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MULTICOLORED LINES IN A FINITE GEOMETRY

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Let $t > 1$, let $P_1, \ldots, P_t$ be pairwise disjoint nonvoid subsets of a finite set P, and let $\mathscr{L}$ be a collection of subsets of P called lines. We say that $(P_1, \ldots, P_t; \mathscr{L})$ is a colored incidence structure if (i) each line meets at least two blocks P_i and P_j, (ii) for arbitrary distinct $x \in P_i$ and $y \in P_j$ the pair $\{x, y\}$ is a subset of exactly one line if $i \neq j$ and at most one line if $i = j$. Extending the work of de Bruijn and Erdös and Meshulam we show that for a coloured incidence structure $(P_1, \ldots, P_t; \mathscr{L})$ with $|P_1| \leqslant \cdots \leqslant |P_t|$ in general $|\mathscr{L}|$ exceeds $1 + |P_1| + \cdots + |P_{t-1}|$. The exceptional cases are exactly: (i) $|\mathscr{L}| = |P_1| + \cdots + |P_{t-1}|$ iff the structure is a truncated projective plane and (ii) $|\mathscr{L}| = 1 + |P_1| + \cdots + |P_{t-1}|$ holds in exactly six cases: a projective plane, the dual of an affine plane, the dual of a modification of an affine plane, a near-pencil, a structure with 2 lines and a 9-element structure with 7 lines.

Introduction

Extending a 1948 paper by de Bruijn and Erdös [1] and a recent paper by Meshulam [5] we study colored incidence structures $(P_1, \ldots, P_t; \mathscr{L})$ from the extremal set theory point of view. Assuming that $|P_1| \leqslant \cdots \leqslant |P_t|$ we derive the bound $|\mathscr{L}| \geqslant |P_1| + \cdots + |P_{t-1}|$ and completely determine the essentially unique extreme case $|\mathscr{L}| = |P_1| + \cdots + |P_{t-1}|$ and the six cases of $|\mathscr{L}| = 1 + |P_1| + \cdots + |P_{t-1}|$ proving much more than it is conjectured in [5]. The main part of the proof treats a slight extension of the dual of $(P_1, \ldots, P_t; \mathscr{L})$ which is a family $\boldsymbol{F}$ of sets partitioned into blocks $\mathscr{F}^1, \ldots, \mathscr{F}^t$ such that $|Y \cap Y'| \leqslant 1$ ($|Y \cap Y'| = 1$, respectively) whenever $Y \in \mathscr{F}^i$, $Y' \in \mathscr{F}^j$, $Y \neq Y'$ ($i \neq j$, respectively).

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1. Incidence structures

1.1. An *incidence structure* (or *finite geometry*) $\boldsymbol{L} = (P, \mathscr{L})$ is a finite set P (of *points*) and a collection $\mathscr{L}$ of non-empty subsets of P (*lines*) such that each pair of distinct points is a subset of exactly one line. Consequently $\mathscr{L}$ is a 0–1 intersecting family (i.e. $|L \cap L'| \leq 1$ for all $L, L' \in \mathscr{L}$ and $L \neq L'$). The incidence structure $(P, \mathscr{L})$ is *nontrivial* if $|\mathscr{L}| > 1$. For $P' \subseteq P$ the induce structure $L \mid P' := (P'; \mathscr{L}')$ with $\mathscr{L}' := \{L \cap P' : L \in \mathscr{L}\}$ is the *restriction* of $(P, \mathscr{L})$ to P'. Clearly $(P', \mathscr{L}')$ is again an incidence structure. For a family $\boldsymbol{F}$ of sets the *degree* of an element x is $d_{\boldsymbol{F}}(x) := |\{Y \in \boldsymbol{F} : x \in Y\}|$.

1.2. *Examples.* Let k be a positive integer.

(1) The (finite) *projective plane of order* k is an incidence structure $(P, \mathscr{L})$ with $|P| = |\mathscr{L}| = k^2 + k + 1$ in which each point has degree $k + 1$ and every line consists of exactly $k + 1$ points.

(2) Fix an arbitrary line L_0 of the projective plane. The restriction of $(P, \mathscr{L})$ to $P \setminus L_0$ is an *affine plane of order* k. It has k^2 points and $k^2 + k$ lines. For each $x \in L_0$ the k lines $L \setminus \{x\}$ with $x \in L \in \mathscr{L}$, $L \neq L_0$ form a *parallel class* of the affine plane.

(3) A *near-pencil of order* r with *center* c is the incidence structure $(P, \mathscr{N})$ with r points and $\mathscr{N} := \{\{c, x\} : x \in P \setminus \{c\}\} \cup (P \setminus \{c\})$.

1.3. In a 1948 paper [1] N.G. de Bruijn and P. Erdös proved:

Let $\mathbf{L} = (P, \mathscr{L})$ *be a nontrivial incidence structure (i.e. for every line* $L \in \mathscr{L}$ *we have* $2 \leq |L| < |P|$*). Let* T *be a set of non-collinear points of cardinality* $t > 2$ *and* $\mathbf{L}'$ *be the restriction of* $\mathbf{L}$ *to* T. *If* $\mathbf{L}'$ *is neither a near-pencil nor a projective plane, then there are more than* t *lines* L *such that* $|L \cap T| > 1$.

In the two exceptional cases there are exactly t lines L with $|L \cap T| > 1$. Recently R. Meshulam [5] extended this as follows. Given $(P, \mathscr{L})$ let $P_1, \ldots, P_t$ be pairwise disjoint nonempty sets of points (i.e. $P_1, \ldots, P_t$ form a partial t-coloration of P). Denote by $\mathscr{L}(P_1, \ldots, P_t)$ the family of lines meeting at least two colors:

$$\mathscr{L}(P_1, \ldots, P_t) := \{L \in \mathscr{L} : L \cap P_i \neq \emptyset \neq L \cap P_j \quad \text{for some } 1 \leq i < j \leq t\}.$$

(i.e. $\mathscr{L}(P_1, \ldots, P_t)$ is obtained from $\mathscr{L}$ by deleting all monochromatic and colorless lines).

1.4. *Example.* Let $(P, \mathscr{L})$ be a projective plane of order k and let $L_1, \ldots, L_{k+1}$ be the lines through a fixed point p. Put $t = k + 1$ and $P_i := L_i \setminus \{p\}$ $(i = 1, \ldots, t)$. Clearly $\mathscr{L}(P_1, \ldots, P_t) = \mathscr{L} \setminus \{L_1, \ldots, L_t\}$ consists of k^2 lines. The restriction of $(P, \mathscr{L}(P_1, \ldots, P_t))$ to $P \setminus \{p\}$ is called a *truncated projective plane of order* k. Setting $m = k$ we have $|\mathscr{L}(P_1, \ldots, P_t)| = k^2 = m(t-1) = |P_1| + \cdots + |P_{t-1}|$.

1.5. Theorem (Meshulam [5]). *Let* $(P, \mathscr{L})$ *be a nontrivial incidence structure,* $P_1, \ldots, P_t$ *pairwise disjoint* m*-element subsets of* P *and* $\mathbf{L}'$ *the restriction of*

$L(P_1, \ldots, P_t)$ to $P_1 \cup \cdots \cup P_t$. If L' is not a truncated projective plane, then $|\mathscr{L}(P_1, \ldots, P_t)| > m(t-1)$.

1.6. Our aim is to extend and complement this theorem. Obviously the points from $P \setminus (P_1 \cup \cdots \cup P_t)$ have no impact on $\mathscr{L}(P_1, \ldots, P_t)$ and so without loss of generality we may assume that $P_1, \ldots, P_t$ forms a partition of P. The structure of $\mathscr{L}(P_1, \ldots, P_t)$ leads to the following definition. Let $t > 1$, let $P_1, \ldots, P_t$ be pairwise disjoint nonvoid finite sets and let $\mathscr{L}$ be a family of subsets of $P := P_1 \cup \cdots \cup P_t$ of cardinality at least 2. Then $(P_1, \ldots, P_t; \mathscr{L})$ is a *colored incidence structure* if

(i) each line meets at least two sets P_i and

(ii) for arbitrary $x \in P_i$ and $y \in P_j$, $x \neq y$, the pair $\{x, y\}$ is a subset of exactly one line if $i \neq j$ and at most one line if $i = j$.

For simplicity we assume that $|P_1| \leq \cdots \leq |P_t|$. We illustrate this concept in the following examples providing the exceptional (extreme) cases in our main theorem.

1.7. *Example.* Let $P = (P, \mathscr{N})$ be a near-pencil with center c (cf. 1.2.3 above) and let $P_1, \ldots, P_t$ be a partition of P with $|P_1| \leq \cdots \leq |P_t|$ and $c \in P_t$. Then $\mathscr{N}(P_1, \ldots, P_t)$ consists of all lines $\{c, x\}$ with $x \in P_1 \cup \cdots \cup P_{t-1}$ and the line $P \setminus \{c\}$ and so

$$|\mathscr{N}(P_1, \ldots, P_t)| = 1 + |P_1| + \cdots + |P_{t-1}|$$

1.8. *Example.* Let $t = 2$, $P_1 := \{z\}$, $\emptyset \neq Z \subsetneq P_2$ and $\mathscr{L} := \{\{z\} \cup Z, \{z\} \cup (P_2 \setminus Z)\}$. Then $(P_1, P_2; \mathscr{L})$ is a colored incidence structure with $|\mathscr{L}| = 2 = 1 + |P_1|$.

1.9. *Example.* Let $L = (P, \mathscr{L})$ be a projective plane of order k, let $t = k^2 + k + 1$ and $|P_1| = \cdots = |P_t| = 1$. Then $L = (P_1, \ldots, P_t; \mathscr{L})$ and $|\mathscr{L}| = k^2 + k + 1 = 1 + |P_1| + \cdots + |P_{t-1}|$.

1.10. *Example.* Let $L_1, \ldots, L_{k+1}$ be the lines through a fixed point p of a projective plane $L = (P, \mathscr{L})$ of order k. Fix $q \in L_1 \setminus \{p\}$ and put $P_1 := L_1 \setminus \{p, q\}$, and $P_i := L_i \setminus \{p\} (i = 2, \ldots, k+1)$. Further let $\mathscr{R}$ be the restriction of $\mathscr{L} \setminus \{L_1, \ldots, L_{k+1}\}$ to $P \setminus \{p, q\}$ and $t := k + 1$. Clearly $(P_1, \ldots, P_t; \mathscr{R})$ is a colored incidence structure and $|\mathscr{R}| = k^2 = 1 + k - 1 + k(k-1) = 1 + |P_1| + \cdots + |P_{t-1}|$ (this is the dual of an affine plane of order k with one line deleted).

1.11. *Example.* Let $L = (P, \mathscr{L})$ be a projective plane of order k and $L_1, \ldots, L_{k+1}$ the lines through a fixed point p. Let $s > 1$, let $P_1, \ldots, P_s$ be a partition of $L_1 \setminus \{p\}$ and $P_{s+i} := L_{i+1} \setminus \{p\} (1 \leq i \leq k)$. Let $\mathscr{S}$ be the restriction of $\mathscr{L} \setminus \{L_1, \ldots, L_{k+1}\}$ to $P \setminus \{p\}$. It is easy to see that $(P_1, \ldots, P_{s+k}; \mathscr{S})$ is a colored incidence structure with $|\mathscr{S}| = k^2 + 1 = 1 + |P_1| + \cdots + |P_{s+k-1}|$.

	1	2	3	4	5	6	7	8	9
L_1	×			×				×	×
L_2	×				×	×	×		
L_3		×	×	×			×		
L_4		×			×			×	
L_5		×				×			×
L_6			×		×				×
L_7			×			×		×	

(a)

	1	2	3	4	5	6	7
	×	×					
$\mathcal{F}^1$			×	×	×		
			×			×	×
	×		×				
$\mathcal{F}^2$		×		×		×	
		×			×		×
		×	×				
$\mathcal{F}^3$	×			×			×
	×				×	×	

(b)

Fig. 1.

1.12. *Example.* Let $P_1\backslash\{1,2,3\}$, $P_2\backslash\{4,5,6\}$, $P_3\backslash\{7,8,9\}$ and

$$\mathcal{L}\backslash\{\{1,4,8,9\},\{1,5,6,7\},\{2,3,4,7\},\{2,5,8\},\{2,6,9\},\{3,5,9\},\{3,6,8\}\}.$$

(Fig. 1a). A direct check shows that $(P_1, P_2, P_3; \mathcal{L})$ is a coloured incidence structure (in fact, by adding the pairs $\{1,2\}$, $\{1,3\}$, $\{4,5\}$, $\{4,6\}$, $\{7,8\}$, $\{7,9\}$ we obtain an incidence structure on $\{1,\ldots,9\}$). Clearly $|\mathcal{L}| = 7 = 1 + |P_1| + |P_2|$.

Now we are ready to state our main result.

1.13. Theorem. *Let* $L = (P_1, \ldots, P_t; \mathcal{L})$ *be a colored incidence structure with* $t > 1$ *and* $|P_1| \leq \cdots \leq |P_t|$. *Then*

(i) $|\mathcal{L}| = |P_1| + \cdots + |P_{t-1}|$ *if and only if* L *is a truncated projective plane (of order* $t-1$; *see Example* 1.4),

(ii) $|\mathcal{L}| = 1 + |P_1| + \cdots + |P_{t-1}|$ *if and only if* $\mathcal{L}$ *is one of the colored incidence structures from Examples* 1.7–1.12.

(iii) *In all other cases* $|\mathcal{L}| > 1 + |P_1| + \cdots + |P_{t-1}|$.

In the particular case $|P_1| = \cdots = |P_t| = m$ we obtain the following sharpening of Theorem 1.5.

1.14. Corollary. *Let $L = (P_1, \ldots, P_t; \mathscr{L})$ be a colored incidence structure with $t > 1$ and $|P_1| = \cdots = |P_t| = m$. Then*

(*i*) $|\mathscr{L}| = m(t-1)$ *if and only if $m = t$ and L is a truncated projective plane of order m.*

(*ii*) $|\mathscr{L}| = 1 + m(t-1)$ *if and only if either*

(a) $m = 1$, *and $\mathscr{L}$ is a projective plane of order k where $t = k^2 + k + 1$, or*

(b) $\mathscr{L} = \mathscr{N}(P_1, \ldots, P_t)$ *for a near-pencil or*

(c) $m = t = 3$ *and L is the structure from Example* 1.12.

(*iii*) *In all other cases* $|\mathscr{L}| > 1 + m(t-1)$.

1.15. Remark. The following example shows that in a colored incidence structure $|P_t|$ may largely exceed $|\mathscr{L}|$. Let n be a positive integer, let $P_1 := \{(0,1)\} P_2 := \{(1,0)\}$ and $P_3 := \{(i,j) : 1 \leq i, j \leq n\}$. Further let $L_0 := \{(0,1), (1,0)\}$ and for $i = 1, \ldots, n$ let $L_i := \{(0,1)\} \cup \{(i,j) : j = 1, \ldots, n\}$, $L_{n+i} := \{(1,0)\} \cup \{(j,i) : j = 1, \ldots, n\}$. Put $\mathscr{L} := \{L_0, \ldots, L_{2n}\}$. It is easy to see that $(P_1, P_2, P_3; \mathscr{L})$ is a colored incidence structure with $|P_3| = n^2 = \frac{1}{4}(|\mathscr{L}| - 1)^2$.

2. 0–1 Intersecting families

2.1. In Theorem 1.13 the sufficiency of both (i) and (ii) has been shown in Examples 1.4 and 1.7–1.12. To prove (iii) and the necessity of (i) and (ii) we investigate the dual of $\mathscr{L}$. As it is more convenient for us to work with the dual formulation, therefore we now propose to reinterpret everything in dual form. As usual, the *dual* of $\mathscr{L}$ is obtained by interchanging the roles of points and lines in the following sense. Let $\mathscr{L} := \{L_1, \ldots, L_n\}$ and let $P := \{p_1, \ldots, p_l\}$. Define a zero-one $n \times l$ matrix $M = (x_{ij})$ be setting $x_{ij} = 1$ if $p_j \in L_i$ and $x_{ij} = 0$ otherwise. Interpreting the jth column of M as the set $Y_j := \{i : x_{ij} = 1\} (j = 1, \ldots, l)$ we obtain a system $F = \{Y_1, \ldots, Y_l\}$ of subsets of $\{1, \ldots, n\}$ (the sets Y_j need not be distinct). Now F is naturally partitioned into blocks $\mathscr{F}^i := \{Y_j : p_j \in P_i\} (i = 1, \ldots, t)$. Since there is at most one line through each pair $\{p_a, p_b\} (1 \leq a < b \leq l)$; the system F is 0–1 intersecting. Next, two points from different color blocks are joined by a unique $L \in \mathscr{L}$ and therefore

(D.1) $\quad |Y \cap Y'| = 1$ for all $Y \in \mathscr{F}^i$, $Y' \in \mathscr{F}^j$, $1 \leq i < j \leq t$.

Finally each line meets at least two blocks P_i and P_j and so:

(D.2) $\quad$ The singletons $Y \cap Y'$ with $Y \in \mathscr{F}^i$, $Y' \in \mathscr{F}^j$, $1 \leq i < j \leq t$ cover $\{1, \ldots, n\}$.

2.2. Consider the duals of the colored incidence structures from Examples 1.4 and 1.7–1.12. The dual of a truncated projective plane of order k is an affine plane of order k. The dual of a projective plane of order k is again a projective plane of order k. The dual of $(P_1, \ldots, P_t; \mathcal{N})$ from Example 1.7 is a near-pencil whose center c' corresponds to the line $P\backslash\{c\}$ augmented by the singleton $\{c'\}$ taken $t-1$ times. The dual of Example 1.8 has $\mathcal{F}^1 = \{\{1,2\}\}$ while $\mathcal{F}^2 = \{\{1\}, \ldots, \{1\}, \{2\}, \ldots, \{2\}\}$ with $\{1\}$ appearing $|Z|$ times. The dual of Example 1.11 is obtained as follows. In an affine plane L of order k fix a parallel class $B_1, \ldots, B_k$ and choose a new point b. Let $\mathcal{F}^1, \ldots, \mathcal{F}^s$ be a partition of $\{B_1 \cup \{b\}, \ldots, B_k \cup \{b\}\}$ and let $\mathcal{F}^{s+1}, \ldots, \mathcal{F}^{s+k}$ be the remaining parallel classes of L. Finally the dual of Example 1.12 is on Fig. 1b.

2.3. As mentioned above and seen in 2.2, a set Y may appear in F several times. If this happens then clearly Y is a singleton, $Y = \{y\}$. Suppose $Y = \{y\} \in \mathcal{F}^i$. Then Y is the intersection of all sets from $F\backslash\mathcal{F}^i$ and consequently the line L corresponding to y contains $P\backslash P_i$. Now $P_i \nsubseteq L$ because otherwise $L = P$, $\mathcal{L} = \{L\}$ in contradiction to $|\mathcal{L}| > 1$. Fix $x \in P_i\backslash L$. Clearly for each $y \in P\backslash P_i$ there is a line containing $\{x, y\}$ and no other point in $P\backslash P_i$ and therefore $l := |\mathcal{L}|$ satisfies

$$l \geq 1 + |P - P_i| = 1 + |P_1| + \cdots + |P_{i-1}| + |P_{i+1}| + \cdots + |P_t|. \tag{2.1}$$

Since $|P_t| \geq |P_j|$ for all j it follows that $l \geq 1 + |P_1| + \cdots + |P_{t-1}|$. Suppose $l = 1 + |P_1| + \cdots + |P_{t-1}|$. Then by (2.1) we have $|P_i| = |P_t|$ and so without loss of generality we may assume that $i = t$. Moreover, each line distinct from L is determined by a pair $\{x, y\}$ with $y \in P\backslash P_t$. If $P_t\backslash L = \{x\}$ then $\mathcal{L} = \mathcal{N}(P_1, \ldots, P_t)$ is a near-pencil (P, N) with center x (Example 1.7) and we are done. Thus assume that $|P_t\backslash L| > 1$. For $y \in P_t\backslash(L \cup \{x\})$ and $z \in P\backslash P_t$ the line through x and z must coincide with the line though y and z. As there is at most one line through x and y, we can conclude that $P\backslash P_t = \{z\}$. Thus $t = 2$, $P_1 = \{z\}$ and $l = 1 + |P_1| = 2$. Put $Z := P_2 \cap L$. Then $L_1 = \{z\} \cup Z$ and $L_2 = \{z\} \cup (P_2\backslash Z)$ obtaining thus the structure from Example 1.8.

2.4. From now on we assume that F is a set containing no singletons. In the context of 0–1 intersecting families the assumptions (D.2) seems to be somewhat arbitrary and restrictive. If we drop it, we need the following additional case.

Example. Let $L = (P, \mathcal{L})$ be an affine plane of order k with parallel classes $\mathcal{L}^1, \ldots, \mathcal{L}^{k+1}$ where $\mathcal{L}^{k+1} = \{Y_1, \ldots, Y_k\}$. Fix $0 \leq j \leq k$ and $X := P \cup \{x\}$ where $x \notin P$ and put $\mathcal{F}^i = \mathcal{L}^i$ $(i = 1, \ldots, k)$ and $\mathcal{F}^{k+1} := \{\{x\} \cup Y_i : i = 1, \ldots, j\} \cup \{Y_i : i = j+1, \ldots, k\}$. It is easy to see that $\mathcal{F}^1, \ldots, \mathcal{F}^{k+1}$ form a 0–1 intersecting family satisfying (D.1) but not (D.2). Moreover,

$$|X| = k^2 + 1 = 1 + |\mathcal{F}^1| + \cdots + |\mathcal{F}^k|.$$

Now we are ready to formulate the following result for 0–1 intersecting families which may be of independent interest.

2.5. Theorem. *Let F be a 0–1 intersecting family of subsets of an n-set X and let $k := \min\{|Y| : Y \in F\} > 1$. Let F be partitioned into blocks $\mathcal{F}^1, \ldots, \mathcal{F}^t$ of sizes $f_1 \leq \cdots \leq f_t$ so that $t > 1$ and*

(D.1) $|Y \cap Y'| = 1$ *for all* $Y \in \mathcal{F}^i$, $Y' \in \mathcal{F}^j$, $1 \leq i < j \leq t$. *Then*

(*i*) $n = f_1 + \cdots + f_{t-1}$ *if and only if* $t = k+1$ *and F is an affine plane of order k with parallel blocks* $\mathcal{F}^1, \ldots, \mathcal{F}^t$.

(*ii*) $n = 1 + f_1 + \cdots + f_{t-1}$ *if and only if one of the following holds*

(*a*) $t = k^2 + k + 1$, *F is a projective plane of order k and* $|\mathcal{F}^1| = \cdots = |\mathcal{F}^t| = 1$,

(*b*) $t = k+1$ *and there is an affine plane* $L = (P; \mathcal{L})$ *of order k such that $\mathcal{F}^1$ is a parallel class of L minus one line while* $\mathcal{F}^2, \ldots, \mathcal{F}^{k+1}$ *are the other parallel classes,*

(*c*) *F is the dual of Example* 1.11 *or* 1.12 *and*

(*d*) *F is isomorphic to Example* 2.4.

(*iii*) *In all other cases* $n \geq 2 + f_1 + \cdots + f_{t-1}$.

2.6. The following is the key to our proof. Our argument may be seen as a modified version of the method of P. Seymour [6]. Let F and $\mathcal{F}^1, \ldots, \mathcal{F}^t$ be as in Theorem 2.5 and let $f_i := |\mathcal{F}^i|$ $(i = 1, \ldots, t)$; however we do not assume $f_1 \leq \cdots \leq f_t$. Let k denote the mimimum cardinality of a member of F. Further let G denote the family of the k-sets from F. Fix $G \in G$ where $G \in \mathcal{F}^i$ and put $H := F \backslash \mathcal{F}$, $h := |H|$, $H' := H \cap G$, $h' := |H'|$, $H'' := H \backslash G$. Further for $x \in X$ let $d(x)$, $d'(x)$ and $d''(x)$ denote the degree of x in H, H' and H'' (see 1.1). Let $x \in G$. The sets of H containing x meet at most in $\{x\}$ and thus are disjoint on $X \backslash G$. It follows that

$$n - k = |X \backslash G| \geq \sum \{|H| - 1 : x \in H \in H\} \geq (k-1)d'(x) + kd''(x) \tag{2.2}$$

Sum up the inequalities (2.2) over all $x \in G$:

$$k(n-k) \geq (k-1) \sum_{x \in G} d'(x) + k \sum_{x \in G} d''(x) \tag{2.3}$$

If we partition H' by putting Y and Y' in the same block whenever $Y \cap G = Y' \cap G$ and observe that the block with $Y \cap G = \{x\}$ has exactly $d'(x)$ sets we obtain that

$$h' = \sum_{x \in G} d'(x).$$

Similarly the second sum in (2.3) equals $|H''| = h - h'$ and therefore

$$k(n-k) \geq (k-1)h' + k(h - h') = kh - h',$$

i.e.

$$n \geq h + (k^2 - h')/k. \tag{2.4}$$

Suppose $k^2-k>h'$. Then from (2.4) we get $n>h+1$ i.e. $n\geqslant h+2$ and in view of $h=|\boldsymbol{H}|=|\boldsymbol{F}\setminus\mathcal{F}^i|$ we have (iii). Thus from now on we assume that

$$k^2-k\leqslant h' \tag{2.5}$$

whenever $G\in\boldsymbol{G}\cap\mathcal{F}^i$ and $h:=|\boldsymbol{G}\cap(\boldsymbol{F}\setminus\mathcal{F}_i)|$. Put $\boldsymbol{K}:=\{G\}\cup\boldsymbol{H}'$. Here $\boldsymbol{K}\subseteq\boldsymbol{G}$ is a family of k-sets and $|\boldsymbol{K}|=1+h'\geqslant k^2-k+1$. Clearly $\boldsymbol{K}$ is 0–1 intersecting. We have two cases:

(a) there are no disjoint sets in $\boldsymbol{K}$ and

(b) $\boldsymbol{K}$ contains a pair of disjoint sets. The case (a) is studied in Section 3 and the case (b) in Section 4.

3. Fully intersecting $\boldsymbol{K}$

3.1. We assume that there is a set $G\in\boldsymbol{G}\cap\mathcal{F}^i$ such that $\boldsymbol{K}:=\{G\}\cup(\boldsymbol{G}\cap(\boldsymbol{F}\setminus\mathcal{F}^i))$ has at least k^2-k+1 elements and $|Y\cap Y'|=1$ for all $Y,\ Y'\in\boldsymbol{K}$, $Y\neq Y'$. From [2] it follows that $\boldsymbol{K}$ is either a projective plane of order $k-1$ or a star (i.e. $\bigcap\mathbf{K}=\{c\}$ where c is called the *kernel* of $\boldsymbol{K}$).

(α) Let $\boldsymbol{K}$ be a projective plane of order $k-1$. We need the following easy fact:

Fact. Let L *be a line of a projective plane of order* $q>1$. *If a set* T *meets every line distinct from* L *in a singleton, then* $T=$ L.

We distinguish two cases $k>2$ and $k=2$.

3.2. Suppose $k>2$. Apply the fact to the line G of $\boldsymbol{K}$ and $T\in\mathcal{F}^i$. Then $T=G$ (since otherwise $|G|=|T\cap G|\leqslant 1$) i.e. $\mathcal{F}^i=\{G\}$. It follows that the set $\boldsymbol{G}$ of k-sets from $\boldsymbol{F}$ has cardinality $g:=k^2-k+1$. Now $G\in\boldsymbol{G}$ was chosen arbitrarily in $\boldsymbol{G}$ and so there are $1\leqslant i_1<\cdots<i_g\leqslant t$ such that $\mathcal{F}^{i_j}=\{G_j\}$ $(j=1,\ldots,g)$ where $\boldsymbol{G}=\{G_1,\ldots,G_g\}$. Applying once more the above fact we obtain easily that $t=g$ and $i_j=j$ for all $j=1,\ldots,t$, thus $\boldsymbol{F}=\boldsymbol{G}$ is a projective plane of order $k-1\geqslant 2$ and we are done.

3.3. Let $k=2$. Then $\boldsymbol{K}$ is the projective plane of order 1 and so we may assume that $G=\{1,2\}\in\mathcal{F}^1$ and $\boldsymbol{H}'=\{\{1,3\},\ \{2,3\}\}$. Now either $\{1,3\}$ and $\{2,3\}$ belong to different blocks (case I) or to the same block (case II).

(I) Let $\{1,3\}\in\mathcal{F}^2$ and $\{2,3\}\in\mathcal{F}^3$. Then $t=3$ because there is no set meeting each of $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ in a singleton. If $\boldsymbol{F}=\boldsymbol{I}:=\{\{1,2\},\ \{1,3\},\ \{2,3\}\}$ we have the degenerated projective plane (of order 1) and we are done. Thus let $\boldsymbol{F}\supset\boldsymbol{I}$ and let $s:=\min\{|Y|:Y\in\boldsymbol{F}\setminus\boldsymbol{I}\}$. Without loss of generality we may assume that there is $Z\in\mathcal{F}^1$ of cardinality s. It is easy to see that $3\in Z$ while $1,2\notin Z$. Without loss of generality we may assume that $f_i:=|\mathcal{F}^i|$ satisfy $f_2\geqslant f_3$. Let $\mathcal{F}^2:=\{\{1,3\},\ Y_2,\ldots,\ Y_{f_2}\}$. Suppose that $Y_i\cap\{1,2\}=\{1\}$ for some i. Then

$Y_i \cap \{1, 3\} = \{1\}$ and $\{2, 3\}$ is disjoint from Y_i. This contradiction shows $2 \in Y_i$ for $i = 2, \ldots, f_2$. Thus $Y_i \cap Y_j = \{2\}$ for all $2 \leqslant i < j \leqslant f_2$. Every Y_i has at least s elements and thus at least $s - 2$ elements outside $Z' := \{1, 2\} \cup Z$. It follows that $(Y_2 \cup \cdots \cup Y_{f_2}) \setminus Z'$ has at least $(f_2 - 1)(s - 2)$ elements and consequently

$$n := |X| \geqslant (f_2 - 1)(s - 2) + s + 2 = f_2(s - 2) + 4. \tag{3.1}$$

On the other hand $2 + f_2 + f_3 \leqslant 2(f_2 + 1)$. If $s \geqslant 4$, then we have the required inequality $2 + f_2 + f_3 \leqslant n$. If $s = 3$, then $f_2 \leqslant s = 3$ and by the inequality (3.1) $n \geqslant f_2 + 4 \geqslant 1 + f_2 + f_3$. If $n \geqslant 2 + f_2 + f_3$ we are done. Let $n = 1 + f_2 + f_3$. Then $f_2 = 3$, $n = 7$ and $f_3 = 3$. The only possibility for $\boldsymbol{F}$ is given by Fig. 1(b).

It remains to consider the case $s = 2$. E.g. $\{1, 2\}$, $\{3, 4\} \in \mathcal{F}^1$. Then for each $Y \in \mathcal{F}^2 - \{1, 3\}$ we have $\{2, 4\} \subset Y$, hence $f_3 \leqslant f_2 \leqslant 2$. Hence in the case $n \geqslant 6$ we are ready. The case $n \leqslant 5$ is covered by the duals of Examples 1.4–1.11.

(II) Let $\{1, 3\}$, $\{2, 3\} \in \mathcal{F}^2$. Put $G := \{1, 3\}$. Then $\boldsymbol{F} \setminus \mathcal{F}^2$ contains at least $2^2 - 2$ pairs which are all in $\mathcal{F}^1$. Thus there is $\{a, b\} \in \mathcal{F}^1$, $\{a, b\} \neq \{1, 2\}$. Suppose $\{a, b\} \cap \{1, 2\} \neq \emptyset$, say $a = 1$ and $b \notin \{1, 2\}$. Then $b \neq 3$ and $\{1, b\}$ does not meet $\{2, 3\}$. This contradiction shows that $a, b \notin \{1, 2\}$. Thus $3 \in \{a, b\}$, say $a = 3$ and $b = 4$. Now each $Y \in \boldsymbol{F} \setminus \mathcal{F}^1$ intersects both $\{1, 2\}$ and $\{3, 4\}$ in a singleton. Suppose there is $Z \in \boldsymbol{F} \setminus (\mathcal{F}^1 \cup \mathcal{F}^2)$. Then Z meets both $\{1, 3\}$ and $\{2, 3\}$ in a singleton. It is easy to see that this is impossible. Thus $t = 2$ and $f_2 \leqslant 4$. If $f_1 = 2$ then $n \geqslant 4 = 2 + f_1$ and we are done. Thus let $f_1 > 2$. Put $\mathcal{U} := \mathcal{F}^1 \setminus \{\{1, 2\}, \{3, 4\}\}$. For $T \in \mathcal{U}$ we have $\{1, 2, 3, 4\} \cap \mathrm{T} = \{3\}$ and therefore $T \cap T' = \{3\}$ whenever T, $T' \in \mathcal{U}$, $T \neq T'$. Considering $\mathcal{F}^2 - \{\{1, 3\}, \{2, 3\}\}$ it follows that besides $1, 2, 3, 4$ there are at least $(f_1 - 2)(f_2 - 2)$ points and therefore

$$n \geqslant 4 + (f_1 - 2)(f_2 - 2) \geqslant 4 + f_2 - 2 = 2 + f_2$$

completing the proof of (II) and thus fully settling the case when $\boldsymbol{K}$ is a projective plane.

3.4. We consider the case

(β) $\boldsymbol{K}$ is a star with kernel c.

We have two subcases.

(i) Suppose there is a k-set $Y \in \boldsymbol{F}$ not containing c. Then $Y \in \mathcal{F}^j$ and $Y \neq G$. For Z, $Z' \in \boldsymbol{H}'$, $Z \neq Z'$ we have $Y \cap Z \neq Y \cap Z'$ whence

$$k = |Y| \geqslant h' \geqslant k^2 - k$$

which yields $2 \geqslant k$, $k = 2$ and $h' = 2$. For simplicity let $G := \{3, 4\} \in \mathcal{F}^2$, $c := 3$ and let $\boldsymbol{H}' := \{\{1, 3\}\{2, 3\}\}$. Then $Y = \{1, 2\}$. Now we are in the situation considered in Section 3.3 above and so we are done.

(ii) We may assume that the set $\boldsymbol{G}$ of the k-sets from $\boldsymbol{F}$ forms a star with kernel c. For $j = 1, \ldots, t$ put $\mathcal{G}^j := \boldsymbol{G} \cap \mathcal{F}^j$, $g_j := |\mathcal{G}^j|$ and let $g := g_1 + \cdots + g_t$.

Choosing $G \in \mathscr{G}^1$ we obtain

$$g_2 + \cdots + g_t = h' \geq k^2 - k \geq 2 \tag{3.2}$$

Put

$$\mathscr{S}^l := \{Y \in \mathscr{F}^l \setminus \boldsymbol{G} : c \in Y\}, \quad s_l := |\mathscr{S}^l| \, (l = 1, \ldots, t)$$

and $\boldsymbol{S} := \mathscr{S}^1 \cup \cdots \cup \mathscr{S}^t$. Clearly $\boldsymbol{S}' := \boldsymbol{G} \cup \boldsymbol{S}$ is a star (with kernel c). Let $\boldsymbol{J} := (\mathscr{F}^2 \cup \cdots \cup \mathscr{F}^t) \setminus \boldsymbol{S}'$, $j := |\boldsymbol{J}|$ and $\alpha := |\bigcup \boldsymbol{F} \setminus \bigcup \boldsymbol{S}'|$. Finally let $s := s_1 + \cdots + s_t$. At each $Y \in \boldsymbol{S}$ has at least $k+1$ elements and every $Z \in \boldsymbol{G}$ is a k-element set we have

$$n \geq \alpha + |\bigcup \boldsymbol{S}'| \geq 1 + (k-1)g + ks + \alpha.$$

On the other hand

$$2 + f_2 + \cdots + f_t = 2 + g_2 + \cdots + g_t + s_2 + \cdots + s_t$$
$$+ j = 2 + g - g_1 + s - s_1 + j$$

and so it suffices to show that

$$j \leq (k-2)g + (k-1)s + g_1 + s_1 + \alpha - 1. \tag{3.3}$$

We have four cases.

A. Let $g \geq 4$ and $k \geq 3$. We prove;

Claim 1. We can rearrange the $\mathscr{F}^i$'s so that

$$g_1 + \cdots + g_j > \tfrac{1}{2}(k^2 - k), \quad g_{j+1} + \cdots + g_t \geq 2 \tag{3.4}$$

for some $1 \leq j < t$.

Proof. Suppose $g_1 \geq \cdots \geq g_p > 0 = g_{p+1} = \cdots = g_t$. By (3.2) (for p instead of 1) we have $g_1 + \cdots + g_{p-1} \geq 2$. If $g_p > 1$ we choose $j = p-1$. Thus assume $g_p = 1$. Again by (3.2) we have $p > 2$. If $p > 3$ we choose $j = p-2$. Thus let $p = 3$. If $g_1 > g_2$ or $g_1 = g_2 > \frac{1}{2}(k^2 - k)$, then we choose $j = 1$. Finally if $g_1 = g_2 = \frac{1}{2}(k^2 - k)$ we have $k^2 - k + 1 = g > 3$ and hence $k > 2$. Now we rearrange the sequence to $\frac{1}{2}(k^2 - k)$, 1, $\frac{1}{2}(k^2 - k)$ and choose $j = 2$. □

Claim 2. Let $\boldsymbol{C} := \{Y \in \mathscr{F}^1 \cup \cdots \cup \mathscr{F}^j : c \notin Y\}$ and $\boldsymbol{C}' := \{Y \in \mathscr{F}^{j+1} \cup \cdots \cup \mathscr{F}^t : c \notin Y\}$. Then $|\boldsymbol{C}| \leq (k-1)^2$ and $|\boldsymbol{C}'| \leq k-1$.

Proof. Put $\boldsymbol{D} := \mathscr{G}^1 \cup \cdots \cup \mathscr{G}^j$ and $\boldsymbol{G}' := \mathscr{G}^{j+1} \cup \cdots \cup \mathscr{G}^t$. According to (3.4) there are two distinct $G, G' \in \boldsymbol{G}'$. To each $Y \in \boldsymbol{C}$ assign $\psi(Y) := (a, a')$ where $\{a\} := Y \cap G'$ and $\{a'\} := Y \cap G'$. The map ψ from $\boldsymbol{C}$ into $(G \setminus \{c\}) \times (G' \setminus \{c\})$ is injective proving the first claim.

Suppose that $\boldsymbol{C}'$ contains k sets $C_1, \ldots, C_k$. Fix $Z \in \boldsymbol{D}$ and for $1 \leq j \leq k$ put

$\psi_Z(i) := a_i$ where $\{a_i\} := C_i \cap Z$. The map $\psi_Z := \{1, \ldots, k\} \to Z \setminus \{c\}$ is clearly non-injective and so there are $1 \leq i < i' \leq k$ such that $\psi_Z(i) = \psi_Z(i') = a$. Put $\phi(i, i') := Z$. Since $C_i \cap C_{i'} = \{a\}$, we have defined an injective map ϕ from a subset E of $R := \{(i, i'): 1 \leq i < i' \leq k\}$ onto D. Clearly $|D| = |E| \leq R$. However R has $\frac{1}{2}(k^2 - k)$ elements while by (3.4) D has more than $\frac{1}{2}(k^2 - k)$ elements. This contradiction shows that $|C'| \leq k - 1$ completing the proof of the claim. □

We prove the inequality (3.3) for $k \geq 3$. By Claim 2 we have $j \leq |C| + |C'| \leq k^2 - k$. It is easy to see that

$$1 \leq (k-3)(k^2 - k) + (k-1)g_1,$$

whence

$$j \leq k^2 - k \leq (k-2)(g_1 + k^2 - k) + g_1 - 1.$$

By (3.2) we have $g_1 + k^2 - k \leq g$ and hence j does not exceed the right hand side of (3.3) and we are done.

B. Let $k = 2$ and $g_1 \geq 2$. If $c = 0$ and $\{0, 1\}, \{0, 2\} \in \mathcal{G}^1$, then $1, 2 \in Z$ for each $Z \in \boldsymbol{J}$ proving $j \leq 1$. In view of $1 \leq g_1 - 1$ we see that (3.3) holds.

C. Let $k = 2$ and $g_1 = \cdots = g_p = 1$, $g_{p+1} = \cdots = g_t = 0$ where $p > 3$. Again $j \leq 1$. If $s + s_1 + \alpha > 0$ clearly (3.3) holds. Thus assume that $j = 1$ and $s = \alpha = 0$. Then $\boldsymbol{F}$ is of the form $\mathcal{F}^i = \mathcal{G}^i = \{\{0, i\}\}(i = 1, \ldots, t-1)$ and either $\mathcal{F}^t = \{\{0, t\}, \{1, \ldots, t\}\}$ or $\mathcal{F}^t = \{1, 2, \ldots, t-1\}$. In both cases $\boldsymbol{F}$ is a near-pencil.

D. Finally let $k = 2$ and $g_1 \cdots g_p = 1$, $g_{p+1} = \cdots = g_t = 0$ where $p \leq 3$. In view of (3.2) we have $p - 1 = g_2 + \cdots + g_p \geq 2$, hence $p \geq 3$, i.e. $p = 3$. Suppose $c = 0$ and $\mathcal{G}^l := \{\{0, l\}\}(l = 1, 2, 3)$. It is not difficult to check that $j \leq 2$. The inequality (3.3) reduces to $j \leq s + s_1 + \alpha$. Without loss of generality we may assume that $s_1 \geq s_2 \geq s_3$. If $s_1 \geq 1$ then $j \leq 2 \leq 2s_1 \leq s + s_1 + \alpha$ and we are done. Thus assume $s_1 = s_2 = s_3 = 0$. The inequality (3.3) becomes $j \leq s_4 + \cdots + s_t + \alpha$.

(a) If $s_4 \cdots + s_t \geq 2$ we are done.

(b) Thus let $s_4 + \cdots + s_t = 1$, say $s_4 = 1$, $s_5 = \cdots = s_t = 0$, and let $j = 2$. Then $t = 4$. Let $\mathcal{G}^4 = \mathcal{F}^4 := \{Y\}$, $\mathcal{F}^2 = \{\{0, 2\}, U\}$ and $\mathcal{F}^3 = \{\{0, 3\}, V\}$. It is easy to see that $1, 3 \in U$, $U \cap Y = \{4\}$, $1, 2 \in V$ and $V \cap Y = \{5\}$. If $f_1 = 1$ then $2 + f_2 + f_3 + f_4 = 6 \leq n$ and we are done. Thus assume $f_1 > 1$. Then $\mathcal{F}^1 = \{\{0, 1\}, W\}$ where $2, 3 \in W$ and $W \cap Y = \{6\}$. Again we have $2 + f_1 + f_3 + f_4 = 7 \leq n$ and we are done.

(c) Finally let $s_1 = \cdots = s_t = 0$. The inequality (3.3) reduces to $j \leq \alpha$. If $\boldsymbol{J} \cap \mathcal{F}^2 = \{Y\}$, then $1, 3 \in Y$ and Y contains an additional element, say 5. If $\boldsymbol{J} \cap \mathcal{F}^3 = \{Z\}$, then $1, 2 \in Z$ and Z has another element which is distinct from 5. Thus either $\boldsymbol{F}$ is a near-pencil on 4 elements or we have $j \leq \alpha$ and the proof of the case D is complete.

4. K Is not fully intersecting

4.1. Finally we consider the case of K with a pair of disjoint sets. Thus we may assume that some block, say $\mathscr{F}^1$, contains two disjoint k-sets G and G'. Again we put $H := F \setminus \mathscr{F}^1$, $h := |H|$, $H' = H \cap G$ and $h' := |H'|$. To prove (iii) from Theorem 2.5 it suffices to show that $|X| = n \geq 2 + h$. Since each $Y \in H$ meets both G and G' in a singleton we have $k^2 = |G|\,|G'| \geq h$, hence

$$k^2 \geq h \geq h' \geq k^2 - k \tag{4.1}$$

and if $n \geq 2 + k^2$ we are done. Thus we assume that $k^2 + 1 \geq n$. Let $d(x)$ denote the degree of x in H. Let $x \in X \setminus G$. To each $Y \in H$ containing x assign $\psi(Y) := a$ where $\{a\} = Y \cap G$. The map ψ is injective and so $d(x) \leq |G| = k$. A similar argument shows

$$\sum_{x \in G} d(x) = h \geq h' \geq k^2 - k. \tag{4.2}$$

Fix $y \in G$. The sets G, G' and $\{Y \setminus (G \cup G') : y \in Y \in H\}$ are pairwise disjoint and $|Y \setminus (G \cup G')| = k - 2$ whence

$$n \geq 2k + (k-2)d(y). \tag{4.3}$$

We have two cases.

(1) Let $d(x) \leq k - 1$ for all $x \in G$. Then

$$k^2 - k \geq \sum_{x \in G} d(x)$$

and from (4.2) we obtain

$$\sum_{x \in G} d(x) = h = k^2 - k.$$

It follows that $d(x) = k - 1$ for all $x \in G$. From (4.3) we get the desired inequality

$$n \geq 2k + (k-1)(k-2) = 2 + k^2 - k = 2 + h.$$

We have the case

(2) $d(x) = k$ for some $x \in G$. For $y = x$ the inequality (4.3) gives $n \geq 2k + k(k-2) = k^2$. Since $k^2 + 1 \geq n$ we have two possibilities: $n = k^2 + 1$ and $n = k^2$. Now clearly $n \geq 2 + h$ holds except in the two cases (α) $n = k^2$ and $k^2 \geq h \geq k^2 - 1$ and (β) $n = k^2 + 1$ and $h = k^2$. Before we start to investigate (α) and (β) we prove the following two claims.

Claim 1. Let $i > 1$ and let $\mathscr{F}^i \cap H' \neq \emptyset$. Then

(i) For each $Y \in \mathscr{F}^i \cap H'$ there are at least $h + k - k^2 - 1$ sets in $\mathscr{F}^i$ disjoint from Y.

(ii) $f_i \geq h + k - k^2$.

Proof. Let $Y \in \boldsymbol{H}' \cap \mathcal{F}^i$ and let a denote the number of sets from $\boldsymbol{H} \setminus \{Y\}$ incident with Y. Since Y is a k-set and $d(y) \leqslant k$ for all $y \in Y$ we have $a \leqslant k(k-1)$. Clearly each $Z \in \boldsymbol{H}$ disjoint from Y belongs to $\mathcal{F}^i$ and so there are at least $h - a - 1 \geqslant h + k - k^2 - 1$ sets $Z \in \mathcal{F}^i$ disjoint from Y. □

Claim 2. If $\boldsymbol{F}$ does not satisfy (iii), then $|\boldsymbol{F}| \geqslant n + h + k - k^2 - 1$.

Proof. Suppose $f_1 \leqslant n + k - k^2 - 2$. In view of $h' \geqslant k^2 - k \geqslant 2$ some $\mathcal{F}^i$ meets $\boldsymbol{H}'$. Applying the assumption and Claim 1 we get

$$2 + f_1 + \cdots + f_{i-1} + f_{i+1} + \cdots + f_t = 2 + h + f_1 - f_i \leqslant n.$$

Thus we may assume $f_1 \geqslant n + k - k^2 - 1$ and $|\boldsymbol{F}| = f_1 + h \geqslant n + h + k - k^2 - 1$. □

4.4. We consider the case (α) $n = k^2$ and $k^2 \geqslant h \geqslant k^2 - 1$. We need

Claim 3. $\boldsymbol{F}$ consists of k-sets (i.e. $\boldsymbol{F} = \boldsymbol{G}$).

Proof. We prove the claim for $h = k^2 - 1$. For $h = k^2$ the proof is similar but simpler. Put $R := X \setminus (G \cup G')$. There is $(a, b) \in G \times G'$ such that each pair $(x, y) \in G \times G'$, $(x, y) \neq (a, b)$ determines a subset R_{xy} of R such that $R_{xy} \cup \{x, y\} \in \boldsymbol{H}$. For a fixed $x \in G \setminus \{a\}$ the family $\{R_{xy} : y \in G'\}$ consists of k pairwise disjoint subsets of R. Since $|R_{xy}| \geqslant k - 2$ and $|R| = k(k-2)$ the set $\{R_{xy} : y \in G'\}$ is a partition of R into k blocks of size $k - 2$ and all $Y \in \boldsymbol{H}$ not containing a have size k. By symmetry the same holds for $Y \in \boldsymbol{H}$ such that $b \notin \boldsymbol{H}$. Since $h = k^2 - 1$, this shows that $\boldsymbol{H} = \boldsymbol{H}'$.

Now let $Y \in \mathcal{F}^1$. If $Y \subset R$ then $|Y| = k$ because $|Y \cap R_{xy}| = 1$ for a fixed $x \neq a$ and all $y \in G'$. Suppose $Y \cap G = \{x\}$ where $x \neq a$. Then Y is disjoint from all R_{xy}'s leading to the contradiction $Y \setminus \{x\} \subseteq G$. Thus by symmetry we may assume that $Y \cap G = \{a\}$, $Y \cap G' = \{b\}$. Now $Y \setminus \{a, b\} \subseteq R \setminus \bigcup_{y \in G' \setminus \{b\}}$ [illegible], and therefore $|Y| \leqslant k$. □

By Claims 2 and 3 the family $\boldsymbol{F}$ is a family of k-sets on a k^2-set with $|\boldsymbol{F}| \geqslant k^2 + k - 2$. First we exclude the case $|\boldsymbol{F}| = k^2 + k - 2$. In this case $h = k^2 - 1$ by Claim 2 and $f_2 = k - 1$. By Claim 1(ii) we have $f_i \geqslant k - 1$ for all $2 \leqslant i \leqslant t$. Hence $t - 1 \leqslant h/(k-1) = k + 1$. If $t \leqslant k + 1$ then there exists an $f_i \geqslant \lceil (k^2 - 1)/(t-1) \rceil = k$ whence $2 + f_1 + \cdots + f_{i-1} + f_{i+1} + \cdots + f_t = 2 + h - f_i + f_1 \leqslant k^2 = n$.

Suppose $t = k + 2$. Then $f_1 = f_2 = \cdots = f_{k+2} = k - 1$ and by Claim 1(i) each $\mathcal{F}^i$ consists of disjoint sets. Now every degree in $\boldsymbol{H} = \mathcal{F}^2 \cup \cdots \cup \mathcal{F}^{k-2}$ is at most k by (2.2). Hence we have for arbitrary $F \in \mathcal{F}^1$

$$k^2 - 1 = |\boldsymbol{H}| = \sum_{x \in F} d_H(x) \leqslant k^2.$$

This implies that exactly one element of F has degree $k - 1$ (in $\boldsymbol{H}$), the others have degree k. Since $\mathcal{F}^1$ consists of disjoint sets, exactly one point of F has degree k in $\boldsymbol{F}$ and the others have degree $k+1$. The same holds for every $1 \leq j \leq k+2$ and $E \in \mathcal{F}^j$. Denote by V the set of elements with degree k. Then

$$|V|k + (n - |V|)(k+1) = \sum_{x \in X} d_{\boldsymbol{F}}(x) = \sum_{E \in \boldsymbol{F}} |E| = |\boldsymbol{F}|k.$$

Hence $|V| = 2k$. But each $F \in \boldsymbol{F}$ intersects V in one element, whence $2k \cdot k = \sum_{x \in V} d_{\boldsymbol{F}}(x) = |\boldsymbol{F}| = k^2 + k - 2$, a contradiction.

From now on we can suppose that $\boldsymbol{F}$ is a family of k-sets on k^2 elements with $|\boldsymbol{F}| \geq k^2 + k - 1$. The following lemma shows that these numerical parameters alone determine $\boldsymbol{F}$. The statement is a consequence of results by Stinson [7] and Dow [3]. Very similar results are in [4].

Lemma 4.4. *Let $|X| = k^2$ and let $\boldsymbol{F}$ be a 0–1 intersecting family of k-subsets of X. If $|\boldsymbol{F}| \geq k^2 + k - 1$, then either $\boldsymbol{F}$ is an affine plane of order k or $\boldsymbol{F}$ is obtained from such a plane by removing one line.*

With the lemma we have finished the case (α) and so can turn to the case (β).

4.5. Let $n = k^2 + 1$ and $h = k^2$. By Claim 2 we may assume $|\boldsymbol{F}| \geq k^2 + k$. We start with:

Claim 4. There are at most k sets $Y \in \boldsymbol{H}$ with $|Y| = k+1$ and all the others are k-sets. The k-sets in $\mathcal{F}^1$ are pairwise disjoint.

Proof. As in the proof of Claim 3 we can show that there are subsets R_{xy} of $X \setminus (G \cup G')$ of cardinality $k-2 \leq |R_{xy}| \leq k-1$ such that (A) $\boldsymbol{H} = \{\{x,y\} \cup R_{xy} : x \in G, y \in G'\}$ and (B) for each $x \in G$ the sets $R_{xy} (y \in G')$ are pairwise disjoint and $|R_{xy}| = k-1$ for at most one $y \in G'$. Let $Y \in \mathcal{F}^1$ with $G \neq Y \neq G'$. Suppose $Y \cap G = \{a\}$ and let $x \in G \setminus \{a\}$. Then Y meets each $\{x,y\} \cup R_{xy} \in \boldsymbol{H}$ in singleton distinct from $\{a\}$ and so $|Y| > k$. By symmetry for a k-set Y we obtain $Y \cap G = Y \cap G' = \emptyset$. Since G and G' can be arbitrary disjoint k-sets in $\mathcal{F}^1$ this proves the second statement. □

Claim 5. $|\mathcal{F}^i \cap G| \geq k$ for some $1 \leq i \leq t$.

Proof. Put $\psi_l := |\mathcal{F}^l \cap G|$ and suppose $\psi_l > 0$ for $l = 1, \ldots, s$ and $\psi_l = 0$ for $l = s+1, \ldots, t$. Assume that $\psi_l \leq k-1$ for $l = 1, \ldots, s$. By Claim 1 we have $f_l \geq k$ for $l = 2, \ldots, s$, hence

$$k^2 = h \geq f_2 + \cdots + f_s \geq (s-1)k \tag{4.4}$$

and therefore $k+1 \geq s$. On the other hand

$$(s-1)(k-1) \geq \psi_2 + \cdots + \psi_s = h' \geq k^2 - k \tag{4.5}$$

proves $s-1\geqslant k$ and so $s=k+1$. Now (4.5) combined with $\psi_l \leqslant k-1$ gives $\psi_2=\cdots=\psi_{k+1}=k-1$. Similarly (4.4) together with $f_l\geqslant k$ yields $f_2=\cdots=f_{k+1}=k$. We show that the sets in $\mathcal{F}^l$ $(l=2,\ldots,k+1)$ are pairwise disjoint. Indeed suppose there are two sets Y, $Y'\in\mathcal{F}^l$ with $|Y\cap Y'|=1$. We may assume that Y is a k-set. Then there are at most $k-2$ sets in $\mathcal{F}^l$ disjoint from Y in contradiction to Claim1(i). Thus the sets in $\mathcal{F}^l$ are pairwise disjoint. There are $k-1$ sets and one set of cardinality $>k$. Since $|X|=k^2+1$ it follows that the sets in $\mathcal{F}^l$ partition X. Now $k+1\geqslant 3$. Pick a set $Y\in\mathcal{F}^3$. It intersects the members of $\mathcal{F}^2$ in exactly one point, hence $|Y|=k$. This contradicts $f_3=k>k-1=\psi_3$.

Now we can complete the proof of Theorem 2.5. By Claims 4 and 5 we may assume that some block, say $\mathcal{F}^1$ contains k pairwise disjoint k-sets $Y_1,\ldots,Y_k$. Set $Y:=Y_1\cup\cdots\cup Y_k$. Then $|Y|=k^2$ and so $X\setminus Y=\{x\}$. Each $Z\in\mathcal{F}^i$ satisfies $|Z\cap Y|=k$ $(i=2,\ldots,t)$. Let F' denote the restriction of F to Y. Clearly F' is a family of at least k^2+k subsets (of cardinality k) of the k^2-set Y. Applying Lemma 4.4 we obtain that F' is an affine plane of order k. Let $\mathcal{L}^1,\ldots,\mathcal{L}^{k+1}$ be the parallel blocks of F'. It is easy to see that for all $Z\in F$ containing x the sets $Z\cap Y$ belong to the same parallel block, say $\mathcal{L}^{k+1}$. Since F' is an affine plane, we have $\mathcal{F}^i=\mathcal{L}^i$ for $i=1,\ldots,k$. Now it is easy to see that F is either the family from Example 2.4 or the dual of Example 1.12. □

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