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Note

On Induced Subgraphs of the Cube

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Consider the usual graph Q^n defined by the n-dimensional cube (having 2^n vertices and $n2^{n-1}$ edges). We prove that if G is an induced subgraph of Q^n with more than 2^{n-1} vertices then the maximum degree in G is at least $(\frac{1}{2}-o(1))\log n$. On the other hand, we construct an example which shows that this is not true for maximum degree larger than $\sqrt{n}+1$. © 1988 Academic Press, Inc.

1. PRELIMINARIES

Denote by Q^n the graph of the *n-dimensional cube*, i.e., the vertex set of Q^n consists of all the $(0, 1)$-vectors of length n, and two vectors $x, y \in \{0, 1\}^n$ are adjacent if they differ from each other in exactly one

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component. For a graph $G=(V, E)$ we denote the *maximum degree* by $\Delta(G)$, i.e.,

$$\Delta(G)=\max_{v \in V(G)} \deg_G(v).$$

The *average degree* $\bar{d}(G)$ is defined to be $\sum_{v \in V(G)} \deg_G(v)/|V(G)|$. We say $G \in Q^n(N)$ if G is an induced subgraph of Q^n with N vertices, i.e., $|V(G)|=N$, $V(G) \subseteq \{0, 1\}^n$, and $E(G)=E(Q^n) \cap (V(G) \times V(G))$.

Q^n is a bipartite graph, so we have a $G \in Q^n(2^{n-1})$ without any edge, namely, G^n_{odd} and G^n_{even}, where $V(G^n_{\text{odd}})=\{x \in \{0, 1\}^n: \sum_{i=1}^n x_i \equiv 1 \pmod 2\}$, $V(G^n_{\text{even}})=\{0, 1\}^n - V(G^n_{\text{odd}})$. Our main result shows that even though the average degree of a graph $G \in Q^n(2^{n-1}+1)$ can be very small (only $2n/(2^{n-1}+1)$), these graphs must have large degree.

THEOREM 1.1. *Let G be an induced subgraph of Q^n with at least $2^{n-1}+1$ vertices. Then for some vertex v of G we have*

$$\deg_G(v) > \tfrac{1}{2} \log n - \tfrac{1}{2} \log \log n + \tfrac{1}{2}. \tag{1.1}$$

On the other hand, there exists a $G \in Q^n(2^{n-1}+1)$ with

$$\Delta(G) < \sqrt{n}+1. \tag{1.2}$$

2. RELATED RESULTS AND PROBLEMS FROM COMPUTER SCIENCE

A (Boolean) function f: $\{0, 1\}^n \to \{0, 1\}$ is said to *depend on coordinate i* if there exists an input vector x such that $f(x)$ differs from $f(x^{(i)})$, where $x^{(i)}$ agrees with x in every coordinate except the ith. In this case x is said to be *critical* for f with respect to i. The function f is called *nondegenerate* if it depends on all n coordinates. For an input vector x, let $c(f, x)$ denote the number of coordinates i such that x is critical for f with respect to i, and let $c(f) := \max\{c(f, x): x \in \{0, 1\}^n\}$. $c(f)$ is called the *critical complexity* of f. This notion is due to Cook and Dwork [3] and Reischuk [5], who showed that $\log_A c(f)$ is a lower bound to the time needed by a parallel RAM to compute the function f (where $A=\frac{1}{2}(5+\sqrt{21})=4.7 \cdots$). (A parallel RAM is a collection of synchronous parallel processors sharing a global memory with no write-conflicts allowed. For precise delfinitions, see [1].) Simon [6] showed that the critical complexity of any nondegenerate Boolean function is at least

$$\alpha(n) := \tfrac{1}{2} \log n - \tfrac{1}{2} \log \log n + \tfrac{1}{2}, \tag{2.1}$$

which implies a $O(\log\log n)$ lower bound for parallel complexity. More results on this topic can be found in [7].

Call a subgraph G of Q^n *nondegenerate* if $E(G)$ contains edges from each of the n directions. Thus, the crucial point of the above problem can be reformulated as follows:

> Let U, V be a partition of $\{0, 1\}^n$ and consider the induced bipartite graph $G(U, V)$. If $G(U, V)$ is nondegenerate then $\Delta(G) > \alpha(n)$. (2.2)

This is completely analogous to our theorem (even the proof is similar). However, we need a slightly more powerful lemma (see Lemma 4.1). Reischuk (see [5]) has a simple example proving that in (2.2), $\Delta(G) = \lfloor \log n \rfloor + 2$ is possible, and it is very likely that this is the right value of

$$b(n) = \min\{\Delta(G): G \text{ as is in } (2.2)\}.$$

Another interesting property of the induced bipartite graphs is proved by Ben-Or and Linial [2] (also dealing with a problem arising in theoretical computer science):

> If U, V is a partition of $\{0, 1\}^n$ then there exists a direction i such that at least $\min\{|U|, |V|\}/n$ edges go from U to V parallel to i. (2.3)

They have an upper bound of $\log n \min\{|U|, |V|\}/n$ and also this seems to be the right order of magnitude.

3. Proof of the Upper Bound

Denote the set of integers $\{1, 2, ..., n\}$ by $[n]$. Since there is a natural bijection between $\{0, 1\}^n$ and $2^{[n]}$, so we will speak about families of finite sets with the underlying set $[n]$. There exists a partition of $[n] = F_1 \cup \cdots \cup F_k$ such that $|k - \sqrt{n}| < 1$ and $|\,|F_i| - \sqrt{n}| < 1$, $1 \leqslant i \leqslant k$. Define the family $\mathbf{X}$ as follows: consider all the even sets (i.e., subsets of $[n]$ with cardinality an even number) which contain some F_i, $1 \leqslant i \leqslant k$, and all the odd sets which do not contain any F_i.

Claim 3.1. $|\mathbf{X}| = 2^{n-1} \pm 1$ according to whether $n + k$ is odd or even.

Claim 3.2. For the subgraphs induced by $\mathbf{X}$ and $2^{[n]} - \mathbf{X}$ we have $\Delta \leqslant k$.

Remark. We can generalize the above construction in the following way. Let $\mathbf{F} \subset 2^{[n]}$ be a collection of finite sets. (Later we will see that it is enough to consider Sperner families, with $\bigcup \mathbf{F} = [n]$.) Define

$$\mathbf{X}(\mathbf{F}) = \{S \subset [n]: |S| = \textit{even}, \text{ and there exists } F \in \mathbf{F} \text{ with } F \subset S\}$$
$$\cup \{S \subset [n]: |S| = \textit{odd}, F \setminus S \neq \emptyset \text{ for all } F \in \mathbf{F}\}.$$

Let $G(\mathbf{F})$ be the induced subgraph of Q^n with vertex set $\mathbf{X}(\mathbf{F})$, and $G'(\mathbf{F})$ the induced subgraph on $2^{[n]} - \mathbf{F}$. The *rank* of $\mathbf{F}$ is the largest size of its edges, i.e., $r(\mathbf{F}) = \max\{|F|: F \in \mathbf{F}\}$. Denote by $t(\mathbf{F})$ the maximum value of t such that one can find $F_1, F_2, \ldots, F_t \in \mathbf{F}$ and $x_i \in F_i$, $1 \leqslant i \leqslant t$, so that for $i \neq j$ we have $x_i \notin F_j$. In other words, $t(\mathbf{F})$ is the largest size of the disjointly representable subsystems of $\mathbf{F}$.

PROPOSITION 3.3. *$\Delta(G(\mathbf{F})) \leqslant \max\{r(\mathbf{F}), t(\mathbf{F})\}$, and the same holds for $\Delta(G'(\mathbf{F}))$.*

Proof. If (S, S') is an edge of Q^n, $S, S' \in \mathbf{X}(\mathbf{F})$, and S is even then $S' \subsetneq S$. Moreover if $F \subset S$, $F \in \mathbf{F}$, then $(S \setminus S') \in F$, so we have

$$\deg(S) \leqslant \left| \bigcap \{F: F \in \mathbf{F}, F \subset S\} \right| \leqslant r(\mathbf{F}). \tag{3.1}$$

On the other hand, if S is odd then $S \subset S'$ so there exists an $F \in \mathbf{F}$, $F \subset S'$, $F \not\subseteq S$. Hence if $S \subset S'_1, S'_2, \ldots, S'_a$ then $F_1, \ldots, F_a$ (where $F_i \subset S'_i$) are disjointly representable, so $a \leqslant t(\mathbf{F})$. The statement $\Delta(G'(\mathbf{F})) \leqslant \max\{r(\mathbf{F}), t(\mathbf{F})\}$ can be proved in the same way. ∎

Now use the sieve method to determine the cardinality of $\mathbf{X}(\mathbf{F})$. Let $F \subset [n]$, and $n \equiv \varepsilon \pmod 2$ ($\varepsilon = 0$ or 1). Then

$$\text{the number of even sets containing } F = \begin{cases} 2^{n-|F|-1} & \text{if } |F| < n, \\ 0 & \text{if } |F| = n, \text{ and } n \text{ is odd}, \\ 1 & \text{if } |F| = n, \text{ and } n \text{ is even}. \end{cases}$$

Similarly,

$$|\{S: F \subset S \subset [n], |S| \text{ odd}\}| = \begin{cases} 2^{n-|F|-1} & \text{if } |F| < n, \\ \varepsilon & \text{if } |F| = n. \end{cases}$$

Let $\mathbf{F} = \{F_1, \ldots, F_N\}$. The cardinality of the first part of $\mathbf{X}(\mathbf{F})$ is

$$\sum_{i \in [N]} (2^{n-|F_i|-1})^* - \sum\sum_{\{i,j\} \subset [N]} (2^{n-|F_i \cup F_j|-1})^* + - \cdots, \tag{3.2}$$

where $(2^A)^*$ means 2^A for $A \geqslant 0$ and $1-\varepsilon$ for $A=-1$. The cardinality of the second part of $\mathbf{X}(\mathbf{F})$ is

$$2^{n-1} - \sum_{i \in [N]} (2^{n-|F_0|-1})^{**} + \sum\sum_{\{i,j\} \subset [N]} (2^{n-|F_i \cup F_j|-1})^{**} - + \cdots, \quad (3.3)$$

where $(2^A)^{**}$ means 2^A for $A \geqslant 0$ and ε for $A=-1$. We have $(2^A)^* - (2^A)^{**} = 0$ or $1-2\varepsilon$ according to whether $A \geqslant 0$ or $A=-1$. So summing up (3.2) and (3.3) we have

$$|\mathbf{X}(\mathbf{F})| = 2^{n-1} + (1-2\varepsilon)\left[\sum_{\substack{F_i \in \mathbf{F} \\ |F_i|=n}} 1 - \sum\sum_{\substack{F_i, F_j \in \mathbf{F} \\ |F_i \cup F_j|=n}} 1 + \sum\sum\sum_{\substack{F_i, F_j, F_l \in \mathbf{F} \\ |F_i \cup F_j \cup F_l|=n}} 1 - + \cdots\right]. \quad (3.4)$$

Denote by $f(\mathbf{F})$ the bracketed expression on the right-hand side of (3.4). It is clear that if $\mathbf{F}$ is a k-partition of $[n]$ (into nonempty parts) then $f(\mathbf{F}) = (-1)^{k+1}$, which implies

$$|\mathbf{X}(\mathbf{F})| = 2^{n-1} + (-1)^{n+k+1},$$

proving Claim 3.2.

In general we are not able to calculate $f(\mathbf{F})$ explicitly since it tends to get complicated. Some properties of f are:

(i) If $F_0 = [n] \in \mathbf{F}$ then $f(\mathbf{F}) = f(\mathbf{F} - \{F_0\})$;

(ii) If $F_0 = \varnothing \in \mathbf{F}$ then $f(\mathbf{F}) = 0$;

(iii) If $\mathbf{F} = \{F_0, F_1, ..., F_N\}$, $\varnothing \neq F_0 \neq [n]$ then

$$f(\{F_0, ..., F_N\} \mid [n]) = f(\{F_1, ..., F_N\} \mid [n]) - f(\{F_1 - F_0, ..., F_n - F_0\} \mid [n] - F_0);$$

(iv) If $F_0 \neq \varnothing$ and for some $F_i \supset F_0$ then $f(\mathbf{F}) = f(\mathbf{F} - \{F_0\})$.

PROPOSITION 3.4. *Suppose* $f(\mathbf{F}) \neq 0$. *Then* $\max\{r(\mathbf{F}), t(\mathbf{F})\} \geqslant \sqrt{n}$.

Proof. Suppose that $|F| < \sqrt{n}$ holds for all $F \in \mathbf{F}$. $f(\mathbf{F}) \neq 0$ implies that $|\bigcup \mathbf{F}| = n$. Let $\{F_1, ..., F_s\}$ be a minimal subfamily of $\mathbf{F}$ with $\bigcup \mathbf{F} = [n]$. Then $\{F_1, ..., F_s\}$ is disjointly representable and $s \geqslant \sqrt{n}$. ∎

However, it may be possible that using a more complicated $\mathbf{F}$ with large $f(\mathbf{F})$ and deleting some members of $\mathbf{X}(\mathbf{F})$ (but fewer than $f(\mathbf{F})$) one can obtain a $G \in \mathbf{G}^n(2^{n-1}+1)$ with $\Delta(G) \ll \sqrt{n}$.

4. Proof of the Lower Bound

We begin with a lemma.

LEMMA 4.1. *Let G be a subgraph of the cube with average degree $\bar{d}$. Then* $|V(G)| \geqslant 2^{\bar{d}}$.

A similar lemma was used in [6], where $|V(G)| \geqslant 2^{\min \deg(v)}$ was proved. We point out that a related result of Kleitman et al. [4] immediately implies Lemma 4.1 in the case that $\bar{d}$ is an integer.

Proof. We use induction on $|V(G)|$. Split Q^n into two $(n-1)$-dimensional subcubes Q_1 and Q_2 such that $V_1 = Q_1 \cap V(G) \neq \emptyset$ and $V_2 = Q_2 \cap V(G) \neq \emptyset$. Suppose that $|V_2| \geqslant |V_1|$ and there are s edges between V_1 and V_2 in G (so that $|V_1| \geqslant s$). The restriction of G to V_i, $i = 1, 2$, is denoted by G_i. The induction hypothesis gives

$$|V_i| \log |V_i| \geqslant \sum \deg_{G_i}(v) = \sum_{v \in V_i} \deg_G(v) - s,$$

so that

$$|V_1| \log |V_1| + |V_2| \log |V_2| + 2s \geqslant \sum_{v \in V(G)} \deg_G(v). \tag{4.1}$$

However,

$$(|V_1| + |V_2|) \log(|V_1| + |V_2|)$$
$$\geqslant |V_1| \log |V_1| + |V_2| \log |V_2| + 2\, |V_1|$$

if $|V_2| \geqslant |V_1|$. (Here we used the fact that the base of the logarithm is 2.) ∎

Of course, Q^n is decomposable into two $(n-1)$-dimensional subcubes Q_1^i, Q_2^i, $1 \leqslant i \leqslant n$, in natural ways according to the n directions. We prove slightly more than (1.1).

LEMMA 4.2. *Suppose* $G \in Q^n(2^{n-1})$ and G contains edges from all the n directions. Then $\Delta(G) > \alpha(n)$.

This immediately implies (1.1). Indeed, let $G \in Q^n(2^{n-1} + b)$ with $\Delta(G) < (n-1)/2$. Delete b vertices from G arbitrarily. In the resulting graph G_0 every direction must occur, since otherwise $\Delta(G) \geqslant (n-1)/2$ would be forced.

Proof of Lemma 4.2. Let $X_i = \{x \in V(G) : x^{(i)} \in V(G)\}$, i.e., the set of endpoints of the edges of G in direction i. Define $Y_i = \{y \notin V(G): y^{(i)} \notin V(G)\}$, $A_i = V(Q^n) - X_i - Y_i$. Then

$$|X_i| = |Y_i| > 0.$$

Let $\Delta = \Delta(G)$ and consider a pair $x, x^{(i)} \in X_i$.

Claim 4.3. x has at most $(2\Delta - 2)$ neighbours in A_i.

Proof. Let us denote the neighbours of x in A_1 by $x^{(j_1)}, ..., x^{(j_s)}$. Then $x^{(j_1)(i)}, ..., x^{(j_s)(i)}$ are neighbours of $x^{(i)}$ in A_i and either $x^{(j_u)}$ or $x^{(j_u)(i)}$ belong to $V(G)$. Thus, $s \leqslant 2(\Delta - 1)$. ▮

Claim 4.3 implies that every $x \in X_i$ has at least $(n - 2\Delta + 1)$ neighbours in Y_i. Hence

$$|E(G(X_i \cup Y_i))| \geqslant \tfrac{1}{2}|X_i| + \tfrac{1}{2}|Y_i| + (n - 2\Delta + 1)|X_i|,$$

implying

$$\bar{d}(G(X_i \cup Y_i)) \geqslant n - 2\Delta + 2.$$

Lemma 4.1 gives

$$|X_i| \geqslant 2^{n - 2\Delta + 1}. \tag{4.2}$$

Counting the degrees in $V(G)$ we have

$$\Delta \cdot 2^{n-1} \geqslant \sum_{v \in V(G)} \deg_G(v) = \sum_{i=1}^{n} |X_i| \geqslant n 2^{n - 2\Delta + 1}.$$

An easy calculation now gives $\Delta \geqslant \alpha(n)$, as desired.

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