectured that

$$f(k, n, \Delta) = |\mathscr{P}(k, n)| \tag{1.4}$$

holds for all $n \geqslant k \geqslant 4$ as well. The authors [FF2] proved this for $2k \geqslant n \geqslant k$. This, by an easy averaging, implies

$$f(k, n, \Delta) \leqslant 2^k \binom{n}{k} / \binom{2k}{k} \quad \text{for all} \quad n \geqslant 2k. \tag{1.5}$$

Recently, Sidorenko [S] proved Bollobás conjecture for $k = 4$. Indeed, he proved the stronger statement

$$f(4, n, \Sigma) = f(4, n, \Delta) = |\mathscr{P}(4, n)|.$$

Actually, both Bollobás and Sidorenko prove that equality holds only for the complete equipartite k-graphs.

Sidorenko [S] provides a new proof for (1.3). For other proofs see [C] and [FF1]. The aim of this paper is to give bounds on $f(k, n, \Sigma)$.

THEOREM 1.1.

$$f(5, n, \Sigma) \leqslant \frac{6}{11^4} n^5 \text{ with equality holding if and only if } 11 | n. \tag{1.6}$$

THEOREM 1.2.

$$f(6, n, \Sigma) \leqslant \frac{11}{12^5} n^6 \qquad \text{with equality holding if and only if } 12|n. \quad (1.7)$$

For general k we prove

THEOREM 1.3. *For all $k \geqslant 7$ one has*

$$\left(1 + O\left(\frac{1}{n}\right)\right) n^k/k! \binom{k}{2} e^{1 + 1/(k-1)} < f(k, n, \Sigma) < n^k/k! \, e \binom{k-1}{2}; \quad (1.8)$$

moreover, for $\binom{k}{2}|n$ the term $O(1/n)$ can be omitted.

Comparing (1.5) and (1.8) shows that, in general, $f(k \, n, \Delta)$ is much smaller than $f(k, n, \Sigma)$, answering a question of de Caen [C]. Actually, for $n > n_0$ we shall determine $f(5, n, \Sigma)$ and $f(6, n, \Sigma)$ exactly (see Theorem 7.1).

For a k-graph $\mathscr{G}$ let $\mathrm{ex}(n, \mathscr{G})$ denote $\max|\mathscr{F}|$, where the maximum is over all $\mathscr{F} \subset \binom{X}{k}$ which contain no subgraph isomorphic to $\mathscr{G}$. Define $\mathscr{G}_k = \{\{1, 2, ..., k\}, \{1, 2, ..., k-1, k+1\}, \{k, k+1, ..., 2k-1\}\}$.

For $k = 2$, $\mathscr{G}_2$ is simply a triangle. For $k = 3$, we proved in [FF2] that for $n > 3000$, $f(3, n, \Sigma) = \mathrm{ex}(n, \mathscr{G}_3)$ holds. Here we prove

THEOREM 1.4. *For all fixed k and $n \to \infty$,*

$$f(k, n, \Sigma) \leqslant \mathrm{ex}(n, \mathscr{G}_k) < \left(1 + O\left(\frac{1}{n}\right)\right) f(k, n, \Sigma) \qquad (1.9)$$

holds.

Actually, Theorem 1.4 will follow from a more general result, Theorem 8.4.

CONJECTURE 1.5. *For all fixed k and $n > n_0(k)$,*

$$f(k, n, \Sigma) = \mathrm{ex}(n, \mathscr{G}_k) \qquad (1.10)$$

holds.

Remark 1.6 (on Turán theory). The number $\mathrm{ex}(n, \mathscr{G})$ is often called the *Turán number* of $\mathscr{G}$. Similarly, we can define $\mathrm{ex}(n, \mathbf{G})$, where $\mathbf{G}$ is a class of k-graphs (the so-called forbidden subgraphs). These numbers were widely investigated, we can refer to [B2] and [ES1] in the case of graphs ($k = 2$), and to [B3] and [FF3] for $k \geqslant 3$.

From an averaging argument of Katona, Nemetz, and Simonovits [KNS] it follows that $\text{ex}(n, \mathbf{G})/\binom{n}{k}$ is monotone decreasing as a function of n. Therefore,

$$\pi(\mathbf{G}) = \lim_{n \to \infty} \text{ex}(n, \mathbf{G}) \bigg/ \binom{n}{k}$$

exists for every class of k-graphs $\mathbf{G}$. If $\mathbf{G}$ has a k-partite member then a theorem of Erdös [E] says that there exists a $\delta = \delta(\mathbf{G}) > 0$ such that

$$\text{ex}(n, \mathbf{G}) = O(n^{k-\delta}).$$

Hence $\pi(\mathbf{G}) = 0$. Otherwise, the example of a complete equipartite k-graphs shows that $\pi(\mathbf{G}) \geqslant k!/k^k$. The value of $\pi(\mathscr{G})$ is unknown for $k \geqslant 3$ for all non-k-partite $\mathscr{G}$ except the above-mentioned four cases (i.e., $\pi(\mathscr{G}_k)$ for $k = 3, 4, 5, 6$).

2. The Lagrange Function of Hypergraphs

Let us fix some notation. For $\mathscr{F} \subset \binom{X}{k}$ and $A \subset X$ define $\mathscr{F}(A) = \{F - A : A \subset F \in \mathscr{F}\}$. For $A = \{y\}$ we write also $\mathscr{F}(y)$ instead of $\mathscr{F}(\{y\})$. Define also $\partial\mathscr{F} = \{H \in \binom{X}{k-1}: H \subset F \in \mathscr{F}$ holds for some $F \in \mathscr{F}\}$, $\partial\mathscr{F}$ is called the *shadow* of $\mathscr{F}$.

The main tool of the proof of Theorems 1.1–1.3–as in Sidorenko [S]—is the Lagrange function of hypergraphs.

With every k-graph $\mathscr{F} \subset \binom{X}{k}$ we associate a homogeneous polynomial in n variables, the *Lagrange polynomial* $\lambda(\mathscr{F}, x_1, ..., x_n)$. We shall write $\lambda(\mathscr{F}, \mathbf{x})$ for short:

$$\lambda(\mathscr{F}, \mathbf{x}) = \sum_{F \in \mathscr{F}} \prod_{i \in F} x_i.$$

The *Lagrange function* $\lambda(\mathscr{F})$ is defined then by

$$\lambda(\mathscr{F}) = \max\{\lambda(\mathscr{F}, \mathbf{x}): x_1 + \cdots + x_n = 1, \ x_i \geqslant 0\}.$$

It has proved very useful in [FR1] for the disproof of a longstanding conjecture of Erdös. Further applications of the Lagrange function are in [FR2].

Setting $x_1 = \cdots = x_n = 1/n$ one obtains the simple but important inequality:

$$|\mathscr{F}| \leqslant n^k \lambda(\mathscr{F}). \tag{2.1}$$

Let us mention that for graphs $\lambda(\mathscr{F}, \mathbf{x})$ was already used by Motzkin and Straus [MS] to give a new proof of Turán's theorem [T].

The Lagrange function can be given a combinatorial interpretation as follows (cf. [FR1]):

For a k-graph $\mathscr{G} \subset \binom{Y}{k}$ and pairwise disjoint subsets $\{X_y: y \in Y\}$, $|X_y| = n_y$, define the blowup $\tilde{\mathscr{G}} = \mathscr{G} \otimes \{n_y: y \in Y\}$ of $\mathscr{G}$ by

$$\tilde{\mathscr{G}} = \left\{ H \in \binom{\bigcup X_y}{k} : \{y: H \cap X_y \neq \varnothing\} \in \mathscr{G} \right\}.$$

Note that the definition implies that for every $H \in \tilde{\mathscr{G}}$ one has $|H \cap X_y| \leqslant 1$ and setting $\mathbf{x} = (x_y: y \in Y)$ with $x_y = n_y/n$,

$$|\tilde{\mathscr{G}}| = n^k \lambda(\mathscr{G}, \mathbf{x}) \text{ holds}, \qquad \text{where} \quad n = \sum_{z \in Y} n_z.$$

Thus

$$\lambda(\mathscr{G}) = \lim \sup |\tilde{\mathscr{G}}| \, n^{-k}, \text{ where the supremum is over } n \text{ and all}$$
blowups of $\mathscr{G}$ with n vertices. $\qquad\qquad$ (2.2)

Let us recall the following result which combines parts of Theorems 2.1 and 2.3 of [FR1].

LEMMA 2.1. *For every non-empty k-graph $\mathscr{F} \subset \binom{X}{k}$ there exists a non-empty subset $Y \subset X$ and an evaluation $x_i = y_i \geqslant 0$, $y_1 + \cdots + y_n = 1$ such that the following hold*:

    (i) $\quad y_i > 0$ *if and only if* $i \in Y$.

    (ii) $\quad \lambda(\mathscr{F}, \mathbf{y}) = \lambda(\mathscr{F})$.

    (iii) $\quad \lambda(\mathscr{F}) = \lambda(\mathscr{G})$ *holds for* $\mathscr{G} = \{F \in \mathscr{F}: F \subset Y\}$.

    (iv) *For every* $i, j \in Y$ *there exists* $G \in \mathscr{G}$ *with* $\{i, j\} \subset G$.

    (v) $\quad (\partial/\partial x_i) \lambda(\mathscr{G}, \mathbf{y}) = k \lambda(\mathscr{G})$ *for all* $i \in Y$.

The following crucial observation is due to Sidorenko [S].

FACT 2.2. *Suppose that* $\mathscr{F} \subset \binom{X}{k}$ *has property* (Σ) *and* $\mathscr{G}$ *is defined by Lemma 2.1. Then*

$$|G \cap G'| \leqslant k - 2 \text{ holds for all distinct, } G, G' \in \mathscr{G}. \qquad (2.3)$$

Proof. If $|G \cap G'| = k - 1$, then G, G' and a set $G'' \in \mathscr{G}$, containing $G \triangle G'$—which exists in view of (iv)—violate (Σ). $\quad\blacksquare$

For illustrating the use of these tools, let us reproduce the proof of Sidorenko [S] showing

$\lambda(\mathcal{F}) = k^{-k}$, $2 \leqslant k \leqslant 4$, for every non-empty k-graph $\mathcal{F}$ with property (Σ). $\hspace{3cm}$ (2.4)

Since $\lambda(\{1, 2, ..., k\}) = k^{-k}$, it is sufficient to show the upper bound part of (2.4). Suppose that $\mathcal{F}$ has property (Σ). Choose, in view of Lemma 2.1 and Fact 2.2, $\mathcal{G} \subset \mathcal{F}$ satisfying (iii), (iv), (v), and (2.3).

Note that (2.3) implies that for $i \neq j$ the polynomials $(\partial/\partial x_i) \lambda(\mathcal{G}, \mathbf{x})$ and $(\partial/\partial x_j) \lambda(\mathcal{G}, \mathbf{x})$ have no common term. Suppose for simplicity $Y = \{1, 2, ..., m\}$ and recall the elementary inequality $\sigma_l(y_1, ..., y_m) \leqslant \binom{m}{l} n^{-l}$ valid for $y_i \geqslant 0$, $y_1 + \cdots + y_m = 1$, where σ_l is the lth elementary symmetric polynomial.

Summing (v) over $i \in Y$ yields $mk\lambda(\mathcal{G}) \leqslant \sigma_{k-1}(y_1, ..., y_m) \leqslant \binom{m}{k-1} m^{-(k-1)}$, that is,

$$\lambda(\mathcal{G}) \leqslant (m-1) \cdots (m-k+2)/m^{k-1}k! \hspace{2cm} (2.5)$$

Remark 2.3. Let us note that if equality holds in (2.5) then equality must hold in the preceding inequality as well. In particular, every $(k-1)$-subset of Y is contained in at least (and consequently exactly) one member of $\mathcal{G}$.

For $k = 2, 3$ the right-hand side is monotone decreasing as a function of m for $m \geqslant k$. Since for $m = k$ the right-hand side is k^{-k}, (2.4) follows.

For $k = 4$ and $m \geqslant 6$ the right-hand side is again monotone decreasing and its value for $m = 6$ is

$$5 \cdot 4/6^3 \cdot 4! = \frac{5}{6^4} < \frac{1}{4^4}.$$

For $m = 4$ its value is 4^{-4}, while the case $m = 5$ is impossible, as (2.3) implies $|\mathcal{G}| \leqslant 1$. Thus (2.4) is proved. ∎

Note that (2.4) together with (2.1) implies $f(n, \Delta) = f(k, \Sigma) = (n/k)^k$ for $k \mid n$ and $2 \leqslant k \leqslant 4$.

To prove the upper bound in Theorem 1.3 we shall simply use the inequality (2.5). However, for Theorems 1.1 and 1.2 we have to refine this argument.

To close this section we shall prove a simple proposition. Let $\mathcal{F} \subset \binom{X}{k}$ be k-graph, Y an m-element set disjoint to X. Define

$$\mathcal{H} = \left\{ H \in \binom{X \cup Y}{k+1} : H \cap X \in \mathcal{F}, |H \cap Y| = 1 \right\}.$$

PROPOSITION 2.4. (i) $|\mathcal{H}| = |\mathcal{F}| m$;

(ii) *if $\mathcal{F}$ has property (Δ) then so does $\mathcal{H}$, too.*

The proof is by inspection and it is left to the reader as well as that of the following.

COROLLARY 2.5. *Suppose that* (1.4) *holds for some k, $k \geqslant 3$. Then it holds for $k - 1$ as well.*

3. LOWER BOUNDS AND THE PROOF OF THEOREM 1.3

Let $|Y| = m$. Recall that a family $\mathcal{R} \subset \binom{Y}{k}$ is called a $(m, k, k-1)$-*packing* or, briefly, a *packing* if

$$|R \cap R'| \leqslant k - 2 \qquad \text{for all distinct } R, R' \in \mathcal{R}. \tag{3.1}$$

It is easy to verify that every blowup of a packing has property (Σ) (cf. the definition of blowups in the preceding section).

Clearly, for every packing $\mathcal{R}$,

$$|\mathcal{R}| \leqslant \binom{m}{k-1} \Big/ k \text{ holds.}$$

In case of equality, $\mathcal{R}$ is called a *perfect packing* or a $(m, k, k-1)$-*Steiner system.*

It is known (cf., e.g., [BJL, N]) that unique $(m, k, k-1)$-Steiner systems exist for $m = 12$, $k = 6$ and $m = 11$, $k = 5$. Let $\mathcal{R}_{12}$ and $\mathcal{R}_{11}$ be such perfect packings. Note that $|\mathcal{R}_{11}| = 66$, $|\mathcal{R}_{12}| = 132$.

This implies

$$\lambda(\mathcal{R}_{11}) \geqslant \lambda(\mathcal{R}_{11}, (\tfrac{1}{11}, ..., \tfrac{1}{11})) = 66/11^5 = 6/11^4, \tag{3.2}$$

$$\lambda(\mathcal{R}_{12}) \geqslant \lambda(\mathcal{R}_{12}, (\tfrac{1}{12}, ..., \tfrac{1}{12})) = 132/12^6 = 11/12^5 \tag{3.3}$$

For arbitrary integers $t \geqslant 1$ considering the blowups $\mathcal{R}_{11} \otimes (t, ..., t)$ and $\mathcal{R}_{12} \otimes (t, ..., t)$ shows

$$f(11t, 5, \Sigma) \geqslant 66t^5,$$

$$f(12t, 6, \Sigma) \geqslant 132t^6,$$

proving the lower bound parts of Theorems 1.1 and 1.2.

Let us recall the following well-known result. (See, e.g., [GS]).

PROPOSITION 3.1. *For every $m \geqslant k$ there exists a packing $\mathscr{R} \subset \binom{Y}{k}$, $|Y| = \{1, \ldots, m\}$ satisfying*

$$|\mathscr{R}| \geqslant \binom{m}{k} \Big/ m. \tag{3.4}$$

Proof. For every $0 \leqslant b < m$ the family $\mathscr{R}^{(b)} = \{R \in \binom{Y}{k}: \sum_{i \in R} i \equiv b \pmod{m}\}$ is a packing and $\Sigma|\mathscr{R}^{(b)}| = \binom{m}{k}$. Thus for some $0 \leqslant b < m$,

$$|\mathscr{R}^{(b)}| \geqslant \binom{m}{k} \Big/ m \text{ holds.} \quad \blacksquare$$

Letting $m = \binom{k}{2}$ and using that for $0 < x < \frac{1}{3}$ one has $1 - x > \exp(-x - \frac{3}{4}x^2)$, we obtain for $k \geqslant 7$,

$$\prod_{1 \leqslant i \leqslant k-1} \frac{m-i}{m} \geqslant \exp\left(-\sum_{1 \leqslant i \leqslant k-1} \frac{i}{\binom{k}{2}} + \frac{3i^2}{(k(k-1))^2} \right)$$

$$= e^{-1 - (2k-1)/(2k(k-1))} > e^{-1 - 1/(k-1)}.$$

Consequently, choosing $\mathscr{R}$ from Proposition 3.1, and $m = \binom{k}{2}$, $k \geqslant 7$, for arbitrary integers $t \geqslant 1$, we obtain

$$f\left(\binom{k}{2} t, k, \Sigma \right) \geqslant |\mathscr{R} \otimes (t, \ldots, t)| > \left(\binom{k}{2} t \right)^k \Big/ k! \binom{k}{2} e^{1 + 1/(k-1)},$$

proving the lower bound part of Theorem 1.3.

On the other hand, examining inequality (2.5) we find

$$(m-1) \cdot \cdots \cdot (m-k+2)/m^{k-1} < m^{-1} \prod_{i=1}^{k-2} e^{-i/m} = e^{-\binom{k-1}{2}/m}/m.$$

Differentiating we find that the maximum of the right-hand side is $1/e\binom{k-1}{2}$, attained for $m = \binom{k-1}{2}$, which yields the upper bound:

$$\lambda(\mathscr{G}) < 1/e \binom{k-1}{2} k!.$$

This together with (2.1) gives the upper bound in Theorem 1.3.

4. SOME BOUNDS ON THE LAGRANGE FUNCTION

In this section we address the following problem. Given n, k and m, $1 \leqslant m \leqslant \binom{n}{k}$, determine or estimate $\lambda(n, k, m) = \max\{\lambda(\mathscr{F}): \mathscr{F} \subset \binom{X}{k}, |\mathscr{F}| = m\}$.

Recall the definition of the *reverse lexicographic* order on $\binom{x}{k}$:

$$F < F' \qquad \text{iff} \max F - F' < \max F' - F.$$

CONJECTURE 4.1. $\lambda(n, k, m)$ *is attained for the family consisting of the smallest m sets in the reverse lexicographic order.*

If this conjecture was proved, then most of our later computations would become superfluous and the proofs of Theorems 1.1 and 1.2 much simpler. Since it is still a conjecture, we shall use the following, much weaker result.

LEMMA 4.2. *Suppose that* $0 \leqslant s \leqslant \binom{n-2}{k-2}$ *and* $m = \binom{n}{k} - s$. *Then*

$$\lambda(n, k, m) \leqslant \max \left\{ \sum_{0 \leqslant i \leqslant k} \binom{k}{i}\binom{n-k}{k-i} x^i y^{k-i} \right.$$

$$\left. - sx^k \colon x, y \geqslant 0, \, kx + (n-k)\, y = 1 \right\}. \tag{4.1}$$

Proof. Suppose that $\lambda(n, k, m) = \lambda(\mathcal{F}, (x_1, ..., x_n))$. We may assume by symmetry that $x_1 \leqslant \cdots \leqslant x_n$ holds.

Then $x_1 x_2 \cdot \cdots \cdot x_k$ is smallest among all $\binom{n}{k}$ products of k terms which implies

$$\lambda(\mathcal{F}) \leqslant \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} x_{i_1} \cdot \cdots \cdot x_{i_k} - sx_1 \cdot \cdots \cdot x_k. \tag{4.2}$$

Thus it will be sufficient to show that the right-hand side of (4.2) is not greater than that of (4.1).

To this effect let us maximize the right-hand side of (4.2). We claim that then necessarily $x_1 = \cdots = x_k$ and $x_{k+1} = \cdots = x_n$ hold—which will prove (4.1).

Indeed, if for some $k < i < j < n$, $x_i < x_j$ holds then replacing both x_i and x_j by $(x_i + x_j)/2$, the right-hand side of (4.2) increases by

$$\left(\frac{x_i - x_j}{2}\right)^2 \sum_{G \in \binom{X - \{i, j\}}{k-2}} \prod_{v \in G} x_v, \qquad \text{a contradiction.}$$

Similarly, if $x_i < x_j$ holds for some $1 \leqslant i < j \leqslant k$ and we replace both x_i and x_j by $(x_i + x_j)/2$ then the RHS of (4.2) increases by

$$\left(\frac{x_i - x_j}{2}\right)^2 \left(\sum_{G \in \binom{X - \{i, j\}}{k-2}} \prod_{v \in G} x_v - s \prod_{\substack{1 \leqslant \mu \leqslant k \\ \mu \neq i, j}} x_\mu \right).$$

However, by monotonicity of the x_ν, each product in the sum is at least as large as the other product on the extreme right, the number of terms in the sum is $\binom{n-2}{k-2} \geqslant s$, thus the change is non-negative. More exactly, it is positive unless $s = \binom{n-2}{k-2}$ and all x_ν with the possible exception of x_i and x_j are equal.

That is, either we get a contradiction straight away or we get one by considering say x_i and x_j, for $j' \neq j, i$. The proof of (4.1) is complete. ∎

Now Lemma 4.2 can be used to derive upper bounds for $\lambda(n, k, m)$ for fixed values of $n, k,$ and m.

PROPOSITION 4.3. *The following three inequalities hold:*

$$\lambda(8, 4, \binom{8}{4} - \binom{6}{2})) < 0.0151 < 8 \cdot 5 \cdot \frac{6}{11^4}, \tag{4.3}$$

$$\lambda(9, 4, \binom{9}{4} - \binom{7}{2})) < 0.0175 < 9 \cdot 5 \cdot \frac{6}{11^4}, \tag{4.4}$$

$$\lambda(10, 4, \binom{10}{4} - \binom{8}{2})) < 0.0196 < 10 \cdot 5 \cdot \frac{6}{11^4}. \tag{4.5}$$

Proof. Since all these inequalities come by straightforward application of (4.1), we shall only prove (4.5). All the calculations summarized in Table I. In view of (4.1) it is sufficient to prove that the maximum of the RHS of (4.1)—call it $p(x, y)$—is less than 0.0195... for $4x + 6y = 1$, $x, y > 0$.

If the maximum is attained for $x = x_0$, $y = y_0$ then necessarily $6(\partial/\partial x) p(x_0, y_0) = 4(\partial/\partial y) p(x_0, y_0)$ holds.

That is,

$$240y_0^3 + 120y_0^2 x_0 - 288y_0 x_0^2 - 744x_0^3 = 0$$

Dividing by $24y_0^3$ and setting $t = x_0/y_0$ we obtain

$$10 + 5t - 12t^2 - 31t^3 = 0$$

This equation has only one root for $t > 0$, namely $t = 0.64325221...$.

The corresponding values of x and y are:

$$y = 1/(6 + 4t) = 0.1166451...,$$

$$x = ty = 0.0750322... .$$

Substituting into $p(x, y)$ we find that its value is $0.0195... < 0.0196$, as desired. ∎

TABLE I

Inequality (4.j)	(4.3)	(4.4)	(4.5)	(4.6)	(4.7)
n	8	9	10	13	14
k	4	4	4	5	5
$s(\leqslant \binom{n-2}{k-2})$	15	21	28	153	28

$$p(x, y) = -sx^k + \sum_{0 \leqslant i \leqslant k} \binom{k}{i}\binom{n-k}{k-i} x^i y^{k-i}$$

$$p_3(x, y) = -14x^4 + 16x^3y + 36x^2y^2 + 16xy^3 + y^4$$

$$p_4(x, y) = -20x^4 + 20x^3y + 60x^2y^2 + 40xy^3 + 5y^4$$

$$p_5(x, y) = -27x^4 + 24x^3y + 90x^2y^2 + 80xy^3 + 15y^4$$

$$p_6(x, y) = -152x^5 + 40x^4y + 280x^3y^2 + 560x^2y^3 + 350xy^4 + 56y^5$$

$$p_7(x, y) = -27x^5 + 45x^4y + 360x^3y^2 + 840x^2y^3 + 630xy^4 + 126y^5$$

$$0 = (n-k)\frac{\partial p(x, y)}{\partial x} - k\frac{\partial p(x, y)}{\partial y} = q_j(x, y)$$

$$q_3(x, y) = -72x^3 - 24x^2y + 24xy^2 + 12y^3$$

$$q_4(x, y) = -480x^3 - 180x^2y + 120xy^2 + 120y^3$$

$$q_5(x, y) = -744x^3 - 288x^2y + 120xy^2 + 240y^3$$

$$q_6(x, y) = -6280x^4 - 1520x^3y - 1680x^2y^2 + 1960xy^3 + 1400y^4$$

$$q_7(x, y) = -1440x^4 - 1980x^3y - 2880x^2y^2 + 2520xy^3 + 2520y^4$$

$t = x/y$. The following equations obtained from $q_j(x, y)$

$$0 = -3t^3 - t^2 + t + 0.5$$

$$0 = -8t^3 - 3t^2 + 2t + 2$$

$$0 = -31t^3 - 12t^2 + 5t + 10$$

$$0 = -157t^4 - 38t^3 - 42t^2 + 49t + 35$$

$$0 = -8t^4 - 11t^3 - 16t^2 + 14t + 14$$

The only root

	(4.3)	(4.4)	(4.5)	(4.6)	(4.7)
$0 < t < 1$	0.625518...	0.636037...	0.643252...	0.692740...	0.905696...
$y = \dfrac{1}{n-k+kt}$	0.15379...	0.13255...	0.116645...	0.08723...	0.07391...
x	0.09619...	0.08430...	0.075032...	0.06042...	0.06694...
max $p(x, y)$	0.01503...	0.01746...	0.019524...	0.003259...	0.003678...

By exactly the same kind of computations we obtain

PROPOSITION 4.4.

$$\lambda(13, 5, (\tbinom{13}{5}) - 153) < 0.00326 < 13 \cdot 6 \cdot \frac{11}{12^5},\qquad(4.6)$$

$$\lambda(14, 5, (\tbinom{14}{5}) - 28) < 0.00368 < 14 \cdot 6 \cdot \frac{11}{12^5}.\qquad(4.7)$$

The next lemma relates these bounds to the originial problem.

LEMMA 4.5. *Suppose that $\mathscr{G}$ is as in Lemma 2.1 and it satisfies (2.3). Then*

$$m \cdot k \cdot \lambda(\mathscr{G}) \leqslant \lambda(m, k-1, k|\mathscr{G}|) \text{ holds.}\qquad(4.8)$$

Proof. Consider the sum over i of the equation (v) from Lemma 2.1; this yields

$$\sum_{H \in \partial \mathscr{G}} \prod_{i \in mH} y_i = m \cdot k \cdot \lambda(\mathscr{G}).$$

Since $y_i \geqslant 0$, $\sum y_i = 1$, the left-hand side is by definition at most $\lambda(m, k-1, k|\mathscr{G}|)$, proving (4.8). ∎

5. The Proof of Theorem 1.1

In view of (2.1), Lemma 2.1, and Fact 2.2 it is sufficient to show that $\lambda(\mathscr{G}) \leqslant 6 \cdot 11^{-4}$ holds for every 5-graph $\mathscr{G}$ satisfying (2.3).

From (2.5) this inequality follows unless $|\bigcup \mathscr{G}| = m = 8, 9$, or 10.

If $|Y| = 8$, then $\mathscr{H} = \{Y - G : G \in \mathscr{G}\}$ is a $(8, 3, 2)$-packing; that is, $|H \cap H'| \leqslant 1$ for all distinct $H, H' \in \mathscr{H}$. This immediately implies that every element of Y is contained in at most three sets in $\mathscr{H}$, yielding $|\mathscr{G}| = |\mathscr{H}| \leqslant 8 \cdot 3/3 = 8$.

This implies for $m = 9$: $|\mathscr{G}| \leqslant 9 \cdot 8/(9 - 5) = 18$, and thus for $m = 10$: $|\mathscr{G}| \leqslant 10 \cdot 18/(10 - 5) = 36$.

Using (4.8) and the above upper bounds for $|\mathscr{G}|$ and $|\partial \mathscr{G}| = 5|\mathscr{G}|$, we obtain for $m = 8, 9, 10$ the following inequalities:

$$8 \cdot 5 \cdot \lambda(\mathscr{G}) \leqslant \lambda(8, 4, 40) \leqslant \lambda(8, 4, (\tbinom{8}{4}) - (\tbinom{6}{2})),$$

$$9 \cdot 5 \cdot \lambda(\mathscr{G}) \leqslant \lambda(9, 4, 90) \leqslant \lambda(9, 4, (\tbinom{9}{4}) - (\tbinom{7}{2})),$$

$$10 \cdot 5 \cdot \lambda(\mathscr{G}) \leqslant \lambda(10, 4, 180) \leqslant \lambda(10, 4, (\tbinom{10}{4}) - (\tbinom{8}{2})).$$

Now $\lambda(\mathscr{G}) < 6 \cdot 11^{-4}$ follows from inequalities (4.3), (4.4), and (4.5), respectively, proving (1.6).

In case of equality, $6 \cdot n^5/11^4$ is an integer, proving $11 \mid n$. ∎

Remark 5.1. From the above proof and Remark 2.3 it is clear that $\lambda(\mathscr{G}) = 6 \cdot 11^{-4}$ holds only if $\mathscr{G}$ is a perfect $(11, 5, 4)$-packing. Since there is only one such packing, then $\mathscr{G}$ is necessarily the unique $(11, 5, 4)$-Steiner system $\mathscr{R}_{11}$.

6. THE PROOF OF THEOREM 1.2

Again, as in the proof of Theorem 1.1, it is sufficient to show that $\lambda(\mathscr{G}) \leqslant 11 \cdot 12^{-5}$ holds for every 6-graph $\mathscr{G}$, satisfying (2.3).

This inequality follows from (2.5) unless $|\bigcup \mathscr{G}| = m = 13$ or 14. Moreover, the inequality is strict, unless $m = 12$ and $\mathscr{G}$ is a perfect $(12, 6, 5)$-packing, that is, $\mathscr{G} = \mathscr{R}_{12}$ the unique $(12, 6, 5)$ Steiner system.

Now we bound $|\mathscr{G}|$ separately in the cases $m = 13, 14$:

(a) $|Y| = |\bigcup \mathscr{G}| = 13$. For $A \in \binom{Y}{4}$ the family $\mathscr{G}(A) = \{G - A : A \subset G \in \mathscr{G}\}$ consists of pairwise disjoint 2-element subsets of the 9-element set $Y - A$. Thus $|\bigcup \mathscr{G}(A)| \leqslant 8$; that is, there exists B, $A \subset B \subset Y$, $|B| = 5$ such that B is not contained in any $G \in \mathscr{G}$.

In other words, considering the family $\mathscr{B} = \binom{Y}{5} - \partial \mathscr{G}$, every $A \in \binom{Y}{4}$ is contained in at least one of its members.

Consequently, for all $D \in \binom{Y}{2}$ one has $|\mathscr{B}(D)| \geqslant \lceil \binom{11}{2}/\binom{3}{2} \rceil = 19$. This, in turn, implies

$$|\mathscr{B}| = \sum_{D \in \binom{Y}{2}} |\mathscr{B}(D)|/\binom{5}{2} \geqslant 19 \cdot \binom{13}{2}/\binom{5}{2} = 148.2.$$

Thus $|\mathscr{B}| \geqslant 149$ holds. From this one derives $6|\mathscr{G}| = |\partial \mathscr{G}| \leqslant \binom{13}{5} - 149$, i.e., $|\mathscr{G}| \leqslant \lfloor (\binom{13}{5} - 149)/6 \rfloor = 189$.

Finally, this implies

$$6|\mathscr{G}| \leqslant 6 \cdot 189 = \binom{13}{5} - 153. \tag{6.1}$$

(b) $|Y| = |\bigcup \mathscr{G}| = 14$. For $D \in \binom{Y}{3}$ the family $\mathscr{G}(D)$ is a $(11, 3, 2)$-packing. Thus $|\mathscr{G}(D)| \leqslant \lfloor \binom{11}{2}/\binom{3}{2} \rfloor = 18$. Consequently, $|\mathscr{G}| \leqslant \lfloor \binom{14}{3} \cdot 18/\binom{6}{3} \rfloor = 327$. We shall only use the weaker inequality

$$6|\mathscr{G}| \leqslant 1974 = \binom{14}{5} - 28. \tag{6.2}$$

Now, as in the case of Theorem 1.1, the inequalities (6.1), (6.2), together with (4.8), imply

$$13 \cdot 6 \cdot \lambda(\mathcal{G}) \leqslant \lambda(13, 5, \binom{13}{5} - 153), \tag{6.3}$$

$$14 \cdot 6 \cdot \lambda(\mathcal{G}) \leqslant \lambda(14, 5, \binom{14}{5} - 28). \tag{6.4}$$

Thus $\lambda(\mathcal{G}) < 11 \cdot 12^{-5}$ follows from the inequalities (4.6) and (4.7), respectively. ∎

7. Exact Bounds for Large Values of n

THEOREM 7.1.   *Suppose that $k = 5$ or 6 and $\mathscr{S}_k$ is the unique $S(k + 6, k, k - 1)$. Let $\mathscr{F} \subset \binom{X}{k}$ have property (Σ); then for appropriate values of $n_1, ..., n_{k+6}$, $n_1 + \cdots + n_{k+6} = n$, $n > n_0(k)$, $|\mathscr{F}| \leqslant |\mathscr{S}_k \otimes (n_1, ..., n_{k+6})|$ holds.*

Moreover, in case of equality,

$$\mathscr{F} = \mathscr{S}_k \otimes (n_1, ..., n_{k+6}).$$

Proof.   Let $\mathscr{F} \subset \binom{X}{k}$ be a k-graph with property (Σ), $x, y \in X$ and suppose that $\{x, y\} \not\subseteq F$ for all $F \in \mathscr{F}$. Define $\mathscr{F}(x) = \{F - \{x\} : x \in F \in \mathscr{F}\}$, the *link* of x.
Sidorenko [S] defines the operator $T_{x, y}$ by

$$T_{x, y}(\mathscr{F}) = \{F \in \mathscr{F} : y \notin F\} \cup \{G \cup \{y\} : G \in \mathscr{F}(x)\},$$

and notes that if $\mathscr{F}$ has property (Σ) then so does $T_{x, y}(\mathscr{F})$ as well. It may be worthwhile to mention that Sidorenko's operator is a direct generalization of Zykov's symmetrization operator; see [Si].
Note that in $T_{x, y}(\mathscr{F})$ the link of x and y is the same, namely, $\mathscr{F}(x)$.
Continuing to apply the operator $T_{x, y}$ for all uncovered pairs x, y with $|\mathscr{F}(x)| \geqslant |\mathscr{F}(y)|$ leads to a new family $\mathscr{H} \subset \binom{X}{k}$ such that whenever $\mathscr{H}(x) \neq \mathscr{H}(y)$ there exists some $H \in \mathscr{H}$ with $\{x, y\} \subset H$, moreover, $|\mathscr{H}| \geqslant |\mathscr{F}|$.
Define an equivalence relation $x \sim y$ on X by $x \sim y$ if $\mathscr{H}(x) = \mathscr{H}(y)$.
Let $X_1, ..., X_m$ be the equivalence classes with $|X_i| = n_i$. Define the k-graph $\mathscr{G} \subset \binom{\{1, ..., m\}}{k}$ by $G \in \mathscr{G}$ if and only if

$$G = \{i : H \cap X_i \neq \varnothing\} \quad \text{for some} \quad H \in \mathscr{H}.$$

It follows that $\mathscr{H} = \mathscr{G} \otimes (n_1, ..., n_m)$. Define now $y_i = n_i/n$. Then $y > 0$, $\sum_{i=1}^{m} y_i = 1$. Set $\mathbf{y} = (y_1, ..., y_m)$. Note that for $x \in X_i$ one has

$$|\mathscr{H}(x)| = \frac{\partial}{\partial y_i} \lambda(\mathscr{G}, \mathbf{y}) \, n^{k-1}. \tag{7.1}$$

Suppose now that $|\mathscr{H}|$ is maximal.

Since $T_{x,y}(\mathscr{H} - \{H \in \mathscr{H}: \{x, y\} \subset H\})$ has property (Σ) and its size is $|\mathscr{H}| + |\mathscr{F}(x)| - |\mathscr{F}(y)| - |\mathscr{F}(x, y)|$, we infer

$$|\mathscr{F}(x)| - |\mathscr{F}(y)| \leqslant |\mathscr{F}(x, y)| \leqslant \binom{n-2}{k-2}. \tag{7.2}$$

Combining (7.1) and (7.2) gives for every $1 \leqslant i \leqslant m$

$$\frac{\partial}{\partial y_i} \lambda(\mathscr{G}, \mathbf{y}) < \binom{n-2}{k-2} + \frac{1}{m} \sum_{i=1}^{m} \frac{\partial}{\partial y_i} \lambda(\mathscr{G}, \mathbf{y}). \tag{7.3}$$

Since $\mathscr{H}$ has property (Σ), $\mathscr{G}$ is a packing, implying

$$\sum_{i=1}^{m} \frac{\partial}{\partial y_i} \lambda(\mathscr{G}, \mathbf{y}) = \lambda(\partial \mathscr{G}, \mathbf{y}) \leqslant \lambda(m, k-1, k|\mathscr{G}|). \tag{7.4}$$

Suppose now that $k = 5$ or 6 and $\mathscr{G}$ is not isomorphic to $\mathscr{S}_k$. Then Propositions 4.3 and 4.4 and the inequality $\lambda(m, k-1, \binom{m}{k-1})) \leqslant \binom{m}{k-1} m^{1-k}$, together with Remark 2.3, imply that for some $\varepsilon > 0$,

$$\lambda(m, k-1, k|\mathscr{G}|) \leqslant m \cdot k \cdot (\lambda(\mathscr{S}_k) - \varepsilon) \text{ holds.} \tag{7.5}$$

Combining inequalities (7.1), (7.3), and (7.5) gives

$$\begin{aligned}
k|\mathscr{H}| &= \sum_{i=1}^{m} n_i \frac{\partial}{\partial y_i} \lambda(\mathscr{G}, \mathbf{y}) \\
&< n \binom{n-2}{k-2} + \frac{n}{m} (\Sigma y_i) \sum_i \frac{\partial}{\partial y_i} \lambda(\mathscr{G}, \mathbf{y}) \, n^{k-1} \\
&\leqslant n \binom{n-2}{k-2} + k n^k (\lambda(\mathscr{S}_k) - \varepsilon),
\end{aligned}$$

i.e., $|\mathscr{H}| < n^k (\lambda(\mathscr{S}_k) - \varepsilon/2)$ holds for $n > n_0(k)$, contradicting the maximality of $|\mathscr{H}|$.

In the remaining case $\mathscr{G} = \mathscr{S}_k$; that is, $\mathscr{H}$ is a blowup of $\mathscr{S}_k$, as desired. To complete the proof of the uniqueness of the optimal families we must show that if $\mathscr{H} = \mathscr{S}_k \otimes (n_1, ..., n_m)$, $|\mathscr{F}| = |\mathscr{H}|$ and $\mathscr{H} = T_{x,y}(\mathscr{F})$ then

$\mathscr{F} = \mathscr{S}_k \otimes (n_1^1, ..., n_m^i)$. However, this is rather easy. Since $\mathscr{F}$ has property (Σ) one has $|A \cap X_i| \leqslant 1$ for all $A \in \mathscr{F}(y)$ and $1 \leqslant i \leqslant m$. Thus for each $A \in \mathscr{F}(y)$ there is an $i = i(A)$ with $A \in \mathscr{F}(z)$ for all $z \in X_i$. If $i = i(A)$ is the same for all $A \in \mathscr{F}(y)$ and $z \in X_i$ then $\mathscr{F}(y) \subset \mathscr{F}(z)$ follows, i.e., $\mathscr{F} \subset \mathscr{S}_k \otimes (n_1', ..., n_m')$.

If i and i' are distinct values attained by $i(A)$, then property (Σ) implies $A \cap (X_i \cup X_{i'}) = \varnothing$ for all $A \in \mathscr{F}(y)$, and $\mathscr{F}(y)$—which corresponds to the blowup of a $(m - 2, k - 1, k - 2)$-packing—turns out to be much smaller than $\mathscr{F}(x)$ and thus $|\mathscr{F}| < |\mathscr{H}|$, a contradiction. ∎

Remark 7.2. Clearly, the same proof works for $k = 4$ as well and gives that for $n > n_0$ every optimal 4-graph with property (Σ) is a complete equipartite 4-graph. However, to obtain the same result for all n requires a little more care. Also, the same proof would work for other values of k if we have the necessary bounds for the Lagrange function. In particular, it works, it for some k there exists a $S(m_0, k, k - 1)$, where m_0 is the *unique* integer for which the right-hand side of (2.5) is maximized. However, nothing is known about the existence of Steiner systems for $k - 1 = t \geqslant 6$.

8. UNFOLDABLE k-GRAPHS

For a k-graph $\mathscr{F}$ define its minimal $(k - 1)$-degree, $\delta_{k-1}(\mathscr{F})$ by $\delta_{k-1}(\mathscr{F}) = \min\{|\mathscr{F}(H)|: H \in \partial\mathscr{F}\}$. In words, $\delta_{k-1}(\mathscr{F}) \geqslant b$ if and only if every $(k - 1)$-set which is contained in some member of $\mathscr{F}$ is contained in at least b members of $\mathscr{F}$.

The following simple lemma is essential for the proof of Theorem 1.4.

LEMMA 8.1. *For every k-graph $\mathscr{F} \subset \binom{X}{k}$ and every positive integer b there exists $\mathscr{F}_0 \subset \mathscr{F}$ such that*

(i) $\delta_{k-1}(\mathscr{F}_0) \geqslant b$

(ii) $|\mathscr{F} - \mathscr{F}_0| \leqslant (b - 1)\binom{n}{k-1}$.

Proof. Define $\mathscr{F}^{(1)} = \mathscr{F}$. We define recursively the families $\mathscr{F}^{(i)}$, $i \geqslant 1$. Suppose that $\mathscr{F}^{(i)}$ was defined.

If $\delta_{k-1}(\mathscr{F}) \geqslant b$ then set $\mathscr{F}_0 = \mathscr{F}^{(i)}$ and stop. Otherwise take $A^{(i)} \in \partial\mathscr{F}^{(i)}$ with $|\mathscr{F}^{(i)}(A^{(i)})| \leqslant b - 1$ and define $\mathscr{F}^{(i+1)} = \{F \in \mathscr{F}^{(i)}: A^{(i)}F\}$.

Note that $|\mathscr{F}^{(i)} - \mathscr{F}^{(i+1)}| \leqslant b - 1$ and $\partial\mathscr{F}^{(i+1)} \subsetneqq \partial\mathscr{F}^{(i)}$. Thus this procedure will end in at most $\binom{n}{k-1}$ steps, proving the lemma. ∎

DEFINITION 8.2 Suppose that $\mathscr{H}$ is a k-graph $\bigcup \mathscr{H} = Y$ and $y \in Y$ is a vertex of degree one, i.e., $\mathscr{H}(\{y\}) = \{K\}$ for some $(k - 1)$-set K. Choose some $y' \in Y - K$. Then the hypergraph $\mathscr{H}' = (\mathscr{H} - \{K \cup \{y\}\}) \cup$

$\{K \cup \{y'\}\}$ is called an *elementary folded* copy of $\mathcal{H}$. The operation φ defined by $\varphi(\mathcal{H}) = \mathcal{H}'$ is called an *elementary folding.*

Now the k-graph $\mathcal{H}'$ is called a *folded* copy of $\mathcal{H}$, if it can be obtained from $\mathcal{H}$ by a series of elementary foldings.

LEMMA 8.3. *Let $\mathcal{H}$ be a k-graph. Suppose that the k-graph $\mathcal{F}$ satisfies $\delta_{k-1}(\mathcal{F}) > m - k$, where $m = |\bigcup \mathcal{H}|$. Moreover, $\mathcal{F}$ contains a folded copy of $\mathcal{H}$. Then $\mathcal{F}$ contains a copy of $\mathcal{H}$ as well.*

Proof. By the definitions it is sufficient to prove the statement if $\mathcal{F}$ contains an elementary folded copy $\mathcal{H}'$ of $\mathcal{H}$, i.e., $\mathcal{H}' \subset \mathcal{F}$. Since $|\bigcup \mathcal{H}'| = |\bigcup \mathcal{H}| - 1 = m - 1$ and $|\mathcal{F}(K)| > m - k = (m-1) - (k-1)$, there is some $F \in \mathcal{F}$ with $F \cap (\bigcup \mathcal{H}') = K$.

Now the k-graph $\mathcal{H}'' = (\mathcal{H}' - \{H \cup \{y'\}\}) \cup \{F\}$ is a copy of $\mathcal{H}$ in $\mathcal{F}$. ∎

The folding defines a partial ordering over k-graphs. More exactly, let $\mathcal{G} <_f \mathcal{H}$ if $\mathcal{H}$ is a folded copy of $\mathcal{G}$. If **G** is a family of k-graphs denote the subfamily of its minimal members with respect to the relation $<_f$ by $\mathbf{G}^0$. In other words, $\mathbf{G}^0$ is the set of the *unfolded* members of **G**.

THEOREM 8.4. *Let **G** be a family of k-graphs and let $m = \max\{|\bigcup \mathcal{G}| : \mathcal{G} \in \mathbf{G}\}$. Denote the subfamily of unfolded members of **G** by $\mathbf{G}^0$. Then*

$$\mathrm{ex}(n, \mathbf{G}) \leqslant \mathrm{ex}(n, \mathbf{G}^0) \leqslant \mathrm{ex}(n, \mathbf{G}) + (m-k)\binom{n}{k-1}.$$

Proof. Since $\mathbf{G}^0 \subseteq \mathbf{G}$, the lower bound is obvious. To prove the upper bound, consider an extremal k-graph $\mathcal{F} \subset \binom{X}{k}$ which contains no copy of a member of $\mathbf{G}^0$ and satisfies $\mathrm{ex}(n, \mathbf{G}^0) = |\mathcal{F}|$. Choose $\mathcal{F}_0 \subset \mathcal{F}$ by Lemma 8.1 applied with $b = m - k + 1$. Then Lemma 8.3 implies that $\mathcal{F}_0$ is **G**-free. Thus we obtain $\mathrm{ex}(n, \mathbf{G}^0) = |\mathcal{F}| \leqslant |\mathcal{F}_0| + (m-k)\binom{n}{k-1}$. ∎

Recall that $\pi(\mathbf{G})$ denotes $\lim_{n \to \infty} \mathrm{ex}(n, \mathbf{G})/\binom{n}{k}$. Let $\mathcal{G}$ be a k-graph and suppose that $x, y \in \bigcup \mathcal{G}$ but $\{x, y\}$ is not contained in any $G \in \mathcal{G}$. Then the family $\mathcal{G}' = \mathcal{G} - \{G : y \in G \in \mathcal{G}\} \cup \{G - \{y\} \cup \{x\} : y \in G \in \mathcal{G}\}$ is called an *elementary compressed* copy of $\mathcal{G}$. (Notice that $|\mathcal{G}'| \leqslant |\mathcal{G}|$.) We say $\mathcal{H}$ is a *compressed* copy of $\mathcal{G}$, if it can be obtained from $\mathcal{G}$ by a series of elementary compressions. Finally, the family of k-graphs **G** is called *closed under compression* if $\mathcal{G} \in \mathbf{G}$, and $\mathcal{H}$ is a compressed copy of $\mathcal{G}$ then $\mathcal{H} \in \mathbf{G}$.

PROPOSITION 8.5. *Let **G** be a family of k-graphs closed under compression. Then*

$$\pi(\mathbf{G})\binom{n}{k} \leqslant \mathrm{ex}(n, \mathbf{G}) \leqslant \pi(\mathbf{G})\frac{n^k}{k!}.$$

Proof. The lower bound is obvious (cf. [KNS]). Let $\mathcal{F}$ be a G-free family over n elements satisfying $|\mathcal{F}| = \text{ex}(n, \mathbf{G})$. Consider the blowup of $\mathcal{F}$, $\mathcal{F} \otimes (t, t, ..., t)$. We obtain another G-free family over nt elemens with cardinality $|\mathcal{F}| t^k$. Hence

$$\frac{\text{ex}(n, \mathbf{G}) t^k}{\binom{nt}{k}} \leqslant \frac{\text{ex}(nt, \mathbf{G})}{\binom{nt}{k}}. \tag{8.1}$$

If $t \to \infty$ we obtain the desired upper bound. ∎

Proof of Theorem 1.4. It follows from Theorem 8.4. We have to note only, that if $\mathcal{H} = \{H_1, H_2, H_3\}$ is a k-graph with $|H_1 \cap H_2| = k - 1$, $H_1 \Delta H_2 \subset H_3$ then necessarily $\mathcal{H}$ is a folded copy of $\mathcal{G}_k$.

Denote the class of k-hypergraphs $\{A, B, C\}$ for which $A \Delta B \subset C$ by Δ_k, and for which $A \Delta B \subset C$ and $|A \cap B| = k - 1$ by Σ_k. (Then $\Sigma^0 = \{\mathcal{G}_k\}$.) These clases are closed under compression, so we can apply to Σ_k and Δ_k both Theorem 8.4 and Proposition 8.5. ∎

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