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Rectangular Dissections of a Square*

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We investigate the problem that how many different ways one can dissect the unit-square into rectangles with prescribed areas $w_1, \ldots, w_n$. One of our answers is the following: If $w_1, \ldots, w_{n-1}$ are algebraicly independent, then the number in the question asymptotically equals to $32(1 + o(1))/\pi\sqrt{3}\,(n!8^n/n^4)$.

1. INTRODUCTION

Thomas Ihringer proposed the following problems [6]:

1. Are there, for every $n \in \mathbb{N}$, only finitely many possibilities to dissect a square into rectangles of equal area?
2. If 'yes', give for every $n \in \mathbb{N}$ the number $f(n)$ of possibilities.

Problem 1 was solved in a more general case. Considering the dissections of the unit square into n rectangles having given areas $w_1, \ldots, w_n$ the same questions can be asked. The finiteness of the number of such dissections was proved in [1, 3, 9], even in higher dimensions, see [1]. In connection with problem 2, an upper bound $O(c^n)$ was given in [3]. But if $w_i \neq w_j$ for $i \neq j$, then dissecting the unit square with lines, parallel to one of the axes gives already $n!$ *different* dissections.

In this paper we give a characterization of the possible dissections and prove, e.g., that the number of dissections for almost all $w_1, \ldots, w_n$ is

$$\frac{32(1 + o(1))}{\pi\sqrt{3}} \frac{n!8^n}{n^4}$$

2. NOTATIONS AND RESULTS

Let $U = \{(x, y) | 0 \leqslant x, y \leqslant 1\}$ denote the unit square, and $\mathbf{D}_n$ denote the set of dissections of U into n rectangles. Here a *dissection* means a finite set D of rectangles, the sides of which are parallel to that of U, such that $\bigcup_{R \in D} R$ covers U, *interior* $(R) \cap$ *interior*$(R') \neq \emptyset$ for $R \neq R' \in D$, and *area*$(R) > 0$ for all $R \in D$.

Consequently $\Sigma_{R \in D}$ *area*$(R) = 1$.

Let us denote W an n-element collection of positive reals with the following properties: W contains s different values $w_1, \ldots, w_s$ with occurrences $n_1, \ldots, n_s$ respectively. $n_i \geqslant 1$, $\Sigma_{i=1}^s n_i = n$ and $\Sigma_{i=1}^s n_i w_i = 1$. $(n_1, \ldots, n_s)$ is called the multiplicity of W.

Let $f(W)$ denote the number of dissections of the unit square into n rectangles having areas prescribed by W. Denote the n-nomial binomial coefficient $n!/\Pi_{i=1}^s n_i!$ by $\binom{n}{n_1, \ldots, n_s}$.

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THEOREM 2.1. *Suppose that W is a set of positive reals with multiplicity $n_1, \ldots, n_s$ satisfying the above properties. Then for the number of distinct dissections we have*

$$f(W) \leqslant \binom{n}{n_1, \ldots, n_s} \frac{2}{n(n+1)^2} \sum_{i=1}^{n} \binom{n+1}{i-1}\binom{n+1}{i}\binom{n+1}{i+1}. \tag{1}$$

Denote by M_n the value of the essential part of the right hand side of (1), i.e., $M_n = [2/n(n+1)^2] \sum_{i=1}^{n} \binom{n+1}{i-1}\binom{n+1}{i}\binom{n+1}{i+1}$. A standard calculation gives:

PROPOSITION 2.2. $M_n = [32(1 + o(1))/\pi\sqrt{3}](8^n/n^4)$ *whenever* $n \to \infty$.

THEOREM 2.3.

$$f(W) \geqslant \binom{n}{n_1, \ldots, n_s} \frac{M_n}{2^{n-1}}.$$

COROLLARY 2.4. *For $f(n)$ introduced in Section 1 we have*

$$4^{n-o(n)} < f(n) \leqslant M_n.$$

CONJECTURE 2.5. *If $n \to \infty$ then*

$$f(W) = (1 - o(1)) \binom{n}{n_1, \ldots, n_s} M_n$$

for every collections W.

Especially we expect that $f(n) = (1 - o(1))M_n$.

We recall that the reals $a_1, \ldots, a_n$ are algebraically independent if for every polynomial $P(x_1, \ldots, x_n) \neq 0$ with integer coefficients $P(a_1, \ldots, a_n) \neq 0$.

THEOREM 2.6. *Let W be an n-element set of positive reals with $\Sigma_{w \in W}\, w = 1$. Suppose that any $n - 1$ of them are algebraically independent. Then*

$$f(W) = n!M_n.$$

This gives

COROLLARY 2.7. *Our Conjecture* (2.5) *is true even with equality for almost all sets W.*

3. THE COMBINATORIAL TYPE

Consider the dissection $D \in \mathbf{D}_n$. We will define its combinatorial type. (It is possible that D has more than one types.) This type $\mathbf{T}(D)$ will be a sequence of pairs $\{\varepsilon_i, T_i\}$, where $\varepsilon_i \in \{(0, 1), (1, 0)\}$ and T_i is a subset of $\{1, 2, \ldots, n\}$ for $2 \leqslant i \leqslant n$.

Let $b(D)$ denote the set of boundary points of rectangles of D.

Let $c(R)$ denote the lower-left corner of the rectangle R and set $C(D) = \{c(R) | R \in D\} \backslash \{(0, 0)\}$.

We will say that a direction $\varepsilon \in \{(0, 1), (1, 0), (0, -1), (-1, 0)\}$ *appears* at the point $c(R) \in C(D)$, if the small open segment $I(R) = \{c(R) + \lambda\varepsilon | 0 < \lambda <$ *min side length in D*$\}$ is contained in $b(D)$. Hence in each $c(R)$ at least 3 directions appear; namely $\{(0, 1), (1, 0)\}$ always and at least one from $\{(0, -1), (-1, 0)\}$. Now choose the direction $\varepsilon(R)$ to a

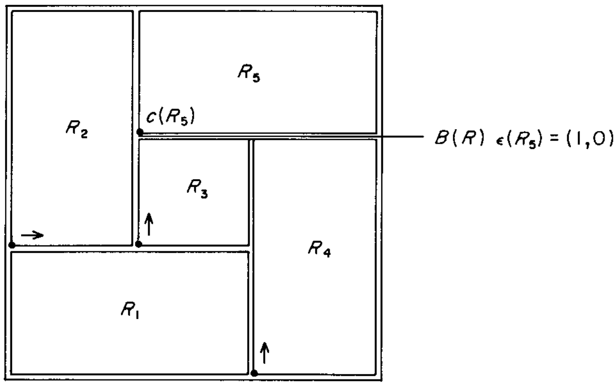

FIGURE 1. Corners, directions and base lines of rectangles in a dissection.

rectangle $R \in D$ arbitrarily from $\{(0, 1), (1, 0)\}$ if all the four directions appear at $c(R)$, otherwise choose the opposite of the missing direction. Then let $B(R)$ denote the longest open segment in the form $B(R, A) = \{(c(R) + \lambda\varepsilon(R)|0 < \lambda < A\}$ having the property that $B(R, A) \cap I(R') = \emptyset$ if $R' \neq R$. $B(R)$ is called the *base* segment of rectangle R, (although it might be longer, than the length of the adjacent side of R). $|B(R)|$ denotes the length of $B(R)$. (see Figure 1).

PROPOSITION 3.1. *The base segments $B(R)$ form a decomposition (so called* base decomposition) *of boundaries inside U, i.e.*

$$\bigcup_{R \in D} B(R) \quad = \quad b(D) \cap int(U).$$

Now we are going to define the *canonical labelling* of a dissection D corresponding to a given base-decomposition $\{B(R)|R \in D\}$. Let R_n be the up-right rectangle in D. If $R_n, \ldots, R_{i+1}$ are already defined, then let $R_i \in D\backslash\{R_n, \ldots, R_{i+1}\}$ that rectangle for which $B(R_{i+1}) \cap R_i \neq \emptyset$ and $B(R_{i+1})\backslash R_i$ is an initial segment of $B(R_{i+1})$. Finally let $T_i(D)$ the set of indices $j < i$ for which $B(R_i) \cap R_j \neq \emptyset$.

Let the sequence $\{\varepsilon(R_i), T_i(D)\}$ be the *combinatorial type* of D. It is unique up to the choice of $\varepsilon(R_i)$'s, hence

PROPOSITION 3.2. *For any given dissection D of n rectangles D has at most 2^{n-1} combinatorial types.*

For example the dissections $2a$ and $2b$ in Figure 2 have different, $2c$ and $2d$ have the same combinatorial types.

A sequence $\{\varepsilon_i, T_i\}$ is a *feasible type* of order n, if there is a dissection $D \in \mathbf{D}_n$ and a labelling of its rectangles $\{R_1, \ldots, R_n\}$ such that $\varepsilon_i = \varepsilon(R_i)$ and $T_i = T_i(D)$.

THEOREM 3.3. *The number of feasible types of order n is M_n.*

Our main result is the following.

THEOREM 3.4. *Let $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ such that $\Sigma_{i=1}^n w_i = 1$, $w_i > 0$, and let $\{\varepsilon_i, T_i\}$ be a feasible type of order n. Then there exists a unique dissection D and a labelling of its rectangles such that it has this type and area $(R_i) = w_i$, for $i = 1, \ldots, n$.*

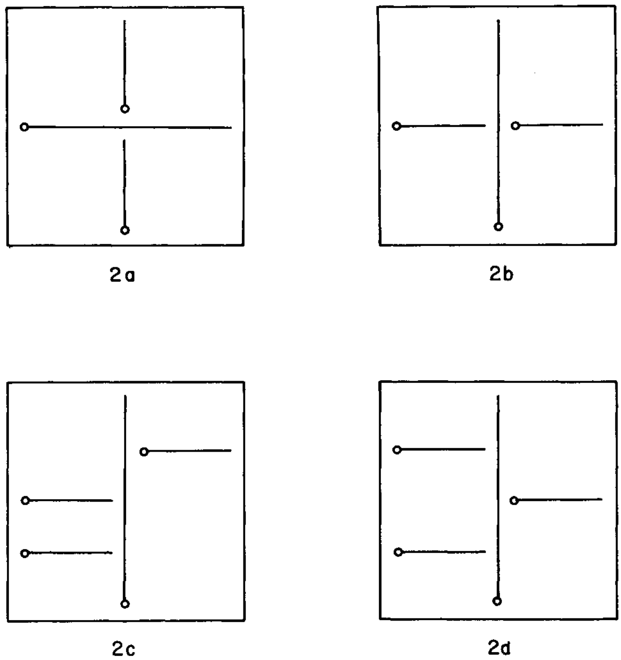

FIGURE 2. Examples of combinatorial types.

The uniqueness of the solution with a given type and areas follows from the earlier results, see [1, 3, 9]. The main point of this theorem is the existence of such a dissection.

Clearly Theorem 2.1 is an easy consequence of Proposition 3.2 and Theorem 3.3, and Theorem 2.3 is a corollary of Propositions 3.1, 3.2 and Theorem 3.3

4. The Existence of a Dissection with Given Areas and Types

In this section we prove Theorem 3.4. For this a few lemmas are needed. An important, but easy to prove lemma is the following.

LEMMA 4.1. *If* $\{(\varepsilon_i, T_i) | i = 2, \ldots, n\}$ *is a feasible type, then* $\{(\varepsilon_i, T_i) | i = 2, \ldots, n - 1\}$ *is also a feasible type.*

PROOF. Let D be a dissection of the given type of n rectangles, and let $\varepsilon \in \{(0, 1), (1, 0)\}$ be orthogonal to ε_n. Then R_n can be eliminated from D by moving the base wall $B(R_n)$ in the direction ε and continuing the rectangles R_i, $i \in T_n$ in this direction. The resulting dissection contains $n - 1$ rectangles and has the type $\{(\varepsilon_i, T_i) | i = 2, \ldots, n - 1\}$.

For example the dissection in Figure 1 has 5 rectangles and has the type $\{((1, 0), \{1\}), ((0, 1), \{2\}), ((0, 1), \{1, 3\})\}$. There $\varepsilon_5 = (1, 0)$, thus moving the base wall $B(R_5)$ in to upward, i.e. in the direction $\varepsilon = (0, 1)$, R_5 can be eliminated from that dissection. The resulted dissection of 4 rectangles is given on Figure 3.

If D is a dissection of $\mathbf{D}_n$, then denote $x_i = x(R_i)$, $y_i = y(R_i)$ the lengths of sides of rectangle $R_i \in D$.

The following lemma can easily be verified, see [1, 3].

LEMMA 4.2. *Let* $D = \{R_1, \ldots, R_n\}$ *and* $D' = \{R'_1, \ldots, R'_n\}$ *be two dissections having the same combinatorial type. Then there is a dissection* $D'' = \{R''_1, \ldots, R''_n\}$ *of the same type, having* $x(R''_i) = \frac{1}{2}(x(R_i) + x(R'_i))$ *and* $y(R''_i) = \frac{1}{2}(y(R_i) + y(R'_i))$ *for* $i = 1, \ldots, n$.

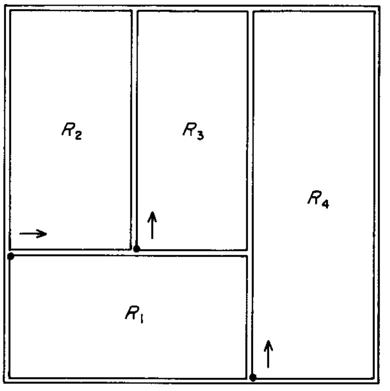

FIGURE 3. The dissection obtained by deleting R_5 from Figure 1.

PROOF. Consider the arithmetical mean of D and D', i.e. let $c(R_i'') = \frac{1}{2}(c(R_i) + c(R_i'))$, for $i = 1, \ldots, n$.

The following lemma states the visible fact that fixing the type of dissection, the side lengths of its rectangles vary continuously with the change of areas.

LEMMA 4.3. *Let $D = \{R_1, \ldots, R_n\}$ and $D' = \{R_1', \ldots, R_n'\}$ be two dissections having the same combinatorial type with areas w_i = area (R_i), w_i' = area (R_i') $i = 1, \ldots, n$, and suppose that $|w_i - w_i'| \leqslant \varepsilon$ for $i = 1, \ldots, n$ with some positive real ε. Then*

$$\sum_{i=1}^{n} (x_i - x_i')^2 \leqslant \frac{\varepsilon n}{aa'} \quad \text{and} \quad \sum_{i=1}^{n} (y_i - y_i')^2 \leqslant \frac{\varepsilon n}{aa'},$$

where x_i, y_i (resp. x_i', y_i') are the side lengths of R_i (resp. R_i') $i = 1, \ldots, n$, and a (resp. a') is the minimal area in D (resp in D').

PROOF. By the symmetry of vertical and horisontal sides, it is sufficient to prove the first inequality.

As D and D' have the same combinatorial type, Lemma 4.2 can be applied and

$$\sum_{i=1}^{n} (y_i - y_i')(x_i - x_i') = 0 \tag{2}$$

follows.

Using the equation $w_i - w_i' = (y_i - y_i')x_i' + (x_i - x_i')y_i$ and the inequalities $-\varepsilon \leqslant w_i - w_i' \leqslant \varepsilon$ we have

$$\frac{\varepsilon - y_i(x_i - x_i')}{x_i'} \geqslant y_i - y_i' \geqslant \frac{-\varepsilon - y_i(x_i - x_i')}{x_i'}. \tag{3}$$

Introducing $I = \{i | x_i \geqslant x_i'\}$ and $J = \{i | x_i < x_i'\}$ and adding up $(x_i - x_i')$ times (3) for $i \in I \cup J$ we get

$$\sum_{i \in I} \frac{\varepsilon(x_i - x_i') - y_i(x_i - x_i')^2}{x_i'} + \sum_{i \in J} \frac{-\varepsilon(x_i - x_i') - y(x_i - x_i')^2}{x_i'} \geqslant \sum_{i=1}^{n} (y_i - y_i')(x_i - x_i').$$

The right hand side here is 0 by (2). From this it follows that

$$\varepsilon \sum_{i=1}^{n} \frac{|x_i - x_i'|}{x_i'} \geqslant \sum_{i=1}^{n} \frac{y_i}{x_i'} (x_i - x_i')^2.$$

Here $|x_i - x_i'| \leqslant 1$, $a' \leqslant x_i' \leqslant 1$ and $y_i \leqslant a$. Substituting these, the lemma follows immediately.

PROOF OF THEOREM 3.4. The uniqueness follows from Lemma 4.2, see [1, 3].

Suppose on the contrary that $D = \{R_1, \ldots, R_n\}$ and $D' = \{R_1', \ldots, R_n'\}$ are two dissections with the same combinatorial type, and with the same areas $w_i = area(R_i) = area(R_i')$ for $i = 1, \ldots, n$. Then Lemma 4.2 can be applied and there is a third dissection $D'' = \{R_1'', \ldots R_n''\}$ with side lengths $x_i'' = (x_i + x_i')/2$ and $y_i'' = (y_i + y_i'/2)$.

By the arithmetic–geometric inequality

$$\frac{x_i y_i' + x_i' y_i}{2} \geqslant \sqrt{x_i y_i' x_i' y_i} = w_i = \frac{x_i y_i + x_i' y_i'}{2}$$

hold for $i = 1, \ldots, n$, hence

$$area(R_i'') = \frac{x_i + x_i'}{2} \times \frac{y_i + y_i'}{2} \geqslant w_i$$

for $i = 1, \ldots, n$. But $\Sigma_{i=1}^n\, area(R_i'') = 1 = \Sigma_{i=1}^n\, w_i$, thus the equality hold in the above inequalities for $i = 1, \ldots, n$. From these equations $x_i = x_i'$ and $y_i = y_i'$ $i = 1, \ldots, n$ follow immediately, and so the identity of D and D'.

The existence will be proved by induction on n, using a fix point argument, and the continuity lemma 4.3.

For $n = 1$ the statement clearly holds. Suppose now that Theorem 3.4 holds for every $n' < n$.

At first we introduce a few necessary notations. If $J \subset \{1, 2, \ldots, n\}$, then let $S_J \overset{\text{def}}{=} \{\alpha = (\alpha_j | j \in J) | \Sigma_{j \in J}\, \alpha_j = 1, \alpha_j \geqslant 0\}$. For any $\alpha \in S_{T_n}$ let w^α denote the vector defined by

$$w_j^\alpha = \begin{cases} w_j, & \text{if } j \notin T_n \\ w_j + \alpha_j w_n, & \text{if } j \in T_n \end{cases}$$

Now for any $\alpha \in S_{T_n}$, $w^\alpha \in \mathbb{R}^{n-1}$ and $\Sigma_{j=1}^{n-1}\, \omega_j^\alpha = 1$.

Moreover $\{(\varepsilon_i, T_i) | i = 2, \ldots, n - 1\}$ is also a feasible type by Lemma 4.1. Therefore by the induction hypothesis there is a dissection D^α having this type and having areas given in w^α.

Define

$$\beta_j(\alpha) \overset{\text{def}}{=} \begin{cases} \dfrac{x(R_j)}{\sum\limits_{k \in T_n} x(R_k)}, & \text{if } \varepsilon_n = (1, 0) \\[2ex] \dfrac{y(R_j)}{\sum\limits_{k \in T_n} y(R_k)}, & \text{if } \varepsilon_n = (0, 1) \end{cases}$$

for $j \in T_n$, $R_j \in D^\alpha$, and let $\beta(\alpha) = (\beta_j(\alpha) | j \in T_n)$.

It is clear that for every $\alpha \in S_{T_n}$: $\beta(\alpha) \in S_{T_n}$.

If for some $\alpha \in S_{T_n}$ it happens that $\beta(\alpha) = \alpha$, then a required dissection can be obtained from D^α by a cut, parallel to α_n along the rectangles of $R_j \in D^\alpha, j \in T_n$. Thus the theorem will proved if we can show a fixpoint, $\alpha \in S_{T_n}$ with $\beta(\alpha) = \alpha$.

The mapping $\alpha \to \beta(\alpha)$ is continuous over the compact set S_{T_n} by Lemma 4.2, since $w_j^\alpha \geqslant w_j > 0$ for any $\alpha \in S_{T_n}$. Hence there exists such a fixpoint, and the theorem is proved.

5. FEASIBLE TYPES

In this section we give a characterisation of feasible types.

Let $\mathrm{T} = \{(\varepsilon_i, T_i) | i = 2, \ldots, n\}$ denote a feasible type and denote t_i the cardinality $|T_i|$.

LEMMA 5.1. *Let* $\mathbf{T} = \{(\varepsilon_i, T_i) | i = 2, \ldots, n\}$ *be a feasible type*, $t_i = |T_i|$. *Then the following inequalities hold for* $k = 2, \ldots, n$.

$$t_k \leqslant \begin{cases} k - 1 - \sum_{\varepsilon_i = (0,1)}^{i<k} t_i, & \text{if } \varepsilon_k = (0, 1), \\ k - 1 - \sum_{\varepsilon_i = (1,0)}^{i<k} t_i, & \text{if } \varepsilon_k = (1, 0). \end{cases} \tag{4}$$

Moreover, if there is a collection $\{(\varepsilon_i, t_i) | i = 2, \ldots, n\}$ *satisfying* (4), *then there is a unique feasible type* $\mathbf{T} = \{(\varepsilon_i, T_i) | i = 2, \ldots, n\}$ *with* $t_i = |T_i|$.

PROOF. Let $\mathbf{T}(k) = \{(\varepsilon_i, T_i) | i = 2, \ldots, k\}$. Then by Lemma 4.1 $\mathbf{T}(k)$, $k = 2, \ldots, n$ are also feasible types. In the proof of Lemma 4.1 we gave a geometrical interpretation of the mapping $\mathbf{T}(k + 1) \to \mathbf{T}(k)$. From this it can be clear that the number of rectangles of $\mathbf{T}(k - 1)$ touching the upper (resp. right) side of U is given by $k - 1 - \Sigma_{\varepsilon_i = \hat{\varepsilon}}^{i<k} t_i$ with $\hat{\varepsilon} = (1, 0)$ (resp. $\hat{\varepsilon} = (0, 1)$). These prove the inequalities (4). Moreover T_k contains exactly the indices of t_k rectangles closest to the up-right corner (1, 1) of U touching the upper (in case of $\varepsilon_k = (1, 0)$) or right (in case of $\varepsilon_k = (0. 1)$) side of U. From this the second part of the lemma follows.

On a planar walk (from (0, 0) to (v, h)) we mean a sequence of $v + h$ vectors of $\{(0, 1), (1, 0)\}$, the sum of which is equals to (v, h). The points formed by the partial sums is considered as points of the walk.

For the proof of Theorem 3.3 we will show that to each feasible type there corresponds a unique triplet of planar walks, that are non-crossing.

Let $\mathbf{T} = \mathbf{T}(n) = \{(\varepsilon_i, T_i) | i = 2, \ldots, n\}$ be given a feasible type. Moreover let $v(\mathbf{T}) = |\{i | \varepsilon_i = (1, 0)\}|$ and $h(\mathbf{T}) = |\{i | \varepsilon_i = (0, 1)\}|$.

We will construct walks $W_0 = W_0(\mathbf{T})$, $W_- = W_-(\mathbf{T})$ and $W_+ = W_+(\mathbf{T})$ all from (0, 0) to $(v(\mathbf{T}), h(\mathbf{T}))$.

Let W_0, the so called *middle walk*, be formed by the steps $\varepsilon_2, \ldots, \varepsilon_n$.

The so called *upper walk*, W_+ is given by the steps $((t_i - 1)$ *times* $(1, 0)) + (0, 1)$ whenever $\varepsilon_i = (0, 1)$, $i = 2, \ldots, n$.

The *lower walk*, W_- will be given similarly, by $((t_i - 1)$ *times* $(0, 1)) + (1, 0)$ whenever $\varepsilon_i = (1, 0)$, $i = 2, \ldots, n$.

The last point of the upper (resp. lower) walk has the form $(\alpha, h(\mathbf{T}))$ (resp. $(v(\mathbf{T}), \beta)$). Finally connect these points to $(v(\mathbf{T}), h(\mathbf{T}))$ by the appropriate number of steps (1, 0) resp.(0, 1)).

For example in the case of the dissection given in Figure 1,

$$\begin{aligned} \varepsilon_2 &= (1, 0), & T_2 &= \{1\}, \\ \varepsilon_3 &= (0, 1), & T_3 &= \{2\}, \\ \varepsilon_4 &= (0, 1), & T_4 &= \{1, 3\}, \\ \varepsilon_5 &= (1, 0), & T_5 &= \{3, 4\}. \end{aligned}$$

moreover $v(\mathbf{T}) = 2$ and $h(\mathbf{T}) = 2$, thus the corresponding walks from (0, 0) to (2, 2) are as follows (see Figure 4):

$$\begin{aligned} W_0 &= \{(1, 0), (0, 1), (0, 1), (1, 0)\}, \\ W_- &= \{(0, 1)\} \cup \{(1, 0), (0, 1)\} \cup \{(1, 0)\}, \\ W_+ &= \{(1, 0)\} \cup \{(0, 1), (1, 0)\} \cup \{(0, 1)\}. \end{aligned}$$

LEMMA 5.2. *If* $\mathbf{T}$ *is a feasible type, then the walks* $W_0(\mathbf{T})$, $W_-(\mathbf{T})$ *and* $W_+(\mathbf{T})$ *are non-crossing. Precisely to each point* (x, y) *of* W_0 *there are points* (η, y) *of* W_- *and* (x, ξ) *of* W_+, *with* $\eta \leqslant x$ *and* $\xi \leqslant y$.

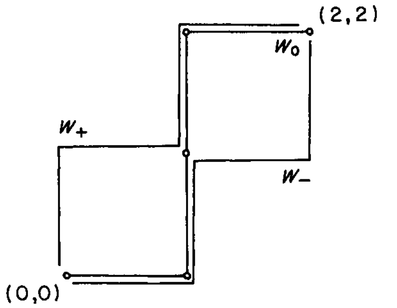

FIGURE 4. The upper, middle and lower walks corresponding to the dissection given in Figure 1.

PROOF. By the symmetry it is sufficient to show this relation between W and W_-.

If $h(\mathrm{T}) = 0$, then these walks coincide and the statement is trivial. Thus consider the cases $h(\mathrm{T}) > 0$, and let (x, y) be an arbitrary point of W_0. Moreover consider the point (η, y) of W_- just after the yth occurrence of $\varepsilon_i = (0, 1)$. Then

$$\eta = \sum_{\varepsilon_i=(0,1)}^{i \leqslant x+y} (t_i - 1)$$

by the definition. On the other hand

$$x - \eta = x + y - \sum_{\varepsilon_i=(0,1)}^{i < x+y+1} t_i \geqslant 0$$

by Lemma 5.1. Therefore $x \geqslant \eta$ as it was stated. In fact this proof holds only when $x + y > 1$ and $x + y < n$. But the remaining cases are trivial.

The converse of this lemma is also true.

LEMMA 5.3. *Let v, h be non-negative integers and let W_0, W_- and W_+ be three non-crossing walks from* (0, 0) *to* (v, h). *Suppose W_- is under, W_+ is over of W_0. Then there is a unique feasible type* T *with* $W_0 = W_0(\mathrm{T})$, $W_- = W_-(\mathrm{T})$ *and* $W_+ = W_+ = (\mathrm{T})$.

PROOF. All these walks consist of steps (0, 1) and (1, 0). Let $\varepsilon_2, \ldots, \varepsilon_n$ be defined as the steps of W_0, starting the indices with 2, and using the notation $n = v + h + 1$.

Let the integers $n_j, j = 1, \ldots, h$ be defined such that the point (n_j, j) is the endpoint of the jth (0, 1) step in W_+. Similarly, denote (k, m_k) $k = 1, \ldots, v$ the endpoints of the kth (1, 0) steps in W_-. Moreover let $m_0 = n_0 = 0$. Clearly $n_0 \leqslant n_1 \leqslant \cdots \leqslant n_h$ and $m_0 \leqslant m_1 \leqslant \cdots \leqslant m_v$. Then define

$$t_i \overset{\text{def}}{=} \begin{cases} 1 + n_j - n_j - 1, & \text{if } \varepsilon_i \text{ is the } j\text{th } (0, 1) \text{ step in } W_0 \\ 1 + m_k - m_{k-1}, & \text{if } \varepsilon_i \text{ is the } k\text{th } (1, 0) \text{ step in } W_0 \end{cases}$$

Using these definitions, it is easy to check that the non-intersecting property of the walks is equivalent to the inequalities (4) for the collection $\{(\varepsilon, t_i) | i = 2, \ldots, n\}$. Thus the statement follows by Lemma 5.1.

PROOF OF THEOREM 3.3. Now the number of feasible types of dissections of n rectangles is equals to the number of non-crossing walk triplets from (0, 0) to (v, h) for non-negative integers v, h with $v + h = n - 1$, by Lemmas 5.2, 5.3. But the number of non-crossing walk triplets from (0, 0) to (v, h) by an old theorem of Mac Mahon [4] (see also in [8]) is

$$\sum_{i=1}^{v} \prod_{j=1}^{h} \prod_{r=1}^{3} \frac{i + j + r - 1}{i + j + r - 2}.$$

This equals to

$$\frac{2}{n(n+1)^2}\binom{n+1}{h}\binom{n+1}{h+1}\binom{n+1}{h+2}.$$

A summation give Theorem 3.3.

PROOF OF PROPOSITION 2.2. We can use the approximation (see [7])

$$\binom{n}{\frac{n}{2}-x} = \frac{2^n}{\sqrt{\frac{\pi n}{2}}}\cdot e^{-2x^2/n}\left(1+o\left(\frac{x}{n}\right)\right),$$

which holds for $x < n^{2/3}$, and the well-known fact that

$$\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}.$$

We omit the details here.

6. DISSECTIONS WITH ALGEBRAICLY INDEPENDENT AREAS

Here we prove Theorem 2.6.

Suppose that $w_1, \ldots, w_{n-1}$ are positive algebraically independent reals, $w_n = 1 - \Sigma_{i=1}^{n-1} w_i > 0$. Let $\{(\varepsilon_i, T_i) | i = 2, \ldots, n\}$ be a feasible type. By Theorem 3.4 there is a unique dissection D with this type and with these areas.

What we really have to prove is the following:

PROPOSITION 6.1. *D has only one type.*

This Proposition 6.1 and Theorem 3.4 imply that in this case $f(W) = n!M_n$, as it was stated in Theorem 2.6.

PROOF OF PROPOSITION 6.1. First we associate a system of linear equalities to D. Denote the lengths of the sides of $R_i \in D$ by $x_i = x(R_i)$ and $x_{n+i} = y(R_i)$ for $i = 1, \ldots, n$. For every base $B(R_i)$ the sum of the lengths of the sides adjacent to $B(R_i)$ is $2|B(R_i)|$. This gives us a linear equality of the form

$$\sum_{j \in U_i} x_j = \sum_{j \in V_i}, \qquad 2 \leqslant i \leqslant n. \tag{5}$$

(Here, of course, $U_i \cap V_i = \emptyset$ and $U_i \neq \emptyset$, $V_i \neq \emptyset$.) Add the following two equalities to (5). Denote the segment $\{(x, 0)|0 \leqslant x \leqslant 1\}$ by I, and the segment $\{(0, y)|0 \leqslant y \leqslant 1\}$ by J.

$$\begin{aligned} 1 &= \sum_{i:B(R_i)\cap I \neq \emptyset,} x_i, \\ 1 &= \sum_{i:B(R_i)\cap J \neq \emptyset} x_i. \end{aligned} \tag{6}$$

One can prove that the system of $n + 1$ linear equalities given by (5) and (6) has rank $n + 1$. So we can find a subset $K \subset \{1, 2, \ldots, 2n\}$, $|K| = n - 1$ and rational coefficients $m_{i,j}$ $i = 1, \ldots, 2n$ $j \in K$ such that

$$x_i = l_i + \sum_{j \in K} m_{i,j} x_j \tag{7}$$

holds for all $i = 1, \ldots, 2n$. (Here l_i is also a rational and of course $i \in K$ implies $l_i = 0$, $m_{i,i} = 1$, $m_{i,j} = 0$ for $j \neq i$.)

Now consider the algebraically independent numbers w_i, $i = 1, \ldots, n-1$. We have from (7) that

$$w_i = area(R_i) = x_i \cdot x_{n+1} = P_i(x_k | k \in K), \qquad i = 1, \ldots, n-1, \tag{8}$$

where P_i is a polynomial over the variables $\{k | k \in K\}$ with rational coefficients. These polynomials depend only on the combinatorial type $\{(\varepsilon_i, T_i) | i = 2, \ldots, n\}$.

The main point is to understand what does it mean that *D has at least two combinatorial types*. It means that there is a corner $c(R)$, $R \in D$ which is covered by 4 rectangles of D. This gives us a new linear equality, independent of (5) and (6). So we can have a $K' \subset \{1, 2, \ldots, 2n\}$, $|K'| \leqslant n-2$ replacing K in (7). Hence the polynomials P_i in (8) have only $n-2$ variables. Then there exists a polynomial $Q(y_1, \ldots, y_{n-1}) \neq 0$ of $n-1$ variables having rational coefficients such that for the composition

$$Q(P_1, \ldots, P_{n-1}) \equiv 0.$$

So (8) implies $Q(w_1, \ldots w_{n-1}) = 0$ which contradicts their algebraic independence. Hence such an extra linear dependency does not exist, i.e., every corner $c(R)$, $R \in D$ is covered by at most 3 times. Then by the definition of combinatorial type D has only one type, which proves the proposition.

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