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Dimension Versus Size

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Abstract. We investigate the behavior of $f(d)$, the least size of a lattice of order dimension d. In particular we show that the lattice of a projective plane of order n has dimension at least $n/\ln(n)$, so that $f(d) = O(d^2 \log^2 d)$. We conjecture $f(d) = \theta(d^2)$, and prove something close to this for height-3 lattices, but in general we do not even know whether $f(d)/d \to \infty$.

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1. Introduction and Results

We will be concerned in this paper with how small a lattice can be relative to its dimension. For our purposes a *linear extension* of a poset P is an order-preserving bijection

$$\sigma: P \to \{1, \ldots, |P|\}.$$

The (order) *dimension* of P, denoted $\dim P$, is the least s for which there exist linear extensions $\sigma_1, \ldots, \sigma_s$ of P such that for all $p, q \in P$ with $p \not< q$ there exists $i \in \{1, \ldots, s\}$ with $\sigma_i(p) > \sigma_i(q)$. For more information on dimension see [4] or [5].

A venerable theorem of Hiraguchi [3] states that if $\dim P \geqslant 3$, the size of P is a least twice its dimension (this bound being attained for $\dim P = d$ by the poset of 1- and $(d-1)$-element subsets of a d-element set, ordered by containment).

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For lattices the situation is completely different, and far more complicated; the problem of finding lattice analogues of Hiraguchi's theorem was recently raised by B. Sands (see [2]). Let us denote by $f(d)$ the least size of a d-dimensional lattice. Sands asked whether $f(d) \geqslant 2^d$, i.e., whether the Boolean algebra of order d is the smallest d-dimensional lattice. This was answered in the negative by Ganter *et al.* [2]. Let π_n denote the lattice of partitions of an n-set and $L_n(q)$ the lattice of subspaces of an n-dimensional vector space over GF(q). It is shown in [2] that

$$\frac{3}{8}\binom{n}{2} \leqslant \dim \pi_n \leqslant \binom{n}{2}, \tag{1.1}$$

$$\frac{2}{n+1}(2^n - 1) \leqslant \dim L_n(2) \leqslant 2^n - 1. \tag{1.2}$$

Either of these shows that $f(d)$ grows much more slowly than 2^d, and in particular (1.2) gives

$$f(d) < c^{\log^2 d}.$$

Ganter *et al.* in turn asked for better bounds on $f(d)$, and specifically whether $f(d)$ is bounded by a polynomial in d. Here we answer this in the affirmative:

THEOREM 1.3. *If $\mathscr{P}_n$ is the lattice of a projective plane of order n, then* $\dim \mathscr{P}_n > n/2 \ln(n)$.

COROLLARY 1.4. $f(d) = 0\,(d^2 \log^2 d)$.

An upper bound of $2n + 2$ on $\dim \mathscr{P}_n$ was shown to us by K. Reuter and appears to us to be closer to the truth. Of course this would give $f(d) = O(d^2)$ and we (somewhat recklessly) propose.

CONJECTURE 1.5. $f(d) = \theta(d^2)$.

In fact we cannot even show $f(d)/d \to \infty$, though this seems certain to be the case. We mention one small step in the direction of the conjecture (recalling that the *height* of a poset is one less than the size of a largest chain).

PROPOSITION 1.6. *If $\mathscr{L}$ is a lattice of height 3 then $\dim \mathscr{L} = 0(|\mathscr{L}|^{1/2}$ $\log |\mathscr{L}|)$.*

2. Proofs

Let us denote by P and L the point and line sets of the projective plane associated with $\mathscr{P}_n$. For the lower bound in Theorem 1.3, note that as there are $(n^2 + n + 1)\,n^2$ nonincident pairs $(p, l) \in P \times L$, it suffices to prove

LEMMA 2.1. *For any linear extension σ of $\mathscr{L}$ there are at most $n^3 \ln(n^2 + n + 1)$ pairs $(p, l) \in P \times L$ for which $\sigma(l) < \sigma(p)$.*

Proof. We need the following useful result of Corradi (see [6, prob. 13.13]).

(2.2) *If $\mathcal{F}$ is a family of subsets of a set X, with $F, G \in \mathcal{F} \Rightarrow |F| \geqslant k$, $|F \cap G| \leqslant \lambda$, then*

$$|X| \geqslant \frac{k^2 |\mathcal{F}|}{k + (|\mathcal{F}| - 1)\lambda}.$$

This implies (taking $\mathcal{F} = L_0$, $X = P \backslash P_0$).

(2.3) *If $P_0 \subset P$ and $L_0 \subset L$ satisfy $p \notin l \, \forall \, p \in P_0$, $l \in L_0$, then* $(|P_0| + n)(|L_0| + n) \leqslant n(n+1)^2$.

Now number the lines of L so that

$$\sigma(l_1) < \cdots < \sigma(l_{n^2+n+1}).$$

If $\sigma(p) > \sigma(l_i)$ then $p \notin \bigcup_{j=1}^{i} l_j$, so by (2.3)

$$|\{p : \sigma(p) > \sigma(l_i)\}| \leqslant \left[\frac{n(n+1)^2}{i+n}\right] - n.$$

The Lemma and Theorem follow after a little calculation for $n \geqslant 5$. For $n \leqslant 4$, $[n/\ln(n)] = 2$ and trivially $\dim \mathcal{P}_n > 2$. □

REMARK. As far as we know the correct upper bound in Lemma 2.1 could be $O(n^3)$, which would give $\dim \mathcal{P}_n = \theta(n)$, in agreement with Conjecture 1.5.

Proof of Proposition 1.6. We denote by 0 and 1 the minimum and maximum elements of $\mathcal{L}$, and by $L_0(L_1)$ the set of elements covering 0 (covered by 1). Obviously we may assume $L_0 \cap L_1 = \emptyset$.

As in [1], to show $\dim \mathcal{L} \leqslant s$ we need only find permutations $\sigma_1, \ldots, \sigma_s$ of L_0 satisfying

(2.4) *for all $p \in L_0$, $l \in L_1$, with $p \not< l$ there exists $i \in \{1, \ldots, s\}$ such that $\sigma_i(p) > \sigma_i(q)$ for all $q < l$.*

Let $n = \max\{|L_0|, |L_1|\}$. If we choose $\sigma_1, \ldots, \sigma_r$, $r = 4n^{1/2} \ln(n)$, at random, then with positive probability (2.4) holds for (p, l) whenever

$$|\{q \in L_0 : q < l\}| < 2\sqrt{n} \tag{2.5}$$

(see e.g. [1]). But this excludes only a small subset of L_1:

$$|\{l \in L_1 : l \text{ violates } (2.5)\}| < 2\sqrt{n}. \tag{2.6}$$

(To see this, note that L_1 may be regarded as a collection of subsets of L_0, no two having more than one element in common, and apply (2.2).) We may thus choose $\sigma_1, \ldots, \sigma_r$ so that (2.4) holds whenever (2.5) is true, and add to these for each l violating (2.5) a permutation σ_l satisfying

$$\sigma_l(q) < \sigma_l(p) \quad \forall q < l,\ p \nless l$$

to obtain the desired set of $O(|\mathscr{L}|^{1/2} \log |\mathscr{L}|)$ permutations.

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