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Matchings and Covers in Hypergraphs

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Abstract. Almost all combinatorial question can be reformulated as either a matching or a covering problem of a hypergraph. In this paper we survey some of the important results.

1. Covers

A *hypergraph* $\mathbf{H}$ is an ordered pair $(X, \mathscr{H})$ where X is a finite set (the set of *vertices*, or points, or elements) and $\mathscr{H}$ is a collection of subsets of X (called *edges*, or members of $\mathbf{H}$). We will often use the notation $X = V(\mathbf{H})$, $\mathscr{H} = E(\mathbf{H})$. The *rank* of $\mathbf{H}$ is $r(\mathbf{H}) = \max\{|E|: E \in \mathscr{H}\}$. If every member of $\mathscr{H}$ has r elements we call it *r-uniform*, or an *r-graph*. The 2-uniform hypergraphs are called *graphs*. In almost all cases we will deal with hypergraphs without multiple edges. 2^X and $\binom{X}{r}$ denote the family of all subsets (all r-subsets) of X, resp.

A set T is called a *cover* (in other words a transversal or a blocking set) of $\mathbf{H}$ if it intersects every edge of $\mathbf{H}$, i.e., $T \cap E \neq \emptyset$ for all $E \in \mathscr{H}$. The minimum cardinality of the covers is denoted by $\tau(\mathbf{H})$, and called the *covering number* of $\mathbf{H}$. E.g., $\tau\left(\binom{X}{r}\right) = |X| - r + 1$, and it is easy to see that $\tau(PG(2, r-1)) = r$, where $PG(2, q)$ denotes the hypergraph having as edges the system of lines of any finite projective plane of order q. (See later in Section 3). If a family $\mathscr{H}$ is *intersecting* (i.e., $H \cap H' \neq \emptyset$ for every $H, H' \in \mathscr{H}$) then $\tau(\mathscr{H}) \leq \min_{E \in \mathscr{H}} |E|$. On the other hand we can get a trivial lower bound for the covering number considering the subfamilies of pairwise disjoint edges. If $E_1, \ldots, E_v \in \mathscr{H}$, $E_i \cap E_j = \emptyset$ $(1 \leq i < j \leq v)$ then $\tau(\mathscr{H}) \geq v$.

The great importance of the covering problem is supported by the fact that apparently *all* combinatorial problem can be reformulated as the determination of the covering number of a certain hypergraph. The calculation of the covering number of an arbitrary hypergraph is an NP-hard problem even in the class of graphs. (For those who are not familiar with the notions of algorithm theory, we remark that NP-hard means, roughly saying, that the solution of the problem seems to be hopeless in general. See [260]). Hence every result which determines $\tau(\mathscr{H})$ for a certain class of hypergraphs, e.g. the König-Hall theorem about bipartite graphs (see [188]) is especially valuable.

Definition 1.1. The family of sets $\mathscr{H}$ is *τ-critical* if each of its subfamilies has a smaller covering number, i.e., $\tau(\mathscr{H} - \{E\}) < \tau(\mathscr{H})$ for all $E \in \mathscr{H}$.

Examples 1.2. The following ones are τ-critical hypergraphs.

(i) The family consisting of τ disjoint edges,

(ii) the complete r-graph $\binom{X}{r}$ where $|X| = \tau + r - 1$,

(iii) the circuit of length $2\tau - 1$, $C_{2\tau-1}$,

(iv) (see Erdös and Lovász [108]). Let S be a set with $2\tau - 2$ elements. For each partition $\pi = \{P, P'\}$ of S where $P \cup P' = S$, $|P| = |P'| = \tau - 1$ take a new element x_π. Let $X = S \cup \{x_\pi\colon \text{for all } \pi\}$ and define $\mathscr{H}$ to consist of all τ-tuples of the form $P \cup \{x_\pi\}$, where $\pi = \{P, P'\}$ is a partition. Then $|\mathscr{H}| = \binom{2\tau-2}{\tau-1}$, $|X| = 2\tau - 2 + \frac{1}{2}\binom{2\tau-2}{\tau-1}$, $\tau(\mathscr{H}) = \tau$.

Theorem 1.3 (Bollobás [44]). *If $\mathscr{H}$ is a τ-critical hypergraph of rank r then* $|\mathscr{H}| \le \binom{r+\tau-1}{r}$.

Here equality holds only in the case when $\mathscr{H}$ is a complete r-graph over $\tau + r - 1$ vertices (in notation $\mathscr{H} \cong \mathbf{K}_r^{\tau+r-1}$). Theorem 1.3 means that every family $\mathscr{H}$ of rank r has a relatively small subfamily $\mathscr{F} \subset \mathscr{H}$, $\left(|\mathscr{F}| \le \binom{r+\tau-1}{r}\right)$ such that $\tau(\mathscr{F}) = \tau(\mathscr{H})$. In other words, if $\mathscr{H}$ is a family of rank r and every subfamily of $\binom{r+t}{r}$ members has a $(t+1)$-cover then $\tau(\mathscr{H}) \le t + 1$. The theorem was proved for the case of graphs ($r = 2$) by Erdös, Hajnal and Moon [104].

Proof. Here we only reformulate the statement, the theorem will follow from Corollary 1.5. Let $\mathscr{H} = \{A_1, \ldots, A_m\}$ be τ-critical, i.e., $\tau(\mathscr{H} - \{A_i\}) < \tau$ for all $1 \le i \le m$. Hence there exists a $(\tau - 1)$-element cover B_i of $\mathscr{H} - \{A_i\}$. This means $A_j \cap B_i \neq \varnothing$ for all $j \neq i$. Moreover B_i does not cover all the edges of $\mathscr{H}$, hence $A_i \cap B_i = \varnothing$ holds. Thus the system $\{A_i, B_i\}_{1 \le i \le m}$ fulfils the constraints of corollary 1.5 with $r = a$, $\tau - 1 = b$, which will yield $m \le \binom{r+\tau-1}{r}$. □

Theorem 1.4. *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be finite sets such that*

$$A_i \cap B_i = \varnothing \quad \text{and} \quad A_i \cap B_j \neq \varnothing \quad \text{for all } i \neq j. \tag{1.1}$$

Then $\sum_i 1 \Big/ \binom{|A_i| + |B_i|}{|A_i|} \le 1$.

Corollary 1.5. *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be finite sets with $|A_i| \le a$, $|B_i| \le b$ satisfying* (1.1). *Then* $m \le \binom{a+b}{a}$. □

Proof of Theorem 1.4. This proof is due to Katona [163] and Jaeger and Payan [159]. Let $X = (\bigcup A_i) \cup (\bigcup B_i)$, $|A_i| = a_i$, $|B_i| = b_i$, $|X| = n$. Consider a permutation π of X. We say it has type "i" if each element of A_i precedes each element of B_i (Fig. 1.1).

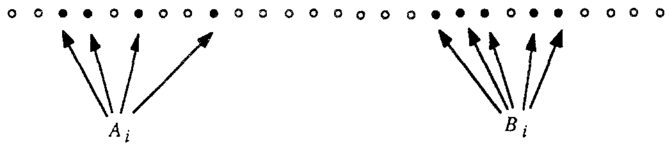

Fig. 1.1

We claim that π has at most one type. Suppose on the contrary that π has two types i and j. Let x_i (and x_j) be the maximum element of A_i (A_j) in π, resp., and suppose that either $x_i = x_j$ or x_i precedes x_j. Then each element of A_i precedes each element of B_j yielding $A_i \cap B_j = \emptyset$, a contradiction. Count how many permutations have type "i". This number is $\binom{n}{a_i + b_i} \times (n - a_i - b_i)!a_i!b_i! = n! \Big/ \binom{a_i + b_i}{a_i}$. Summing up these we get all permutations at most once

$$\sum_i n! \Big/ \binom{a_i + b_i}{a_i} \leq n! \qquad \square$$

Because of its importance Corollary 1.5 and Theorem 1.4 were several times rediscovered (Jaeger and Payan [159], Katona [163], Tarjan [233], Griggs, Stahl and Trotter [146]). Even it was stated as a conjecture several years later than Bollobás proved it (Ehrenfeucht and Mycielski [92]). However, only the original proof yields that equality holds in Corollary 1.5 iff the sets A_i and B_i are all the a-element and b-element subsets of a given $(a + b)$-set. (Bollobás used induction on $n = |(\bigcup A_i) \cup (\bigcup B_i)|$.)

In the rest of this chapter first we mention 3 simple applications of Theorem 1.4 and a nice new version of it (which also has interesting applications.)

Minimal $\mathbf{K}_r^k$*-saturated r-graphs* (Bollobás [44] for all r, Erdös, Hajnal and Moon [104] for $r = 2$). Call the hypergraph $\mathbf{H}$ $\mathbf{K}_r^k$*-saturated* if it does not contain a copy of $\mathbf{K}_r^k$ as subgraph, but $E(\mathbf{H}) \cup \{F\}$ does it for all $F \in \binom{X}{r} \setminus E(\mathbf{H})$. An example is: $\mathscr{H} = \left\{E \in \binom{X}{r} : E \cap K \neq \emptyset\right\}$, where K is a fixed $(k - r)$-set.

Theorem 1.6. *If* $\mathbf{H}$ *is a* $\mathbf{K}_r^k$*-saturated r-graph over n vertices then* $|E(\mathbf{H})| \geq \binom{n}{r} - \binom{n - k + r}{r}$. *Here equality holds only for the above example.* $\square$

Helly families of maximal size. We say that $\mathscr{H}$ has the *k-Helly property* if in every subfamily $\mathscr{H}' \subset \mathscr{H}$ with empty intersection one can find a subfamily $\mathscr{H}'' \subset \mathscr{H}'$ consisting of at most k members whose intersection is also empty. In other words, if every k members of $\mathscr{H}'$ have a common vertex than $\bigcap \mathscr{H}' \neq \emptyset$.

Theorem 1.7 (Bollobás and Duchet [52], for a generalization see [53], Mulder [204] and Tuza [239]). *If $k < r$ and $\mathscr{H} \subset \binom{X}{r}$ is a k-Helly family on n vertices, then $|\mathscr{H}| \leq \binom{n-1}{r-1}$. If $|\mathscr{H}| = \binom{n-1}{r-1}$ then all edges have a common point.* □

Maximum number of unrelated chains (Griggs, Stahl, and Trotter [146]). The famous theorem of Sperner [226] states that if $\mathscr{H} \subset 2^X$ is a collection of subsets, $|X| = n$, no two ordered by inclusion, (i.e., for $H, H' \in \mathscr{H}$ $H \subsetneqq H'$ is impossible), then $\mathscr{H}$ has at most $\binom{n}{\lfloor n/2 \rfloor}$ edges. A *chain* in 2^X of length $k+1$ is a collection of sets $E_0 \subsetneqq E_1 \subsetneqq \cdots \subsetneqq E_k (\subset X)$. Denote by $f_k(n)$ the maximum number of unrelated chains of length $k+1$. I.e., we seek the maximum m such that there exist subsets $E_j^i \subset X, 0 \leq j \leq k, 1 \leq i \leq m$, satisfying $E_0^i \subset E_1^i \subset \cdots \subset E_k^i$ and for all $i \neq i'$

$$E_j^i \not\subset E_{j'}^{i'}.$$

We can obtain such a collection of $\binom{n-k}{\lfloor (n-k)/2 \rfloor}$ unrelated chains of length $k+1$ as follows: Let $X = A \cup B$, $|A| = k$, $|B| = n-k$, $A = \{a_1, \ldots, a_k\}$. The sets E_0^i are the $[(n-k)/2]$-subsets of B, and for $j \geq 1$, $E_j^i = E_0^i \cup \{a_1, \ldots, a_j\}$. The following theorem states that this example is optimal:

Theorem 1.8. $f_k(n) = \binom{n-k}{\lfloor (n-k)/2 \rfloor}$. □

Griggs et al. use this theorem to determine the dimension on some partially ordered sets. On dimension theory of posets see [165]. Note that Sperner's theorem is a corollary of Theorem 1.8 ($k = 0$). The following theorem which is due to Tuza gives a very useful variant of Theorem 1.4. Although, this result was first formulated by Tuza, it appeared in implicit form (at least the case $p = q = 1/2$) much earlier in [164], and in [15], too.

Theorem 1.9 ([241]). *Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be finite sets such that*

$$A_i \cap B_i = \varnothing \text{ and for all } i \neq j \text{ either} \tag{1.2}$$

$A_i \cap B_j = \varnothing$ or $A_j \cap B_i \neq \varnothing$ holds. Let $p, q \geq 0$ be reals, $p + q = 1$. Then $\sum_i p^{|A_i|} q^{|B_i|} \leq 1$ holds.

Proof. We are going to use the so called probabilistic method. (Actually, here it is easy to replace it by a double counting.) Let $X = (\bigcup A_i) \cup (\bigcup B_i)$. Define a random set $S \subset X$ as follows:

$$prob(a \in S) = p \quad \text{for all } a \in X.$$

Then consider the events $E_i = \{A_i \subset S, B_i \cap S = \varnothing\}$. Clearly $prob(E_i) = p^{|A_i|}(1-p)^{|B_i|}$. These events are disjoint by (1.2), so we obtain $\sum_i p^{|A_i|} q^{|B_i|} = \sum_i prob(E_i) \leq 1$. □

Corollary 1.10 ([15], [164] and [241]). *Let A_i, B_i $(1 \le i \le m)$ finite sets satisfying* (1.2) *and* $|A_i| + |B_i| \le k$. *Then* $m \le 2^k$. □

Local Ramsey-number of odd cycles. (Katona and Szemerédi [164]). Consider $E(\mathbf{K}_2^n)$, the edge-set of the complete graph on n vertices, and color it by k colors, $E(\mathbf{K}_2^n) = G_1 \cup \cdots \cup G_k$. Then it is immediate that $n > 2^k$ implies that there exists a G_i which has chromatic number at least three and therefore contains an odd cycle. A generalization of this is the following.

Theorem 1.11 ([164]). *Suppose* $n > 2^k$ *and let* $E(\mathbf{K}_2^n) = G_1 \cup G_2 \cup \cdots$ *be a partition of the complete graph in such a way that for every vertex* $x \in V(\mathbf{K}_2^n)$ *the edges* $\{x, y\}$ *belong to at most k different* G_i – *s. (I.e., every point gets at most k colors.) Then there exists a* G_i *which contains an odd cycle.* □

Touching simplices in $\mathbb{R}^d$. Two simplices S_1 and S_2 in the d-dimensional space are touching if they do not have common interior point, $int\, S_1 \cap int\, S_2 = \varnothing$, but they have facets F_1, F_2 ($d-1$-dimensional faces) such that $int\, F_1 \cap int\, F_2 \ne \varnothing$. In other words there exists a hyperplane $H \subset \mathbb{R}^d$ which separates $int\, S_1$ and $int\, S_2$ but for a point $p \in H$ and a small ball $B(p)$ around p we have $B(p) \subset S_1 \cup S_2$. Using induction on d one can construct 2^d pairwise touching simplices in $\mathbb{R}^d$. (See Zaks [248]).

Denote the maximum number of pairwise touching simplices in $\mathbb{R}^d$ by $t(d)$. It has been repeatedly conjectured that $t(d) = 2^d$ (see e.g., Bagemihl, 1956 [18]). The following theorem is due to Perles and was published in [209].

Theorem 1.12. $2^d \le t(d) \le 2^{d+1}$.

Proof. Let $S_1, \ldots, S_m$ be a family of touching simplices, and consider all the hyperplanes of their facets: $H_1, \ldots, H_v$ $(v \le (d+1)m)$. Every hyperplane H_i divides the space into two halfspaces, call one of them the positive, the other the negative side of H_i. Define the following sets A_i, B_i $(1 \le i \le m)$:

$$A_i = \{H_j\colon H_j \text{ contains a facet of } S_i \text{ and } int\, S_i \text{ lies on the positive side of } H_j\}$$

$$B_i = \{H_j\colon H_j \text{ contains a facet of } S_i \text{ and } int\, S_i \text{ lies on the negative side of } H_j\}.$$

Clearly $A_i \cap B_i = \varnothing$, $|A_i \cup B_i| = d + 1$, and (1.2) holds. Then Corollary 1.10 implies $m \le 2^{d+1}$. □

Previously the best hound was $f(d) \le \frac{2}{3}(d+1)!$ (Zaks [247]). Baston [23] wrote a book on this topic, proving $t(3) \le 9$. It was recently established that $f(3) = 8$ (Zaks [249]). More about this problem, see [250].

Note that the above argument gives that if $\mathscr{P}$ is a family of polytopes in $\mathbb{R}^d$, each of them having at most k facets and any two of them are touching then $|\mathscr{P}| \le 2^k$. Moreover, the proof uses a weaker assumption: it is sufficient to state that for any two simplices, say P and Q, there exists a hyperplane $H \subset \mathbb{R}^d$ containing a facet of both P and Q, such that H separates $int\, P$ and $int\, Q$.

One could think that a stronger version of Corollary 1.10 would supply an improvement of the upper bound of $t(d)$. Indeed, we know that for all $i \ne j$ either $|A_i \cap B_j| = 1$ or $|A_j \cap B_i| = 1$ holds in the proof of Theorem 1.12. But the following example, due to Tuza [271], shows that this approach is not so simple.

Example 1.13. For every d there exists a family $\{A_i, B_i\}$ where $1 \le i \le 2^d$ with the following properties: $A_i \cap B_i = \varnothing$, and for all $i \ne j$ either $A_i \cap B_j = \varnothing$ and $|A_j \cap B_i| = 1$ or $|A_i \cap B_j| = 1$ and $A_j \cap B_i = \varnothing$ hold. Moreover, $|A_i| + |B_i| = d$ for all $1 \le i \le 2^d$. It is easy to construct such a family for $d = 1$. Then we can use induction on d. Consider $\{A_i, B_i\}$ $(1 \le i \le 2^d)$ and take another (disjoint) copy of it $\{A_i', B_i'\}$. Now add a new element x to all A_i and B_i'.

Tuza has other examples with equality in Theorem 1.9. We can ask in general:

Problem 1.14. Describe the families $\{A_i, B_i\}$ for which (1.2) and $\sum_i p^{|A_i|} q^{|B_i|} = 1$ hold.

Problem 1.15. Let $s(a, b) = \max\{m$: there exists a family $A_1, \ldots, A_m, B_1, \ldots, B_m$ satisfying (1.2) and $|A_i| \le a, |B_i| \le b\}$. Determine $s(a, b)$.

Tuza proved

$$s(a,b) \le \min\left\{\frac{(a+b)^{a+b}}{a^a b^b}, \sum_{i=0}^{b} a^i \binom{a+b-i}{a}, \sum_{i=0}^{a} b^i \binom{a+b-i}{b}\right\},$$

and $s(a, 1) = 2a + 1$.

Some further results and problems. Alon [12] found the following generalization of Corollary 1.5.

Theorem 1.16 ([12]). *Suppose that $V_1, \ldots, V_r$ are disjoint sets and let $a_1, \ldots, a_r, b_1, \ldots, b_r$ be positive integers. Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be finite sets satisfying (1.1) and $|A_i \cap V_j| \le a_j$, $|B_i \cap V_j| \le b_j$ for every $1 \le i \le m$, $1 \le j \le r$. Then*

$$m \le \prod_{i=1}^{r} \binom{a_i + b_i}{a_i}.$$

A special case of this $(r = 2, a_1 = a_2 = 1)$ was conjectured by Erdös, Hajnal and Moon [104] and was proved by Bollobás [45] and Wessel [243]. One can prove Theorem 1.16 by the method of the proof of Theorem 1.4. But Alon also proved that Theorem 1.16 remains valid if we suppose that (1.1) holds only for $1 \le i \le j \le m$. The proof uses multilinear techniques (see the next chapter.)

Call an r-graph $\mathscr{H}$ *r-partite* if there exists a partition $V(\mathscr{H}) = V_1 \cup \cdots \cup V_r$ such that $|V_i \cap E| = 1$ holds for all $E \in \mathscr{H}$, $1 \le i \le r$.

Problem 1.17. Determine $f(t, \tau) =: \max\{|\mathscr{H}|: \mathscr{H}$ is τ-critical, r-partite$\}$.

By Theorem 1.3 this maximum is smaller than $\binom{\tau + r - 1}{r}$. The König-Hall theorem imples that $f(2, \tau) = \tau$. Another possible generalization is to consider three or more families instead of 2. Using the above method one can prove

Theorem 1.18. *Suppose we have set A_i^α, $1 \le i \le m$, $1 \le \alpha \le k$. Suppose further that for every $1 \le i \le m$*

$$A_i^\alpha \cap A_i^\beta = \varnothing \quad \text{for } 1 \le \alpha < \beta \le k$$

and for every $i \ne j$ there exist $\alpha \ne \beta$ such that

$$A_i^\alpha \cap A_j^\beta \ne \varnothing.$$

Then for arbitrary reals satisfying $p_1, \ldots, p_r > 0$, $\sum p_i = 1$ we have

$$\sum_i \prod_\alpha p_\alpha^{|A_i^\alpha|} \le 1.$$

This theorem (in a slightly weaker form) appeared in [15]. A corollary of this is a generalization of Theorem 1.11:

Theorem 1.19. *Suppose $n > r^k$ and $E(\mathbf{K}_2^n) = G_1 \cup G_2 \cup \cdots$ is a coloring of the edge-set of the complete graph in such a way that for every vertex $x \in V(\mathbf{K}_2^n)$ the edges $\{x, y\}$ belong to at most k $G_i - s$. Then there exists a G_i whose chromatic number is at least $r + 1$.* □

The following question emerged in computer science: Suppose that the family $\{A_i, B_i\}$ $(1 \le i \le m)$ satisfies the assumptions of Corollary 1.10 (i.e., $A_i \cap B_i = \varnothing$, $|A_i| + |B_i| \le k$, for all $i \ne j$ either $A_i \cap B_j \ne \varnothing$ or $A_j \cap B_i \ne \varnothing$). Suppose further that $A_i \ne \varnothing$ for all i, and denote $f(k) = \max\{\tau(\mathscr{A})\colon \mathscr{A} = \{A_1, \ldots, A_m\}$ as above$\}$. It is easy to see that

$$2k - 1 \le f(k) \le \binom{k+1}{2}$$

Conjecture 1.20. $f(k) = 2k - 1$.

Even the inequality $f(k) < k^{1+\varepsilon}$ would have some applications. (See [261]). Another problem concerning Corollary 1.10 arises from the study of comma-free codes (see [172]). Suppose that the family $\{A_i, B_i\}$ $(1 \le i \le m)$ satisfies the following assumptions: $A_i \cap B_i = \varnothing$, A_i, $B_i \subset X$ where X is a k-element set, and for all $i \ne j$ exactly one of the following two conditions holds, either $A_i \cap B_j \ne \varnothing$ or $A_j \cap B_i \ne \varnothing$. This family is called a $\{0, 1, *\}$ *tournament code.* Let $t(k)$ denote the size of the largest $\{0, 1, *\}$ tournament code of length k. Corollary 1.10 implies $t(k) \le 2^k$. The best upper bound known for $t(k)$ is due to Graham [144], who has shown that there is a constant $c > 0$ such that $t(k) \le k^{c \log k}$ for k sufficiently large. It is easy to see that $t(k) \ge 0(k)$ (see [172]) and Collins, Shor and Stembridge [74] gave a construction proving $t(k) \ge (1 - o(1))k^{3/2}$.

Problem 1.21. What is the order of magnitude of $t(k)$? Is $t(k) = O(k^2)$ true?

Maximum spanned cycle in the Kneser graph. Let $A_1, A_2, \ldots, A_m$ be k-sets with the property that $A_i \cap A_{i+1} = \varnothing$ $(1 \le i \le m)$, $A_m \cap A_1 = \varnothing$ but $A_i \cap A_j \ne \varnothing$ in any other case. Denote the maximum value of m by $g(k)$. Corollary 1.5 implies that

$$g(k) \le 1 + 2\binom{2k}{k}$$

P. Alles [11] proved by induction on k that $g(k) \ge 2^k + 2$.

Conjecture 1.22 ([11]). $g(k) = 2^k + 2$.

2. Geometric Hypergraphs

The most fruitful generalization of Theorem 1.3 was given by Lovász [181, 185]. He introduced the notion of *geometric hypergraph.* This means that the vertices of $\mathbf{H}$ are embedded into a (real) projective space. (Instead of real space one can consider other spaces over arbitrary fields, or even matroids). We illustrate his method showing an improvement of Corollary 1.5.

The following theorem was conjectured by Bollobás [46, 50] and Pin [211] and was proved (independently) by Frankl and Kalai using a slightly modified version of Lovász' geometrical method.

Theorem 2.1 ([116], [162]). *Let $A_1, \dots, A_m$ and $B_1, \dots, B_m$ be finite sets with $|A_i| \le a$, $|B_i| \le b$ satisfying*

$$A_i \cap B_i = \varnothing \quad \text{and} \quad A_i \cap B_j \neq \varnothing \quad \text{for } 1 \le i < j \le m. \tag{2.1}$$

Then $m \le \binom{a+b}{a}$.

Proof. First we recall some definitions. Consider the vectorspace V^d of multilinear polynomials over $\mathbb{R}$. That is, every member of V^d is a polynomial of the form $f(\mathbf{x}) = \sum_{I \subset \{1,2,\dots,d\}} c_I \left(\prod_{i \in I} x_i \right)$, where $c_I \in \mathbb{R}$. This is a 2^d-dimensional vectorspace, so we have two operations on the elements of V^d, multiplication by a real number $\left(af(\mathbf{x}) = \sum_I ac_I x_I \right)$ and addition $\left(f(\mathbf{x}) + f'(\mathbf{x}) = \sum_I (c_I + c_I')x_I \right)$. Here $\prod_{i \in I} x_i$ is denoted by x_I.

Define a noncommutative but associative and distributive operation, the so-called *wedge product* $\wedge$. If $f(\mathbf{x}) = \sum_I c_I x_I$, and $f'(x) = \sum c_I' x_I$ then

$$f(\mathbf{x}) \wedge f'(\mathbf{x}) = \sum_I \sum_J c_I c_J' (x_I \wedge x_J).$$

Moreover

$$x_I \wedge x_J = \begin{cases} 0 & \text{if } I \cap J \neq \varnothing \\ x_{I \cup J} & \text{or } -x_{I \cup J} \text{ if } I \cap J = \varnothing \\ & \text{according as } \{I, J\} \text{ is an even} \\ & \text{or odd permutation of } I \cup J \end{cases}$$

e.g., $f \wedge f = 0$, $c \wedge x_I = cx_I$, $x_2 \wedge x_{\{1,3\}} = x_2 \wedge x_1 \wedge x_3 = -x_{\{1,2,3\}}$. For the proof of the existence and uniqueness of $\wedge$ see, e.g., [190]. Now we are ready to prove Theorem 2.1. We can suppose that $|A_i| = a$, $|B_i| = b$ holds for all i. Consider $P = (\bigcup A_i) \cup (\bigcup B_i)$. We can suppose that P is finite. Choose a vector $\mathbf{v}(p) \in \mathbb{R}^{a+b}$ for every element $p \in P$ in such a way that every $a + b$ of them are linearly independent. Associate a polynomial $f_i(\mathbf{x})$ with every A_i:

$$f_i(\mathbf{x}) = \bigwedge_{p \in A_i} \left(\sum_{j=1}^{a+b} v(p)_j x_j \right),$$

similarly let

$$g_i(\mathbf{x}) = \bigwedge_{p \in B_i} \left(\sum_{j=1}^{a+b} v(p)_j x_j \right).$$

(2.1) implies that $f_i \wedge g_j = 0$ for $i < j$, but $f_i \wedge g_i \neq 0$. We claim that the system

$\{f_i(\mathbf{x})\colon 1 \le i \le m\}$ is linearly independent over $\mathbb{R}$. Suppose on the contrary that $\sum_i \alpha_i f_i(\mathbf{x}) = 0$, and let s be the maximal index for which $\alpha_s \ne 0$, i.e., $s = \max\{j\colon \alpha_j \ne 0\}$. Then

$$0 = \left(\sum_{i=1}^{s} \alpha_i f_i(\mathbf{x})\right) \wedge g_s(\mathbf{x}) = \alpha_s(f_s \wedge g_s)$$

which leads to $\alpha_s = 0$, a contradiction. All f_i belong to a $\binom{a+b}{a}$-dimensional subspace of V^{a+b}, namely $f_i \in W_a^{a+b} = \{f(\mathbf{x})\colon f(x) = \sum \alpha_I x_I \text{ where } \alpha_I = 0 \text{ if } |I| \ne a\}$. They are linearly independent in W_a^{a+b}, hence $m \le \binom{a+b}{a}$. □

With the same method Lovász proved the following geometric versions:

Theorem 2.2. *Let $A_1, \ldots, A_m$ be a-dimensional and let $B_1, \ldots, B_m$ be b-dimensional subspaces of a linear space with the property $\dim(A_i \cap B_j) = 0$ iff $i = j$. Then $m \le \binom{a+b}{a}$.* □

Theorem 2.3. *Let $A_1, \ldots, A_m$ be a-dimensional subspaces of a linear space and $B_1, \ldots, B_m$ be b-element point-sets with the property $A_i \cap B_j = \varnothing$ iff $i = j$. Then $m \le \binom{a+b}{a}$.* □

The first step of the proofs of these theorems is a projection into a $(a+b)$-dimensional hyperplane of general position. Frankl and Stečkin [126] conjectured that the following generalization is also true:

Theorem 2.4 (Füredi [132]). *Suppose that t is a nonnegative integer, $a, b \ge t$, $A_1, \ldots, A_m$ is a collection of a-sets, $B_1, \ldots, B_m$ is a collection of b-sets such that $|A_i \cap B_i| \le t$ and $|A_i \cap B_j| > t$ for $i \ne j$. Then $m \le \binom{a+b-2t}{a-t}$.*

The case $t = 0$ corresponds to the previous theorems. The bound $\binom{a+b-2t}{a-t}$ is best possible. Let S be an $a+b-2t$, T a t-element set $S \cap T = \varnothing$. Let $\binom{S}{a-t} = \{D_1, \ldots, D_m\}$ and define $A_i = D_i \cup T$, $B_i = (S - D_i) \cup T$. One of the geometric generalizations is also true.

Theorem 2.5 ([132]). *Let $A_1, \ldots, A_m$ be a-dimensional and let $B_1, \ldots, B_m$ be b-dimensional subspaces of the real Euclidean space. Suppose that $\dim(A_i \cap B_j) \le t$ iff $i = j$. Then $m \le \binom{a+b-2t}{a-t}$.*

The generalization of Theorem 2.3 leads to new problems (see Problem 2.13.) In each statement we can eplace the assumptions $|A_i| = a$, $|B_i| = b$, $\dim A_i = a \cdots$

by $|A_i| \le a$, $|B_i| \le b$, $\dim A_i \le a$ and so on. All the above Theorems 2.2–2.5 are valid even if we suppose our assumptions only for $1 \le i \le j \le m$.

Proof of Theorem 2.4. It follows from Theorem 2.5 in the same way as Theorem 2.2 implies Theorem 2.1. That is, let $P = (\bigcup A_i) \cup (\bigcup B_i)$. Let us assign a vector $\mathbf{v}(p) \in \mathbb{R}^P$ to each $p \in P$ so that $\{\mathbf{v}(p): p \in P\}$ forms a basis of $\mathbb{R}^P$. Let $\bar{A}_i$ (and $\bar{B}_i$) be the subspaces generated by $\{\mathbf{v}(a): a \in A_i\}$. Then Theorem 2.5 can be applied.

Proof of Theorem 2.5. Suppose that $A_i, B_j \subset \mathbb{R}^N$. We can suppose that N is finite. For a subspace C let us define $C^\perp = \{y \in \mathbb{R}^N: (c, y) = 0 \text{ for each } c \in C\}$ the orthogonal complement of C. Two subspaces D and C of dimensions d and c are in *general position* if $\dim(D \cap C) = \max\{0, d + c - N\}$.

There exists a subspace C of dimension $N - t$ which is in general position with respect to each A_i, B_i and $A_i \cap B_i$. Let $A_i' = A_i \cap C$ and $B_i' = B_i \cap C$. Then $\dim A_i' = a - t$, $\dim B_i' = b - t$, $\dim(A_i' \cap B_i') = 0$ and for $i \ne j$ we have $\dim(A_i' \cap B_j') = \dim((A_i \cap B_j) \cap C) \ge 1$. Now Theorem 2.2 can be applied to $\{A_i', B_i'\}$. □

An application of Theorem 2.1. (Disjointly representable sets).

Theorem 2.6 (Füredi and Tuza [139]). *Let $\mathscr{F}$ be a family of sets of rank r. If $|\mathscr{F}| > \binom{r+k}{k}$, then there exist $F_0, F_1, \ldots, F_{k+1} \in \mathscr{F}$, and points $p_1, p_2, \ldots, p_{k+1}$ such that $p_i \in F_i$ but $p_i \notin F_j$ for $i \ne j$.*

Proof. Suppose that $\mathscr{F}$ does not contain such a subsystem. Without loss of generality we can suppose that $\mathscr{F} = \{F_1, \ldots, F_m\}$ where $|F_i| \le |F_j|$ for $i \le j$. We claim that there exists an A_i, $|A_i| \le k$, $A_i \cap F_i = \varnothing$ but $A_i \cap F_j \ne \varnothing$ for all $i < j$. Indeed, consider the family $\mathscr{H} = \{F_j - F_i: j > i\}$. $\varnothing \notin \mathscr{H}$ whence $\tau(\mathscr{H})$ is finite. Let A_i be a minimal cover of $\mathscr{H}$, $|A_i| = \tau(\mathscr{H})$. Then for every $p \in A_i$ there exists an $F_{j(p)}$ $(j > i)$ such that $A_i \cap F_{j(p)} = \{p\}$. By the indirect assumption the family $\{F_i\} \cup \{F_{j(p)}: p \in A_i\}$ has at most $k + 1$ members, i.e., $|A_i| \le k$. Finally, we can apply Theorem 2.1 for the family $\{A_i, F_i\}$. □

The following examples show that the bound in Theorem 2.6 is exact.

Example 2.7. $\mathbf{H} = \bigcup \mathbf{K}_i^{i+k-1}$ $(0 \le i \le r)$.

Example 2.8. Let $Y_i = \{y_1, \ldots, y_i\}$, $X_j = \{x_1, \ldots, x_j\}$, $Y_r \cap X_{r+k-2} = \varnothing$. Set $\mathscr{H}_{i,j} = \left\{H: H = Y_i \cup S \text{ where } S \in \binom{X_{j+k-2}}{j}\right\}$, and $\mathscr{H} = \bigcup \mathscr{H}_{i,j}$ where $i, j \ge 0$, $i + j \le r$.

A simple proof of the upper bound theorem. This application of Theorem 2.1 is due to Alon and Kalai [14]. Let P be a convex polytope in d-dimensional Euclidean space. Denote the numbers of i-dimensional faces of P by $f_i(P)$, i.e., $f_0(P) = \#$ of vertices, $f_{d-1}(P) = \#$ facets, put $f_{-1}(P) = f_d(P) = 1$.

Theorem 2.9 (McMullen [192]). *If $P \subset \mathbb{R}^d$ is a convex polytope with n vertices then*

$$f_j(P) \le \sum_{i=0}^{j+1} \binom{d-i}{d-j-1} a(n, d, i)$$

where

$$a(n,d,i) = \begin{cases} \binom{n-d+i-1}{i} & \text{for } i \le d/2 \\ \binom{n-i-1}{d-i} & \text{for } i \ge d/2. \end{cases}$$

This was conjectured by Motzkin in 1957 [200]. The following example (the so-called cyclic d-polytope) shows that the bounds in Theorem 2.9 are sharp.

Example 2.10 (Gale [140]). For $x \in \mathbb{R}$ let $p(x) \in \mathbb{R}^d$ be the following point $p(x) = (x, x^2, x^3, \dots, x^d)$. Set $P = conv\{p(i): i = 1, 2, \dots, n\}$.

Using Theorem 2.12 one can easily obtain that for this P equality holds in Theorem 2.9. Indeed, it is easy to see that $f_i(P) = \binom{n}{i+1}$ for $i + 1 \le [d/2]$, so we can calculate the $g_i - s$ and hence the remaining $f_i(P) - s$.

The first step of the proof of Theorem 2.9 is that without loss of generality we may restrict our attention to *simplicial* polytopes. This means we can suppose that every face of P is a simplex. Indeed, a slight perturbation of the vertices of P makes it simplicial and does not decrease $f_i(P)$.

With the simplicial polytope P we associate a hypergraph $\mathscr{C}$, the *simplicial complex* of P. Suppose $P = conv\{p_i: 1 \le i \le n\}$ and let $V(\mathscr{C}) = \{1, 2, \dots, n\}$, $\mathscr{C} = \{S \subset \{1, 2, \dots, n\}: \{p_s: s \in S\}$ is a face of $P\}$. Clearly $f_i(P) = \#$ $(i+1)$-element members of $\mathscr{C}$. In general, a hypergraph $\mathscr{H}$ is called a *simplicial complex* if $H \in \mathscr{H}$, $H' \subset H$ implies $H' \in \mathscr{H}$ for all H, H'. A member S of a simplicial complex $\mathscr{H}$ is *free* if S is contained in a unique maximal face M of $\mathscr{H}$. The operation of deleting S and all faces that contain it is called an *elementary collapse*. If the size of S is s and the size of M is m, it is called an *elementary* (s, m)-*collapse*. A *collapse process* on $\mathscr{H}$ is a sequence $\mathscr{H} = \mathscr{H}_0 \supset \mathscr{H}_1 \supset \cdots \supset \mathscr{H}_t$ of simplicial complexes such that for $1 \le i \le t$ $\mathscr{H}_i$ is obtained by an elementary collapse.

We say that $\mathscr{H}$ is *shellable* if its maximal faces are all of dimension $d - 1$, and they can be ordered $H_1, H_2, \dots, H_t$ so that for all $1 \le k \le t - 1$

$$2^{H_k} \cap \left(\bigcup_{i>k} 2^{H_i} \right) = \bigcup_{j=1}^{s_k} 2^{G_{k,j}}$$

where $G_{k,1}, \dots, G_{k,s_k}$ are some distinct $d - 1$ subsets of H_k. In this case define $\mathscr{H}_i = \bigcup_{j>i} 2^{H_j}$ (i.e., $\mathscr{H}_t = \varnothing$). For $1 \le i \le t - 1$ let $S_i = F_i - \bigcap_{1 \le j \le s_i} G_{i,j}$, and let $S_t = \varnothing$. One can easily check that S_i is a free face of $\mathscr{H}_{i-1}$ and $\mathscr{H}_i$ is obtained from $\mathscr{H}_{i-1}$ by deleting S_i and all faces containing it, i.e., by an elementary $(|S_i|, d)$-collapse.

Theorem 2.11 (Bruggesser and Mani [65]). *Every simplicial polytope $P \subset \mathbb{R}^d$ is shellable.*

Proof (outline). Let H_1 and H_t be two arbitrary facets of P and $x \in int\, H_1$, $y \in int\, H_t$. Consider the line l through x and y. Let $x(H)$ be the point where l meets the hyperplane containing the facet H. We can choose x and y such that these points are all distinct. The points $x(H)$ have a natural ordering if we go along l beginning at $x = x(H_1)$ and going further from y and returning to the other end of l and ending the process at $y = x(H_t)$. This ordering defines the desired order of the faces $H_1, H_2, \ldots, H_t$. □

The above defined shelling has the property that its reverse order is a shelling, too. If in the i-th step in the shelling we had an (S_i, H_i) collapse then in the second shelling in the $(t-i)$-th step we have an $(H_i - S_i, H_i)$ collapse. This implies the following: Denote by g_i the number of elementary (i, d) collapses of the shelling defined first.

Theorem 2.12. (*Dehn-Somerville equations see* [145], Section 9.2]). $g_i = g_{d-i}$.

Proof. Indeed, the number of j-dimensional faces deleted in an elementary (i, d) collapse is $\binom{d-i}{j+1-i}$, hence

$$f_j(P) = \sum_{i=0}^{j+1} \binom{d-i}{j+1-i} g_i. \tag{2.2}$$

The matrix $\left(\binom{d-i}{j+1-i}\right)_{-1 \le j \le d-1, 0 \le i \le d}$ is regular, so the sequence $(g_0, g_1, \ldots, g_d)$ is uniquely determined by the sequence $(f_i(P))_{-1 \le i \le d-1}$. So the value of g_i is independent of the actual shelling process. Denote by g_i' the number of elementary (i, d)-collapses in the second shelling defined above. Then $g_i = g_i' = g_{d-i}$. So the above formula holds for *every* shelling of $\mathscr{C}_P$. □

Returning to the proof of Theorem 2.9, we can rewrite (2.2) using Theorem 2.12 and the notation $\tilde{g}_i = \sum_{j=0}^{i} g_j$.

$$f_j = \sum_{i=0}^{[d/2]} \left[\binom{d-i-1}{d-j-2} - \binom{i}{d-j-2}\right] \tilde{g}_i \text{ for odd } d,$$

$$f_j = \sum_{i=0}^{d/2-1} \left[\binom{d-i-1}{d-j-2} - \binom{i}{d-j-2}\right] \tilde{g}_i + \binom{d/2}{d-j-1} \tilde{g}_{d/2} \text{ for even } d.$$

In order to prove Theorem 2.9 it is enough to show that

$$\tilde{g}_i \le \binom{n-d+i}{i}. \tag{2.3}$$

Proof of (2.3). Let S_l and M_l be the free and the maximal face corresponding to the l-th elementary collapse, $1 \le l \le t$. Let (S_{l_j}, M_{l_j}) be the subsequence of $(S_l, M_l)_{1 \le l \le t}$ consisting of those pairs with $|S_l| \le i$, $1 \le j \le \tilde{g}_i$. One can easily check that $A_j = S_{l_j}$, $B_i = \{1, 2, \ldots, n\} - M_{l_j}$ satisfy the hypothesis of Theorem 2.1. □

Intersection patterns of convex sets. Using the above method Alon and Kalai [14] gave also a simple proof of a theorem conjectured by Katchalski and Perles and proved independently by Eckhoff [87] and Kalai [162]. This theorem asserts that if $\mathscr{K}$ is a family of n convex sets in $\mathbb{R}^d$ and $\mathscr{K}$ has no intersecting subfamily of size $d + r + 1$, then the number of intersecting k-subfamilies of $\mathscr{K}$ for $d < k \le d + r$ is at most

$$\sum_{i=0}^{d} \binom{n-r}{i} \binom{r}{k-i}.$$

Equality holds, e.g., if $\mathscr{K} = \{K_1, K_2, \ldots, K_n\}$ where $K_1 = \ldots = K_r = \mathbb{R}^d$ and $K_{r+1}, \ldots, K_n$ are hyperplanes in general position in $\mathbb{R}^d$.

The generalization of Theorem 2.3 leads to the following problem.

Problem 2.13. Let $A_1, \ldots, A_M$ be a-dimensional subspaces and $B_1, \ldots, B_M$ be b-element sets with the property $|A_i \cap B_i| \leq t$ and $|A_i \cap B_j| > t$ for $i \neq j$. Determine $M_t(a, b)$, the greatest number m such that such a family exists.

Clearly, $\binom{a+b-2t}{a-t} \leq M_t(a,b) \leq \binom{a+b-t}{a}$, but there is no equality in general. E.g., construction of Burr, Grünbaum and Sloane [66], which consists of $b+3$ points and $1 + [b(b+3)/6]$ line such that every line contains exactly 3 points, shows that $M_1(2, b) \geq 1 + [b(b+3)/6]$. An elementary construction can be found in Füredi, Palásti [138]. On the other hand, trivially $M_1(2,b) \leq 1 + \binom{b}{2} < \binom{a+b-t}{a}$. More generally we can ask:

Problem 2.14. Let $A_1, A_2, \ldots, A_m$ and $B_1, \ldots, B_m$ be finite sets with $|A_i| \leq a$, $|B_i| \leq b$ satisfying $|A_i \cap B_i| \leq t$ and $|A_i \cap B_j| \geq l$ for $i \neq j$, where $0 \leq t \leq l$. Determine $m_{t,l}(a,b)$, the greatest number m such that such a family exists.

A generalization for hypersurfaces (Deza and Frankl [80]). A more advanced generalization of Theorem 2.2 is due to Deza and Frankl [80]. Let $\mathbb{F}$ be a commutative field (finite or infinite) and let $\mathbb{P} = \mathbb{P}(n, \mathbb{F})$ be the n-dimensional projective space over $\mathbb{F}$. Every point $\mathbf{x} \in \mathbb{P}$ can be expressed by $n+1$ homogeneous coordinates $\mathbf{x} = (x_1, \ldots, x_{n+1})$ not all zero and $(x_1, x_2, \ldots, x_{n+1}) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_{n+1})$ for $0 \neq \lambda \in \mathbb{F}$. By a *hypersurface* of degree d we simply mean the set of points $\mathbf{x} \in \mathbb{P}$ with $f(\mathbf{x}) = 0$, where f is a homogeneous polynomial of degree d of the variables $x_1, \ldots, x_{n+1}$.

Theorem 2.15. *Suppose that $H_1, \ldots, H_m$ are hypersurfaces of degree at most d in $\mathbb{P}$ so that the intersection of any choice of $\binom{n+d}{d}$ of them is non-empty. Then $\bigcap_{i=1}^{m} H_i \neq \varnothing$.*

The bound in Theorem 2.15 is best possible. The above theorem is a Helly type result. (About Helly type theorems in geometry, see [76]). Theorems 2.1–2.6 all have Helly type reformulations, for example Theorem 2.2 is equivalent to the following.

Corollary 2.16. *Let $\mathscr{A} = \{A_1, A_2, \ldots\}$ be a family of a-dimensional subspaces. Suppose that for every subfamily $\mathscr{B} \subset \mathscr{A}$, with $|\mathscr{B}| \leq \binom{a+b}{a}$ there exists a b-dimensional* *subspace C such that* $\dim(C \cap B) \geq 1$ *for all* $B \in \mathscr{B}$. *Then there exists a b-dimensional subspace D for which* $\dim(D \cap A) \geq 1$ *for all* $A \in \mathscr{A}$.

The structure of τ-critical graphs and hypergraphs. The investigation of τ-critical graphs was initiated by Erdös and Gallai. In 1961 they proved [100] that if G is a τ-critical graph without an isolated vertex then $|V(G)| \leq 2\tau(G)$, and here equality holds iff G is the disjoint union of τ edges. Gallai suggested to classify τ-critical graphs according $\delta(G) = 2\tau(G) - |V(G)|$. Denote by $deg_G(x)$ the number of edges through x in the (hyper)graph G. Hajnal proved [150] that in every τ-critical graph G (without isolated vertices), $deg_G(x) \leq \delta(G) + 1$ holds for every vertex. This implies that if G is τ-critical and $\delta(G) = 1$, then G consists of an odd cycle and disjoint edges.

Without loss of generality we can investigate only connected τ-critical graphs. Gallai conjectured and Andrásfai [16] proved that all the connected τ-critical graphs with $\delta(G) = 2$ can be obtained from $\mathbf{K}^4$ by subdividing each edge by an even number of new points.

It is easy to see that we can eliminate vertices of degree 2 from a τ-critical graph keeping its δ. Namely if G is a τ-critical graph and $x \in V(G)$, $deg(x) = 2$, $\{x, y\}$, $\{x, z\} \in E(G)$ then delete x and $\{x, y\}$, $\{x, z\}$ and identify y and z. The inverse of this operation also keeps τ-criticality (see, e.g., Plummer [213]). So the above result of Andrásfai can be reformulated as follows: if G is a τ-critical connected graph with $\delta(G) = 2$, every vertex of which has degree at least 3, then $G = \mathbf{K}^4$. Gallai conjectured the following finite basis theorem:

Theorem 2.17 (Lovász [183]). *Let* $\delta \geq 2$. *Then the number of connected τ-critical graphs with minimum degree* ≥ 3 *and with* $\delta(G) = \delta$ *is finite.*

Using the above operation to generate vertices of degree 2 we can obtain all τ-critical graphs. The case $\delta = 3$ was proved by L. Surányi [230] (actually he proved that in this case there exist at most 12 such graphs.) He also has a nontrivial construction showing that there are at least $\delta^2/4$ such graphs [231]. (Lovász' upper bound is huge, about 2^{δ^2}). In the proof of 2.17 Lovász uses geometric graphs. The following theorem has an important role also. For $A \subset V(\mathbf{G})$ we call A *independent* if it does not contain an edge of G. $\Gamma(A) =: \{y \colon y \in V(G) - A$, there exists an edge $\{x, y\} \in E(G)$ with $x \in A\}$.

Theorem 2.18 (Lovász [183], Surányi [230]). *If G is a τ-critical graph (without isolated vertices) and* $A \subset V(G)$ *is an independent set then for all* $a \in A$ *we have*

$$deg_G(a) \leq |\Gamma(A)| - |A| + 1. \tag{2.4}$$

This is a generalization of another theorem of Hajnal [150].

Much less is known about τ-critical hypergraphs. Call a set $A \subset V(\mathbf{H})$ *strongly independent* if no edge intersects it in more than one element. Moreover, for $B \subset Y(\mathbf{H})$ define $\Gamma(B) =: \{E - \{b\} \colon b \in B, b \in E \in E(\mathbf{H})\}$. Then Gyárfás, Lehel and Tuza [148] proved that (2.4) holds for every τ-critical r-graph if A is a strongly independent set. Using this result they proved that $|V(\mathbf{H})| \leq \tau\binom{\tau + r - 2}{r - 2} + \tau^{r-1}$. Recently, Tuza determined $\max|V(\mathbf{H})|$ appart from a constant factor.

Theorem 2.19 ([274]). *Suppose that* **H** *is a* τ*-critical r-graph* (*without isolated vertices*). *Then* $|V(\mathbf{H})| \le \binom{\tau+r-1}{r-1} + \binom{\tau+r-2}{r-2}$.

Let $v(r,\tau)$ be the maximum number of (nonisolated) vertices of a τ-critical r-graph (with covering number τ). Theorem 2.19 implies the theorem of Erdös and Gallai ($v(2,\tau) = 2\tau$) and improves a result of Petruska and Szemerédi [210] on 3-graphs (they proved $v(3,\tau) \le 8\tau^2 + 2\tau$). The following construction shows that Theorem 2.19 yields the right order of magnitude of $v(r,\tau)$ for all r and τ.

Example 2.20 (Tuza [240]). Consider the complete $(r-i)$-graph $\mathbf{K}_{r-i}^{\tau+r-1-i}$ on $(\tau+r-1-i)$ vertices. Add i new vertices to each edge. We obtain a τ-critical hypergraph with $i\binom{\tau+r-1-i}{\tau-1} + \tau + r - 1 - i$ vertices.

It is conjectured that this construction is optimal.

Conjecture 2.21 (Gyárfás Lehel, Tuza [148]).

$$v(r,\tau) = \left\lceil \frac{r}{\tau} \right\rceil \binom{\left\lfloor \frac{r(\tau-1)}{\tau} \right\rfloor + \tau - 1}{\tau - 1} + \left\lfloor \frac{r(\tau-1)}{\tau} \right\rfloor + \tau - 1.$$

Tuza [240] proved (see Section 4) that $v(r,\tau) < \binom{\tau+r}{r}$. The ratio of the upper bound in Theorem 2.19 and the lower bound given by Example 2.20 is always less than 4.

Erdös and Gallai [100] solved the case $\tau = 2$ proving $v(r,2) = \lfloor (r+2)^2/4 \rfloor$.

Intersecting and k-wise intersecting families. A hypergraph is *k-wise intersecting* if any k edges of it have a non-empty intersection. It is obvious that for an intersecting family $\mathscr{F}$ we have

$$\tau(\mathscr{F}) \le \min_{F \in \mathscr{F}} |F|.$$

For k-wise intersecting families we have (see, e.g. [186])

$$\tau(\mathscr{F}) \le 1 + (\min_{F \in \mathscr{F}} |F| - 1)/(k-1). \tag{2.5}$$

Hence if $\mathscr{F}$ is at least $(\min|F| + 1)$-wise intersecting then $\tau(\mathscr{F}) = 1$.

Multi-transversals. Let $s \ge 1$ be an integer, **H** a hypergraph. The set A is an *s-multitransversal* of **H** if either $|A \cap H| \ge s$ or $A \supset H$ holds for all $H \in E(\mathbf{H})$. A 1-multitransversal is a cover. Define $\tau'_s(\mathbf{H}) =: \min\{|A|: A \text{ is a } s\text{-multitransversal of } \mathbf{H}\}$, the *s-multitransversal number* of **H**. Lehel [170, 171] introduced the following functions:

$$m(s,\tau) = \max\{\tau'_s(\mathbf{H}): \mathbf{H} \text{ is } \tau\text{-critical with } \tau(\mathbf{H}) = \tau\},$$

$$m_k(s,\tau) = \max\{\tau'_s(\mathbf{H}): \mathbf{H} \text{ is } \tau\text{-critical, } \tau(\mathbf{H}) = \tau \text{ and } \mathbf{H} \text{ is } k\text{-wise intersecting}\}.$$

Theorem 2.22 (Lehel [170, 171]).

(i) *Every* τ*-critical hypergraph has a 2-multitransversal of at most* $[(\tau+2)^2/4]$ *points.*

(ii) *Every τ-critical hypergraph with $\tau = 2$ has an s-multitransversal with cardinality at most* $[(s+2)^2/4]$. *Both estimates are sharp, i.e.,* $m(2,\tau) = [(\tau+2)^2/4]$ *and* $m(s,2) = [(s+2)^2/4]$.
(iii) $m(s,\tau) \leq \tau^s$,
(iv) $m_2(2,3) = 5$, $m_2(3,3) = 9$,

$$m_k(s,2) = \begin{cases} s+1 & \text{if } s \leq k \\ (k+1)(s-k+1) & \text{if } k \leq s \leq 2k \\ [(s+2)^2/4] & \text{if } s \geq 2k. \end{cases}$$

Tuza [240] has additional results (see later in Theorem 4.14) and it looks hopeful to determine $m_k(s,\tau)$ for other values of τ. Another variation of the above problem is the following:

Define $u(r,s) = \max\{\tau_s'(\mathbf{H})\colon H \text{ is intersecting of rank } r, \tau(\mathbf{H}) \geq s\}$.

Problem 2.23. Determine $u(r,s)$.

It is easy to see that $u(r,2) = 3r - 3$ for $r \geq 2$ (see Lovász [186]).

Problem 2.24 (Roudneff [219]). What is the maximum number of edges of a τ-critical linear hypergraph? (Linear means that $|E \cap E'| \leq 1$ for all $E, E' \in E(\mathbf{H})$, $E \neq E'$.) Let $m(\tau) = \max|E(\mathbf{H})|$: $\mathbf{H}$ is linear, τ-critical and $\tau(H) \leq \tau$. Then $\binom{\tau+1}{2} \leq m(\tau) \leq \tau^2 - \tau + 1$. The conjectured value of $m(\tau)$ is the lower bound.

3. Nontrivial Coverings and Designs

Nontrivial coverings in symmetric designs. Suppose that the hypergraph $\mathbf{H} = (X, \mathscr{E})$ is λ intersecting. Then clearly, every set T which contains an edge of $\mathbf{H}$ is a λ-multitransversal. So we call T a *nontrivial* λ-multitransversal (for $\lambda = 1$, non-trivial cover) if

$$\lambda \leq |T \cap E| < |E|$$

holds for every $E \in \mathscr{E}$.

An (r,λ)-*design* is a pair $(X, \mathscr{B})$ where X is a set of v elements, $\mathscr{B}$ is a collection of subsets of X called *blocks*, for every $x \in X$ we have $deg_{\mathscr{B}}(x) = r$ and every pair $\{x,y\} \subset X$ is contained in exactly λ members of $\mathscr{B}$. It is well known (see, e.g., Ryser [220]) that $|\mathscr{B}| \geq |X|$. In the case $|\mathscr{B}| = |X|$ we say $(X, \mathscr{B})$ is a *symmetric* (r,λ)-design. Then $v = (r^2 - r + \lambda)/\lambda$, and two members of $\mathscr{B}$ intersect in exactly λ elements, and for every $B \in \mathscr{B}$ has r elements.

A *projective plane of order n*, $PG(2,n)$, is a symmetric $(n+1,1)$ design. The Desarguesian projective plane, $DPG(2,q)$, is obtained from the finite field $\mathbb{F}_q$. The points of this plane are the equivalence classes in $\mathbb{F}_q^3 - \{(0,0,0)\}$ of the relation "$\sim$" defined by $(x,y,z) \sim (x',y',z')$ if there exists $c \neq 0$ in $\mathbb{F}_q$ such that $(x',y',z') = (cx,cy,cz)$. The lines of $DPG(2,q)$ have equations of the form $ax + by + cz = 0$ in $\mathbb{F}_q$ with $(a,b,c) \neq (0,0,0)$.

A more general example of symmetric (r,λ)-designs is the *t-dimensional finite* *projective space* of order q, $PG(t,q)$, where q is a primepower and $r = q^t + q^{t-1} + \cdots + 1$, $\lambda = q^{t-1} + \cdots + q + 1$. This hypergraph is t-wise intersecting (and i-*wise* $(q^{t-i} + \cdots + q + 1)$-intersecting for $2 \le i \le t$).

The investigation of the covers and non-trivial covers of block designs was initiated by Pelikán [208]. Beyond other results he observed that there is no non-trivial cover T of the projective plane of order 3, and for $n \ge 4$ $|T| \ge n + 1 + \sqrt{n/2}$. The following theorem which sharpens and generalizes this result was proved for $\lambda = 1$ by Pelikán (unpublished) and Bruen [60], [61], for all λ by deResmini [214], and the case of equality (for $\lambda > 1$) was characterized by Drake [83].

Theorem 3.1. *Let $(X, \mathscr{B})$ be a symmetric (r,λ)-design, and suppose that $1 \le |T \cap B| < |B|$ holds for all $B \in \mathscr{B}$. Then $|T| \ge (r + \sqrt{r-\lambda})/\lambda$. Moreover if $|T| = (r + \sqrt{r-\lambda})/\lambda$ holds then T induces a Baer subdesign (i.e., $\{T \cap B: |T \cap B| > 1, B \in \mathscr{B}\}$ is a symmetric $(1 + \sqrt{r-\lambda}, \lambda)$-design).*

It is easy to check that a Baer subdesign is always a (non-trivial) cover, so Theorem 3.1 yields a characterization of Baer subdesigns. In the case if T is a non-trivial cover in a projective plane of order n Bierbrauer [253] improved 3.1 to $|T| > n + \sqrt{n} + 2$. C. Kitto [166] (a student of D. Drake) has proved $|T| > n + \sqrt{n} + 3$ if n is not an integer square, $n > 36$ and if T is of *maximal type.* A non-trivial cover T is of maximal type if $|T| = n + k$ and if some line of the projective plane contains k points of T.

Proof of 3.1. For an arbitrary set $T \subset X$ we have

$$\sum_B |T \cap B| = r|T|, \tag{3.1}$$

and

$$\sum_B \binom{|T \cap B|}{2} = \lambda \binom{|T|}{2}. \tag{3.2}$$

Using linear combinations of (3.1) and (3.2) we can express any polynomial of degree 2 of the form $\sum_B (\alpha|T \cap B|^2 + \beta|T \cap B| + \gamma)$. For example we have ($t = |T|$, $v = (r^2 - r + \lambda)/\lambda$)

$$\sum_B (|T \cap B| - 1) = tr - v, \tag{3.3}$$

$$\sum_B (|T \cap B| - 1)|T \cap B| = \lambda t^2 - \lambda t. \tag{3.4}$$

We need the following.

Proposition 3.2. *For every $B \in \mathscr{B}$ we have $|T \cap B| \le t\lambda - r + 1$.*

Proof. Indeed, let $x \in B \backslash T$ (it is non-empty!). Consider the blocks $B = B_1, B_2, \ldots, B_r$ through x. These blocks cover $X \backslash \{x\}$ exactly λ-times, i.e.,

$$\lambda t = \lambda |T| = \sum_{1 \le i \le r} |B_i \cap T| \ge |T \cap B| + r - 1. \qquad \square$$

Every term of (3.3) and (3.4) is non-negative, hence

$$\lambda t^2 - \lambda t = \sum_B (|T \cap B| - 1)|T \cap B| \le \left(\sum_B (|T \cap B| - 1)\right) \max |T \cap B|$$
$$\le (tr - v)(\lambda t - r + 1) \tag{3.5}$$

Rearranging we obtain

$$0 \le \lambda t^2 - 2rt + v$$

which gives that either $t \ge (r + \sqrt{r - \lambda})/\lambda$ or $t \le (r - \sqrt{r - \lambda})/\lambda$. But Proposition 3.2
implies $t \ge r/\lambda$, hence $t \ge (r + \sqrt{r - \lambda})/\lambda$ must hold.

In the case of equality (3.5) implies that for all B either $|T \cap B| - 1 = 0$ or $|T \cap B| = t\lambda - r + 1 = \sqrt{r - \lambda} + 1$ holds, implying the second part Theorem 3.1. □

Using the above method we can prove.

Theorem 3.3. *Let $(X, \mathscr{B})$ a symmetric (r, λ)-design, and suppose that $i \le |T \cap B| < |B|$ holds for all $B \in \mathscr{B}$, $i \le \lambda$. Then*

$$|T| \ge \frac{i}{\lambda}\left(r + \frac{r - i}{(i-1)/2 + \sqrt{(i-1)^2/4 + i(r-i)(r-1)/(r-\lambda)}}\right).$$

This is a slight improvement on the trivial $|T| \ge ir/\lambda$ by a term of about $\sqrt{ir}/\lambda$. Theorem 3.1 is a consequence of 3.3 (with $i = 1$). For $i = \lambda$ we have

Corollary 3.4. *Let T be a non-trivial λ-multitransversal of a symmetric (r, λ)-design. Then*

$$|T| \ge r + \frac{r - \lambda}{(\lambda - 1)/2 + \sqrt{(\lambda - 1)^2/4 + \lambda(r - 1)}}. \qquad \square$$

Corollary 3.5. *A symmetric (r, λ)-design is a maximal λ-intersecting family of r-sets.*

This means that if T is an arbitrary set satisfying $|T| \le r$, $|T \cap B| \ge \lambda$ (for all $B \in \mathscr{B}$) then T is a member of $\mathscr{B}$.

Proof of 3.3. We can proceed as in the proof of 3.1. Instead of Proposition 3.2 we have

$$\max |T \cap B| \le t\lambda - ir + i$$

Using the notation $t = (ir + x)/\lambda$ (3.1) and (3.2) imply

$$\sum_B (|B \cap T| - i) = rx + ir - i\lambda$$

$$\sum_B (|B \cap T| - i)^2 = x^2 + (r - \lambda)x + i(r - i)(r - \lambda),$$

so we obtain

$$0 \le (r - 1)x^2 + (r - \lambda)(i - 1)x - i(r - i)(r - \lambda)$$

This implies 3.3. □

Drake [83] found a generalization of Theorem 3.1 for some non-symmetric designs. His result was extended by Jungnickel and Leclerc [161] to all (r, λ)-designs. The proofs are very similar to the proofs of Theorems 3.1 and 3.3. Another generalization of Theorem 3.1 is due to Beutelspacher:

Theorem 3.6 ([39]). *Suppose that T is a subset of points of a d-dimensional projective space of order q, which intersects all the $(d-t)$-dimensional subspaces but does not contain a t-dimensional subspace $(d > t \geq 1)$. Then*

$$|T| \geq q^t + \cdots + q + 1 + q^{t-1}\sqrt{q}.$$

He also characterized the case of equality. Bruen and Rothschild [64] proved that the covering number of the Möbius plane of order q is $2q - 1$ (if $q > 3$).

If $\mathbb{P}$ is a Desarguesian projective plane of order q, and q is a square then its vertices has a partition into $q - \sqrt{q} + 1$ Baer subplanes. (See e.g., [157].) The union of k disjoint Baer subplanes intersects every line in k or $k + \sqrt{q}$ elements. Hence $\tau'_k(\mathbb{P}) \leq k(q + \sqrt{q} + 1)$. Lasker and Sherk have the following:

Conjecture ([169]). Let $\mathbb{P}$ be an arbitrary projective plane of order q, and suppose that $|T \cap L| \geq 2$ for every line L. Then $|T| \geq 2(q + \sqrt{q} + 1)$ holds $(q \geq 8)$.

A very special case was proved in [215], but the conjecture is still open. deResmini [215] constructed 2-multitransversals on the Hughes-planes of size $2(q + \sqrt{q} + 1)$ which do not split into two disjoint Baer subplanes. She [216] also has a 3-multitransversal of 36 points in the Ostrom-Rosati plane of order 9 which does not contain any line.

The covering number of an affine plane. An affine plane of order n is a hypergraph $(X, \mathscr{A})$ where $|X| = n^2$, $|\mathscr{A}| = n^2 + n$, $\mathscr{A} \subset \binom{X}{n}$ and any two members of $\mathscr{A}$ (any two lines) have at most one common element. The Desarguesian affine plane of order q is denoted by $DAG(2, q)$, and obtained as follows: $X = \mathbb{F}_q^2$ and the lines have the form $\{(x, y): ax + by + c = 0\}$, $(a, b) \neq (0, 0)$.

The disjointness is an equivalence relation over the lines, so $\mathscr{A}$ can be decomposed into $n + 1$ n-element set $\mathscr{A} = \mathscr{L}_1 \cup \cdots \cup \mathscr{L}_{n+1}$ where $|\mathscr{L}_i| = n$, $\bigcup \mathscr{L}_i = X$, $A \cap A' = \varnothing$ for $A, A' \in \mathscr{L}_i$ $(1 \leq i \leq n + 1)$. These classes are called *parallel classes.* It is easy to prove that

$$\text{Suppose } |T| \leq n,\ T \cap A \neq \varnothing \text{ for all } A \in \mathscr{A} - \mathscr{L}_i. \text{ Then } T \in \mathscr{L}_i. \tag{3.6}$$

Trivially, $\tau(\mathscr{A}) \geq n$. Theorem 3.1 implies that

$$\tau(\mathscr{A}) \geq n + \sqrt{n} \tag{3.7}$$

holds for every affine plane $\mathscr{A}$ of order n. Jamison [160] and independently, Brouwer and Schrijver [59] proved that for $DAG(2, q)$ much more is true, proving a conjecture of J. Doyen. The case $q = 7$ was proved by Hansen and Lorea [151] using an extensive computer search.

Theorem 3.7 ([160], [59]). *For the Desarguesian affine plane of order q, $DAG(2, q)$ we have $\tau(DAG(2, q)) = 2q - 1$.*

Proof. $\tau(\mathscr{A}) \le 2q - 1$ is trivial (consider a line $A \in \mathscr{L}_i$ together with $q - 1$ additional elements, one from each line $L \in \mathscr{L}_i - \{A\}$). Suppose that T is a cover of $DAG(2, q)$, $T \subset \mathbb{F}_q^2$. Without loss of generality we can suppose that $(0,0) \in T$. Define the polynomial $P(x_1, x_2)$ over $\mathbb{F}_q$ as follows

$$P(x_1, x_2) = \prod_{(t_1, t_2) \in T \setminus \{(0,0)\}} (t_1 x_1 + t_2 x_2 - 1)$$

As T intersects every line, $P(x_1, x_2) = 0$ for all $(x_1, x_2) \in \mathbb{F}_q^2$, except $P(0,0) = (-1)^{|T|-1}$. We claim that every polynomial with these properties has degree at least $2(q - 1)$, so we obtain

$$|T| - 1 = deg\, P \ge 2q - 2$$

proving the theorem. P has a decomposition

$$P(x_1, x_2) = (x_1^q - x_1)P_1(x_1, x_2) + (x_2^q - x_2)P_2(x_1, x_2) + J(x_1, x_2),$$

where for every term $cx_1^i x_2^j$ of J one has $0 \le i, j \le q - 1$. Moreover $deg\, P \ge deg\, J$. Clearly

$$J(x_1, x_2) = 0 \text{ for all } \mathbb{F}_q^2 \setminus \{(0,0)\},\ J(0,0) \ne 0$$

holds. Hence J is divisible by the polynomials $x_1^{q-1} - 1$ and $x_2^{q-1} - 1$, so $deg\, J \ge 2q - 2$. □

The above method gives that the covering number of the hypergraph consisting of the hyperplanes of $AG(d, q)$ is $d(q - 1) + 1$. Actually Jamison proved much more. Namely, that if $A_1, \ldots, A_m$ are k-dimensional subspaces of $AG(d, q)$, and $\bigcup A_i = AG(d, q) - \{\mathbf{0}\}$ then $m \ge q^{d-k} - 1 + k(q - 1)$ (for $0 < k < d$).

Problem 3.8. Determine $\tau(\mathscr{A})$ for other affine planes. Describe the minimal ($(2q - 1)$-element) covers of $DAG(2, q)$.

Bruen and deResmini [63] showed that Theorem 3.7 is not valid for arbitrary affine planes. They constructed 16-element covers on all the three known non-Desarguesian affine planes of order 9.

Problem (P. Erdös). It is true that there exists an absolute constant C such that in any finite projective plane there is a cover T such that $|T \cap L| \le C$ for every line of $\mathbb{P}$?

Using a simple probabilistic argument Erdös, Silverman and Stein [257] proved that there exists a cover T with $|T \cap L| \le (2e + o(1)) \log q$, where q is the order of the plane. It was later improved for $|T \cap L| \le (2/\log 2 + o(1)) \log q$ by Abbott and Liu [6] for Desarguasian $PG(2, q)$, but this upper bound still tends to infinity. The only known construction is valid only a small class of the Desarguesian planes.

Example (Bruen and Fisher [62]). Let q be a power of 3. Set

$$T = \{(x, x^3, 0): x \in \mathbb{F}_q\} \cup \{(x, -x^3, 0): x \in \mathbb{F}_q\} \cup \{(0, 0, 1)\}.$$

Then T is a cover of the Desarguesian projective plane $\mathbb{P}$ of order q and no line of $\mathbb{P}$ contains more than 4 points of T.

E. Boros [254] generalized the above example proving the existence of a cover T

on the $DPG(2, p^s)$ such that $|T \cap L| \leq p + 1$ holds for each line L, whenever p is an odd prime.

Maximal intersecting families. A hypergraph $(V, \mathscr{E})$ is called an *r-clique*, or a *maximal intersecting family* (of rank r) if any two edges intersect in at least one point and it cannot be extended to another intersecting family by adding a new r-set. In other words the intersecting r-graph $\mathscr{E}$ is an r-clique if it does not have a non-trivial cover of size at most r. For example the complete hypergraph $\mathbf{K}_r^{2r-1}$, the finite projective plane of order $(r - 1)$ (see Corollary 3.5) are r-cliques. Erdös and Lovász [108] have given bounds for the minimum number $m(r)$ of edges in a r-clique. In particular they proved, that

$$m(r) \geq \frac{8}{3}r - 3,$$

which was improved by Dow, Drake, Füredi and Larson [81],

$$m(r) \geq 3r \quad \text{for } r \geq 4. \tag{3.8}$$

So (3.8) and Example 3.9 give $m(4) = 12$. It is easy to check that $m(2) = 3$, $m(3) = 7$.

The determination of the value of $m(r)$ is one of the few questions dealing with the problem of determination of the minimal cardinality of set-families satisfying certain restrictions in which no set can be added to it without violating these restrictions. This type of problems were raised by Erdös and Kleitman [106].

It was conjectured that $m(r) \geq r^2 - r + 1$, and equality holds whenever a projective plane of order $r - 1$ exists (Meyer [193], [194], Erdös [96]). The following example disproved this conjecture.

Example 3.9 ([128]). ($2n$-clique of size $3n^2$, see Fig. 3.1). Let $(X, \mathscr{A})$ be a finite

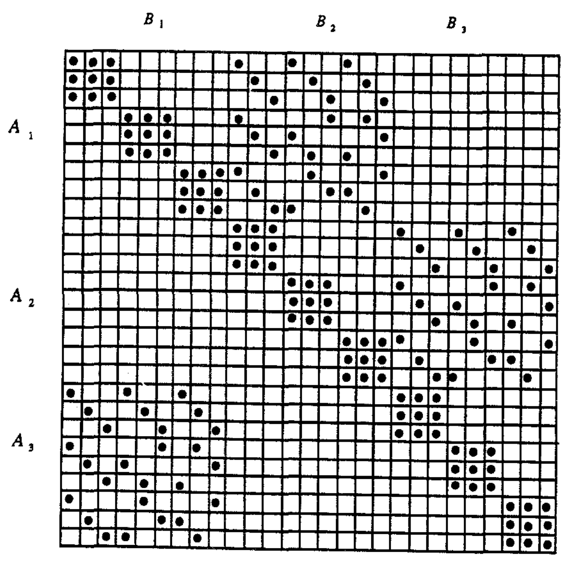

Fig. 3.1. Incidence matrix of a maximal clique of order 6 (Redrawn from Page 285 [128])

affine plane of order n with parallel classes $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_{n+1}$. Let $L_1, \ldots, L_n$ be the lines of $\mathscr{L}_{n+1}$. Consider three disjoint copies $(X^1, \mathscr{A}^1)$, $(X^2, \mathscr{A}^2)$, $(X^3, \mathscr{A}^3)$. Let $V = X_1 \cup X_2 \cup X_3$ and $\mathscr{E} = \{L_j \cup L: L_j \in \mathscr{L}^i_{n+1}, L \in \mathscr{L}^{i+1}_j$ for $i = 1, 2, 3, j = 1, 2, \ldots, n\}$. Then $|\mathscr{E}| = 3n^2$ and $(V, \mathscr{E})$ is a $2n$-clique.

(To prove that this example gives an r-clique we can use (3.6).) S. Sane [221] showed that the 253 blocks of the Witt design $S(23, 7, 4)$ form a 7-clique. r-cliques with few edges seem to be deeply associated with finite projective planes, all the known (and small) constructions use affine or projective geometries. The presently known further 3 classes of r-cliques with size less than r^2 are the following:

Example 3.10 (For $n = 2$ Babai and Füredi [128], for all n Drake and Sane [85]). An n-uniform projective Hjelmslev plane of order q is a $(q^n + q^{n-1})$-clique. This implies that if q is the order of a projective plane then

$$m(q^n + q^{n-1}) \le q^{2n} + q^{2n-1} + q^{2n-2}.$$

Example 3.11 (Blokhuis [43]). If q is an odd primepower $(q \ge 7)$ then $m(q + 1) \le \frac{3}{4}q^2 + 3q + \frac{5}{4}$.

His idea is as follows: Start with $PG(2, q)$, add some new r-sets $(r = q + 1)$. Throw away lines that do not intersect the new sets. Finally add more r-sets (if necessary) until the family is maximal intersecting.

This construction was the first counterexample of the conjecture of the author [128], that $|\mathscr{E}| \ge |\bigcup \mathscr{E}|$ holds for all r-cliques. Using the above idea Boros, Füredi and Kahn [56] obtained.

Example 3.12 ([56]). (*$(q + 1)$ – clique of size* $\frac{1}{2}q^2 + O(q)$). Suppose q is a primepower $q + 1 \equiv 0 (mod\, 6)$, $q > 20$. Consider $DPG(2, q)$ and let C be a conic, e.g., $C =: \{(x, y, z): x^2 + y^2 + z^2 = 0\}$. Let K be an affine regular hexagon inscribed to C (i.e., $K \subset C$). Then there exists a 6 element set $L(K)$, $L(K) \cap C = \varnothing$ with the property that a line through l and k where $l \in L(K)$, $k \in K$, is either a tangent of C or intersects K in a second point. Define

$$\mathscr{E}_0 = \{\text{all the lines which intersect } C \cup L(K)\} \cup \{(C \backslash K) \cup L(K)\}$$

Clearly, $|\mathscr{E}_0| = \dfrac{q^2}{2} + \dfrac{9}{2}q + O(1)$. Using a classification theorem of Wettl [244] about affine regular k-gons in $PG(2, q)$, one can show that every $(q + 1)$ element cover of $\mathscr{E}_0$, which is different from the members of $\mathscr{E}_0$, can be obtained from an inscribed regular k-gon into C. Hence, adding less than $q + 1$ new sets to $\mathscr{E}_0$ we will provide a $(q + 1)$-clique. (Especially, if $(q + 1)/6$ is a prime we do not have to add any new members.)

There exists a small r-clique for all r. Until this point the best upper bound for $m(r)$, valid for all r, supplied by the complete r-graph, implying $m(r) \le \binom{2r-1}{r} < 4^r$. From an r-clique $\mathscr{E}$ we can obtain an $(r + 1)$-clique in the following way: $\mathscr{E}' = \{E \cup \{x\}: E \in \mathscr{E}, x \in E'\} \cup \{E'\}$ where E' is an $(r + 1)$-set disjoint from all members of $\mathscr{E}$. Hence

$$m(r + 1) \le rm(r) + 1. \tag{3.9}$$

Using the current best result from number theory, which says
(3.10) ([263]) there is at least one prime between r and $r - r^\theta$ for all sufficiently large integers r, with $\theta = \frac{23}{42}$,
and repeatedly using (3.9) and the fact that $m(q+1) \le q^2 + q + 1$ for q a prime one obtains that

$$m(r) \le r^{cr^\theta}$$

holds for all r. Surprisingly, to prove a polynomial upper bound for $m(r)$ is really elementary.

Theorem 3.13 (Blokhuis [43]). *For every r there exists an r-clique of size at most r^5.*

Proof. This proof is based on a significantly simplified idea of Drake [84]. We will use induction on r. r can be written as $r = q + s$ where q is a prime and $1 \le s \le q^{0.6}$. The case $s = 1$ is covered by the projective plane, so we suppose $s \ge 2$. Let $(V, \mathscr{E})$ be an s-clique of size $m(s) \le s^5$. Consider $DAG(2,q) = (X, \mathscr{A})$ where $\mathscr{A} = \mathscr{L}_1 \cup \cdots \cup \mathscr{L}_{q+1}$ is a partition of parallel classes. Consider $q+1$ disjoint copies of $(V, \mathscr{E})$ over the underlying sets $V_1, V_2, \ldots, V_{q+1}$, suppose $V_i \cap X = \varnothing$ for all i. Finally define a $(q+s)$ clique $\mathscr{F}$ over $Y = X \cup V_1 \cup \cdots \cup V_{q+1}$ as follows:

$$\mathscr{F} = \{L \cup E: L \in \mathscr{L}_i, E \in \mathscr{E}_i, 1 \le i \le q+1\}$$

Clearly $\mathscr{F}$ is an intersecting family of $(q+s)$-sets of size

$$|\mathscr{F}| = (q^2 + q)m(s) < (q+s)^2 \cdot s^5 < (q+s)^5.$$

It remains to be shown that it is maximal. Let T be a cover of $\mathscr{F}$ of size $|T| \le q + s$. As $|T| < 2q - 1$ by Theorem 3.7 there exists a line L_1, say $L_1 \in \mathscr{L}_1$, such that $L_1 \cap T = \varnothing$. Then considering the sets $\{L_1 \cup E: E \in \mathscr{E}_1\}$ we obtain that $|T \cap V_1| \ge s$. As $|T| < s(q+1)$ there exists a V_i, say V_2, such that $|V_2 \cap T| < s$. Then T avoids a member E_2 of $\mathscr{E}_2$, so considering the edges $\{L \cup E_2: L \in \mathscr{L}_2\}$ we obtain $|T \cap X| \ge q$. Now $|T| \le q + s$ implies $|T \cap X| = q$, $|T \cap V_1| = s$ and thus $T \cap V_i = \varnothing$ for $i > 1$. Now $T \cap X$ intersects all the lines of $\mathscr{A} - \mathscr{L}_1$, hence $T \cap X \in \mathscr{L}_1$, by (3.6). Similarly, $T \cap V_1$ is a blocking set of $\mathscr{E}_1$ of size s, hence $T \cap V_1 \in \mathscr{E}_1$, i.e. $T \in \mathscr{F}$. □

Remark. If we knew that for two consecutive primes $p_{k+1} - p_k = 0(\log p_k^2)$ holds, then the above proof would give

$$m(r) \le r^2(\log r)^{4+\varepsilon}.$$

But this is still far from the lower bound (3.8).

Problem 3.14. Determine the right order of magnitude of $m(r)$. Is it true that $m(r)/r \to \infty$, or $m(r) = o(r^2)$? The author believes that $m(r)$ is closer to r^2.

Proposition 3.15. [128]. *Let $(X, \mathscr{F})$ be a r-clique. Then either $|\mathscr{F}| > r^2$ or $|X| > r^2/2\log r$.*

Proof. Suppose $|X| \le r^2/2\log r$. Count in two ways the number of pairs (F, A) where $F \in \mathscr{F}$, $A \in \binom{X}{r}$, $F \cap A = \varnothing$. □

A family is a (r, λ)-*clique* if it is a λ-intersecting r-graph without any non-trivial λ-multitransversal of size r. Denote by $m_\lambda(r)$ the minimum cardinality of an (r, λ)-clique. By a slight modification of the above construction of Blokhuis one can obtain that

$$m_\lambda(r) < r^5$$

holds for all $r \geq \lambda$. (We simply add $(\lambda - 1)$ common vertices to a $(r - \lambda + 1)$-clique $\mathscr{F}$ defined in the proof of 3.13.)

Minimal intersecting r-graps with covering number r. Erdös and Lovász [108] introduced the function

$$n(r, \tau) = \min\{|\mathscr{F}|: \mathscr{F} \text{ is an intersecting family with } \tau(\mathscr{F}) = \tau\}.$$

Set $n(r) = n(r, r)$. Of course, this function is defined when $1 \leq \tau \leq r$. If we drop the intersecting property we obtain a trivial problem (the smallest example is τ disjoint r-sets.) As $n(r) \leq m(r)$ we have that

$$n(r) \leq r^5 \quad \text{for all } r, \tag{3.11}$$

and

$$n(r) \leq r^2 - r + 1 \quad \text{for } r = q + 1. \tag{3.12}$$

Theorem 3.16 (Erdös and Lovász [108]). *For all r*

$$\tfrac{8}{3}r - 3 \leq n(r)$$

and if q is a primepower then

$$n(q + 1) \leq 4q\sqrt{q}\log q.$$

Proof (sketch). *Lower bound*: Let x_1 be a vertex of maximum degree of $\mathscr{F}$, let x_2 be a vertex of maximum degree of $\mathscr{F} - \{F \in \mathscr{F}: x_1 \in F\}$, and so on. If $|\mathscr{F}|$ is too small we obtain a cover of size less than r.

Upper bound: Consider the family of lines of the finite projective plane $PG(2, q)$. Set $t = 4q\sqrt{q}\log q$. We can choose t lines in $\binom{q^2 + q + 1}{t}$ ways. One can show that all but $o\left(\binom{q^2 + q + 1}{t}\right)$ choices of t lines cannot be covered by fewer than $q + 1$ points. □

Conjecture 3.17 ([108]). $n(r)/r \to \infty$ whenever $r \to \infty$.

P. Erdös offered \$500 [97] for the proof (or disproof) of this conjecture. They also believe that the following is true.

Conjecture 3.18 ([108]). One can choose less than $O(q \log q)$ lines of $PG(2, q)$ which cannot be covered by q points.

Erdös and Duke [86] defined the *property* $C(s)$. The intersecting r-graph $\mathscr{F}$ has property $C(s)$ if $\tau(\mathscr{F}) = r$ and each $(r - 1)$ subset misses at least s edges of $\mathscr{F}$. (If $\tau(\mathscr{F}) = r$, and $\mathscr{F}$ is intersecting, then it has at least property $C(1)$.)

Example 3.19 (Frankl [unpublished]). (r – *graph with property* $C(r)$). Suppose r is even, let X_1, X_2, X_3 be three disjoint $(r-1)$-sets and take as an edge each r-set of $X_1 \cup X_2 \cup X_3$ having exactly $r/2$ vertices in two of the $X_i - s$.

Example 3.20 (Duke and Erdös [86]). (r – *graph with property* $C(r)$). Suppose $r = 2q + 2$ where q is an order of a finite projective plane $(X, \mathscr{P})$. Let X_1, X_2, X_3 be three disjoint $(q^2 + q + 1)$-sets and $\mathscr{P}_1, \mathscr{P}_2, \mathscr{P}_3$ three copies of $\mathscr{P}$ on the $X_i - s$. Form a $(2q+2)$-graph with vertex-set $X_1 \cup X_2 \cup X_3$ by taking as an edge each set which is the union of two lines from two different $X_i - s$.

Problem 3.21 ([86]). What is the minimum cardinality of an r-graph with property $C(r)$? (They conjecture that it is less than r^4 for all r.)

Problem 3.22 ([86]). What is the maximum value of s (for a given r) such that there exists an r-graph with property $C(s)$?

Cliques of maximal size. Lovász [177] proved that the number of r-cliques is finite for any given r, by showing that for every maximal intersecting r-graph $\mathscr{F}$ we have $|\mathscr{F}| \le r^r$. His result was generalized by Gyárfás [147]:

Theorem 3.23 ([147]). *Let $\mathscr{H}$ be an r-graph (not necessarily intersecting) with $\tau(\mathscr{H}) = \tau$. Then the number of covers of $\mathscr{H}$ with τ elements is at most r^τ.*

Proof. We will prove by backward induction on i that every i-element set I is contained in at most $r^{\tau-i}$ τ-element covers. It is obvious for $i = \tau$ and the case $i = 0$ gives the Theorem. If $i < \tau$ then there exists an edge $E \in \mathscr{H}$ such that $E \cap I = \varnothing$. Now apply the inductional hypothesis for the sets $E \cup \{x\}$, $x \in E$. □

The bound r^τ is best possible as it is shown by τ disjoint r-sets. But for intersecting families one can expect a smaller value.

Example 3.24 ([108], [177]). Let $1 \le \tau \le r$ and consider the disjoint sets $S_{r-\tau+1}, \ldots, S_r$ where $|S_i| = i$ $(r - \tau + 1 \le i \le r)$. Define the intersecting r-graph $\mathscr{E}(r, \tau)$ as follows:

$$\mathscr{E}(r,\tau) = \{E: |E| = r, \text{ for some } i: E \supset S_i \text{ and } |E \cap S_j| = 1 \text{ for } i < j \le r\}.$$

Conjecture 3.25 (Lovász [177] for $r = \tau$, Frankl [115] in general). Suppose $\mathscr{H}$ is an intersecting r-graph, with $\tau(\mathscr{H}) = \tau$. Then $\mathscr{H}$ has at most

$$r(r-1)\cdots(r-\tau+1) \qquad \text{for } \tau < r$$

$$\sum_{i=1}^{r} \left(r(r-1)\cdots(r-i+1)\right) = [(e-1)r!] \quad \text{for } \tau = r$$

different τ-element covers.

The case $\tau = 1$ is obvious, the case $\tau = 2$ was proved by Frankl [115].

The following result is a bit more general than Lovász' theorem and 3.23.

Theorem 3.26 ([128]). *Let us suppose for the hypergraph $\mathscr{H}$ that for every $E_1, \ldots, E_{k+1} \in \mathscr{H}$ we have $|\bigcup_{i \ne j}(E_i \cap E_j)| \ge \max_{E \in \mathscr{H}} |E| =: r$. Then $|\mathscr{H}| \le k^r$.*

In the case of equality we can find pairwise disjoint k-element sets $S_1, \ldots, S_r$ such that

$$\mathscr{H} = \{A: |A| = r, |A \cap S_i| = 1 \text{ for } i = 1, \ldots, r\}.$$

Proof. The cases $k = 1$ or $r = 1$ are tricial. Apply induction on k. Once k is fixed apply induction on r. Let $E_0 \in \mathscr{H}$ fixed and define $\mathbf{H}(X) =: \{E - E_0 : E \cap E_0 = X\}$ for all $X \subset E_0$. Then $|\mathbf{H}(X)| \leq (k-1)^{r-|X|}$. □

Although, it was easy to prove, Theorem 3.26 is one of the key lemmas for the celebrated result of Razborov [267] about the complexity of monotone Boolean functions.

4. Matchings

Let $\mathscr{F}$ be a family of sets. A subsystem $\mathscr{M} \subset \mathscr{F}$ is called a *matching* if it consists of pairwise disjoint members. $\nu(\mathscr{F})$, or briefly ν, denotes the maximum number of disjoint edges in $\mathscr{F}$ and it is called the *matching number* of $\mathscr{F}$. Clearly

$$\nu \leq \tau \tag{4.1}$$

holds for every hypergraph $\mathscr{F}$, because in order to cover all the edges of $\mathscr{F}$ we have to choose at least one element from every member of the matching with maximum cardinality.

The computation of the matching number of a given hypergraph is not so hopeless as the calculation of the covering number. E.g., for graphs there exists a minimax formula due to Berge [26] and a polynomial algorithm given by Edmonds [88]. The literature of matchings of graphs is very broad, a comprehensive book about this topic was recently written by Lovász and Plummer [188].

All the elements of the members of a maximum matching obviously form a cover, hence we have

$$\tau \leq r\nu, \tag{4.2}$$

where r denotes the rank of the hypergraph. In both of (4.1) and (4.2) equality can hold.

Examples 4.0. (i) We have $\nu(\mathscr{F}) = 1$ for every intersecting family $\mathscr{F}$.
(ii) For the complete r-graph on $\nu r + r - 1$ vertices $\nu(\mathbf{K}_r^{r\nu+r-1}) = \nu$, $\tau(\mathbf{K}_r^{r\nu+r-1}) = r\nu$.
(iii) The famous theorem of König (see, e.g., in [188]) says that $\nu(\mathscr{G}) = \tau(\mathscr{G})$ holds for every bipartite graph $\mathscr{G}$.

For a set S define the restriction $\mathscr{F}|S$ of $\mathscr{F}$ to S by

$$\mathscr{F}|S = \{F \cap S : F \in \mathscr{F}\}.$$

The aim of the notion of τ-critical hypergraphs and the investigation of their maximum cardinality is to find a small part of the hypergraph which forces the covering number to be big. Analogously, now we are looking for a small part S of the vertices of the hypergraph $\mathscr{F}$ such that $\mathscr{F}|S$ show that $\nu(\mathscr{F})$ cannot be big. Our first step in this direction is to investigate intersecting hypergraphs.

The kernel of intersecting families. In 1964 Calczynska-Karlowicz [67] proved that for every r there exists an $n(r)$ such that to every intersecting family $\mathscr{F}$ of rank r there is a set S of cardinality $n(r)$ such that $\mathscr{F}|S$ is also intersecting. This means that if $\mathscr{F}$ is intersecting then its members intersect each other on a small part of the underlying set. The first explicit upper bound for $n(r)$ was given by Ehrenfeucht and Mycielski [92]. This was improved by Erdös and Lovász [108]. The current best bounds are due to Tuza [240].

Theorem 4.1 [240]. $(2r-4)+2\binom{2r-4}{r-2} \le n(r) \le \binom{2r-1}{r-1}+\binom{2r-4}{r-2}$. A slightly weaker upper bound follows from Theorem 4.8. Tuza's example giving the lower bound is the following.

Example 4.2 ([240]). Let X be a $(2r-4)$-element set, and for each partition $\{E,E'\}$ of X with $|E|=|E'|=r-2$, $E\cup E'=X$ take four new elements x, x', y, y' and set $E\cup\{x,y\}$, $E\cup\{x',y'\}$, $E'\cup\{x,x'\}$, $E'\cup\{y,y'\}$. The obtained family $\mathscr{F}$ has $2\binom{2k-4}{k-2}$ members, and if $\mathscr{F}|S$ is intersecting then $S\supset(\bigcup\mathscr{F})$ holds.

Conjecture 4.3 ([240]). For $r\ge 4$, $n(r)=(2r-4)+\frac{1}{2}\binom{2r-4}{r-2}$.

Hansen and Toft [152] proved that $n(2)=3$, $n(3)=7$, $n(4)=16$. For t-wise s-intersecting families the following generalization of the Calczynska-Karlowicz theorem was proved in [121]:

$$\text{If } \mathscr{F} \text{ is a } t-\text{wise } s-\text{intersecting family of } r-\text{sets then there exists a set } S,$$
$$|S|\le r^{2r}, \text{ such that } |F_1\cap\cdots\cap F_t\cap S|\ge s \text{ still holds for every } F_1,\ldots,F_t\in\mathscr{F}. \quad (4.3)$$

Denote by $n(r,t,s)$ the smallest integer n such that replacing r^{2r} by n (4.3) remains true. With this notation $n(r)=n(r,2,1)$. The existence of $n(r,t,s)$ was also proved by Frankl [114] (in an implicit form) and by Kahn and Seymour [264].

Define an *edge-contraction* as the following operation on a family $\mathscr{F}$: we substitute an edge $E\in\mathscr{F}$ by a smaller, non-empty $E'\subsetneq E$, and thus we get the set-system $\mathscr{F}-\{E\}\cup\{E'\}$. A t-wise s-intersecting family is *critically t-wise s-intersecting* (or briefly, *(t,s)-critical*) if it has no multiple edges and any hypergraph obtained by contracting any of its edges is not t-wise s-intersecting. We can obtain a (t,s)-critical family from any t-wise s-intersecting family $\mathscr{F}$ by contracting its edges as far as possible and deleting all but one copy of the appearing multiple edges. The obtained (smaller) family $\mathscr{K}$ is called the *(t,s)-kernel* of $\mathscr{F}$. (Of course, this $\mathscr{K}$ is not necessarily unique). The following reformulation is obvious:

$$n(r,t,s)=\max\{|\textstyle\bigcup\mathscr{K}|: \mathscr{K} \text{ is } (t,s)-\text{critical of rank at most } r\}. \quad (4.4)$$

For $t\ge 3$ only a few (r,s)-critical hypergraphs are known, and most of them have less than r^2 vertices. The following Example shows that there are exponentially large (t,s)-critical hypergraphs.

Example 4.4 (Alon, Füredi [13]). Define $l=[(r-s)/3(t-1)]$ and suppose that $l\ge 1$. We are going to construct a (t,s)-critical hypergraph $\mathscr{H}$ of rank $s+3(t-1)l$ on $3^l+3lt+s-1$ vertices and with $(3t)^l$ edges. Let the vertex-set consist of all the 3^l sequences $\mathbf{a}=(a_1,\ldots,a_l)$, $a_i\in\{0,1,2\}$ together with an $(s-1)$-element set S and the disjoint union of l $3t$-sets $C^1,\ldots,C^l$, where $C^i=\{x_0^i,x_1^i,\ldots,x_{3t-1}^i\}$. For each sequence $\mathbf{j}=(j_1,\ldots,j_l)$ with $0\le j_i<3t$ we define an edge $E(\mathbf{j})$ of $\mathscr{H}$ by setting

$$E(\mathbf{j}) = S \cup \Bigl(\bigcup_{1 \le i \le l} (C^i - \{x_{j_i}, x_{j_i+1}, x_{j_i+2}\}) \Bigr) \cup \{\mathbf{r}\}$$

where $\mathbf{r} = (r_1, \ldots, r_l)$ is defined by $r_i \equiv l_i \pmod 3$. Then $\mathscr{H}$ is a (t,s)-critical family.

Theorem 4.5 ([13]). *Let $\mathscr{H}$ be a (t,s)-critical family of sets having size at most r. Then*

$$|\bigcup \mathscr{H}| \le r \binom{r - s + 1 + \left[\frac{r-s}{t-1}\right]}{\left[\frac{r-s}{t-1}\right]}.$$

Proof. Let x be a vertex of $\mathscr{H}$. Then $\mathscr{H} - \{x\}$ is not t-wise s-intersecting, so there exist $H_1, \ldots, H_t \in \mathscr{H}$, such that $x \in \bigcap H_i$, $|\bigcap H_i| = s$, $(1 \le i \le t)$. Hence there exists a j $(1 \le j \le t)$ such that $\left| \bigcap_{i \ne j} H_i - \bigcap H_i \right| \le [(r-s)/(t-1)]$. Define

$$A(x) =: H_t,$$
$$B(x) = \bigcap \{H_i : i \ne j, 1 \le i \le t-1\} - \{x\}.$$

Define a sequence $x_1, \ldots, x_m$. Choose x_1 arbitrarily from $\bigcup \mathscr{H}$, and if $x_1, \ldots, x_{i-1}$ are chosen then let $x_i \in (\bigcup \mathscr{H}) - (\bigcup \{H(x_j): j < i\})$. Stop if $\bigcup \{H(x_j): j \le i\} = \bigcup \mathscr{H}$. Then we can use (the sharpened version) of Theorem 2.5 with $A_i = A(x_i)$, $B_i = B(x_i)$, $a = r$, $b = s - 1 + [(r-s)/(t-1)]$, $c = s - 1$. We have

$$|\bigcup \mathscr{H}| = \left| \bigcup_i H(x_i) \right| \le \sum_i |H(x_i)| \le r \binom{a + b - 2c}{a - c},$$

implying Theorem 4.5. □

To obtain a simpler form of Theorem 4.5 we can use the following well-known inequality. For every $a, b \ge 1$ we have

$$\binom{a+b}{a} \le \frac{(a+b)^{a+b}}{a^a b^b} \cdot \sqrt{\frac{a+b}{2\pi ab}}. \tag{4.5}$$

Theorem 4.5 and Example 4.4 imply

$$\frac{1}{3}\left(3^{1/3}\right)^{(r-s)/(t-1)} \le n(r,t,s) \le r \left(\frac{t^t}{(t-1)^{t-1}} \right)^{[(r-s)/(t-1)]} < r(et)^{(r-s)/(t-1)}. \tag{4.6}$$

In many extremal problems using the kernel of an intersecting family is very fruitful, so it would be interesting to narrow the gap between the lower and upper bounds of $n(r,t,s)$. The most important case is when $s = 1$.

Conjecture 4.6 [13]. If t is fixed then $\lim_{r \to \infty} \sqrt[r]{n(r,t,1)}$ exists.

By (4.6) the value of this limit is between $3^{1/3(t-1)}$ and $(t/t-1) \cdot t^{1/(t-1)}$, $\lim \sqrt[r]{n(r,2,1)} = 4$.

ν-critical hypergraphs. A hypergraph $\mathscr{H}$ is *ν-critical* if it has no multiple edges and contracting any of its edges increases ν. We can obtain a ν-critical hypergraph

from any family of sets by contracting its members and deleting all but one copies of the appearing multiple edges as far as ν does not increase.

Examples 4.7. The following ones are ν-critical hypergraphs.
(i) The complete r-graph on $\nu r + r - 1$ vertices.
(ii) $(r = 2)$ a circuit $C_{2\nu+1}$.
(iii) a projective plane of order $r - 1$ (with $\nu = 1$)
(iv) Tuza's example $\mathscr{H}\left(\text{with } \nu = 1, |\bigcup \mathscr{H}| = 2r - 4 + 2\binom{2r-4}{r-2}\right)$ (Example 4.2).
(v) Erdös and Lovász's example $\mathscr{F}\left(\text{with } \nu = 1, |\bigcup \mathscr{F}| = 2r - 2 + \frac{1}{2}\binom{2r-2}{r-1}\right)$ (Example 1.2(iv)).
(vi) (Lovász [177]). Let X be partitioned into disjoint sets $X_1, \ldots, X_r$ where $|X_i| = \nu + i - 1$, $1 \le i \le r$. The edges of $\mathscr{F}$ are all the r-element sets E containing j elements from X_j (for some $1 \le j \le r$) and one element from each X_k $(j < k \le r)$. Here $\nu(\mathscr{F}) = \nu$, and

$$|\mathscr{F}| = \sum_{1 \le j \le r} \binom{\nu + i - 1}{i}(\nu + i)\cdots(\nu + r - 1) = \binom{\nu + r - 1}{r}[r!(e - 1)].$$

Theorem 4.8 ([240]). *If $\mathscr{F}$ is a ν-critical hypergraph of rank r then* $|\bigcup \mathscr{F}| \le \binom{r\nu + r}{r}$.

This theorem was proved by Lovász [177] in a slightly weaker form $\left(|\bigcup \mathscr{F}| \le \frac{r}{2}\binom{r\nu + r - 1}{r}\right)$ using the permutation method (see the proof of Theorem 1.5). In the next section, in the proof of Theorem 4.14(i) we will give Tuza's proof. Theorem 4.8 implies a weaker upper bound in Theorem 4.1 $\left(\text{i.e., } n(r) \le \binom{2r}{r}\right)$. For $r = 2$ Gallai [141] proved that

$$\text{every } \nu\text{-critical connected graph has exactly } 2\nu + 1 \text{ vertices.} \tag{4.7}$$

Conjecture 4.9 (Lovász [177]). There exists a $c(r)$, depending only on r such that every ν-critical hypergraph of rank r can have at most $c(r)\nu$ vertices. According to (4.7) $c(2) = 3$.

Intersecting set-pair systems. Investigating critical hypergraphs Tuza [240] introduced the following function

$$n_1(a, b) = \max\{|\bigcup A_i| : |A_i| \le a, |B_i| \le b, A_i \cap B_i = \varnothing, A_i \cap B_j \ne \varnothing \text{ for } i \ne j\}.$$

Such a system $\{A_i, B_i\}$ $(1 \le i \le m)$ we will call *(a, b)-system.* It is easy to prove that

Proposition 4.10 ([240]). $n_1(a, 0) = a$, $n_1(a, 1) = [(a + 2)^2/4]$ *for* $a \ge 1$.

Examples 4.11 ([240]). The followings are (a, b)-systems with a large number of vertices.

(i) Suppose $0 \le u \le a$ and consider a $(b + u)$-set Y, $m = \binom{b + u}{b}$. Let $\{B_1, \ldots, B_m\} =$

$\binom{Y}{b}$ and let A_i be the union of $Y - B_i$ and $(a - u)$ new elements. Then

$$|\bigcup A_i| = b + u + (a - u)\binom{b + u}{u}.$$

(ii) Suppose $0 \leq v \leq b$ and consider a $(a + v - 1)$-element set $Y, m = (b + 1 - v) \times \binom{a + v - 1}{a - 1}$. Every $(a - 1)$-element set of Y will be completed to a-element sets in $(b - v + 1)$ different ways such that all the m points $(A_i \backslash Y)$ are distinct. Then let $B_i = (Y - A_i) \cup \{A_j - Y: A_j \cap Y = A_i \cap Y\}$. Then

$$|\bigcup A_i| = a + v - 1 + (b + 1 - v)\binom{a + v - 1}{a - 1}.$$

The above constructions imply the lower bound in the following theorem.

Theorem 4.12 (Tuza [240].) *For all* $a \geq 1, b \geq 0$

$$\frac{1}{4}\binom{a + b + 1}{b + 1} < n_1(a, b) < \binom{a + b + 1}{b + 1}.$$

Proof of the upper bound. Clearly. $n_1(0, b) = 0 < \binom{b + 1}{b + 1}$, so using induction on a the following lemma implies the desired upper bound.

Lemma 4.13. $n_1(a, b) \leq \binom{a + b}{b} + n_1(a - 1, b)$.

Proof. Let $\{A_i, B_i\}$ $(i \in \Pi)$ be an (a, b)-system with $|\bigcup A_i| = n_1(a, b)$. We can choose an $\Pi_0 \subset \Pi$ such that $\bigcup \{A_i: i \in \Pi\} = \bigcup \{A_i: i \in \Pi_0\}$, but the same does not hold for $\Pi_0' \subsetneqq \Pi_0$. This implies that there exists an $x_i \in A_i$ for all $i \in \Pi_0$ such that $x_i \notin A_j$ for $i \neq j \in \Pi_0$. We have

$$|\Pi_0| \leq |\Pi| \leq \binom{a + b}{b}$$

by Corollary 1.5. Consider the sets $A_i' = A_i - \{x_i\}$ $(i \in \Pi_0)$ and delete all but one copy of the appearing multiple edges. We obtain a $\Pi_1 \subset \Pi_0$. If $x_i \in B_j$ (for $i \neq j$, $i, j \in \Pi_1$) then choose an element $x_i^j \in A_i' - A_j'$. Finally, set $B_j' = B_j - \{x_i: x_i \in A_i, i \in \Pi_1\} \cup \{x_i^j: x_i \in A_i \cap B_j, i \in \Pi_1\}$. Then $\{A_i', B_i'\}$ $(i \in \Pi_1)$ is an $(a - 1, b)$-system and $\bigcup \{A_i: i \in \Pi\} = \{x_i: i \in \Pi_0\} \cup (\bigcup \{A_i': i \in \Pi_1\})$. □

We can apply Theorem 4.12 in several cases

Theorem 4.14 ([240]).
(i) *If $\mathscr{F}$ is a v-critical hypergraph of rank r than* $|\bigcup \mathscr{F}| \leq n_1(rv, r - 1)$.
(ii) *For the maximal number of vertices* $v(r, \tau)$ *of a* τ*-critical hypergraph of rank r we have* $v(r, \tau) \leq n_1(r, \tau - 1)$ (c.f. Theorem 2.19, Example 2.20, Conjecture 2.21).
(iii) *For the s-multitransversal of a k-wise intersecting* τ*-critical hypergraph,* $m_k(s, \tau)$

we have

$$\frac{1}{k}n_1(s,\tau-1) \le m_k(s,\tau) \le n_1(s,\tau-1).$$

(Especially, $n_1(s,\tau-1) = m(s,\tau)$. Cf. Theorem 2.22).

Proof. We give only the proof of (i). The other proofs are similar. Let $\mathscr{H}$ consist of the sets that can be obtained as the union of ν pairwise disjoint members of $\mathscr{F}$, e.g., if $\nu = 1$ then $\mathscr{H} = \mathscr{F}$. If there is an $H \in \mathscr{H}$ which is contained in the union of the other members of $\mathscr{H}$, delete it. Repeat this step with the new family until it is possible. Finally we obtain a subfamily $\mathscr{H}_0 = \{H_1, \ldots, H_m\} \subset \mathscr{H}$ with the following properties. $\bigcup \mathscr{H}_0 = \bigcup \mathscr{H} = \bigcup \mathscr{F}$ and there exists an $x_i \in H_i$ for each i such that $x_i \notin H_j$ holds for all $j \ne i$, $1 \le j \le m$. The ν-critical property of $\mathscr{F}$ implies that there exists an $F_i \in \mathscr{F}$ such that $H_i \cap F_i = \{x_i\}$. Then $\{H_i, F_i - \{x_i\}\}$ is an $(r\nu, r-1)$-system. □

Problems about intersecting set-pair systems. Tuza [240] also introduced the following related function

$$n(a,b) = \max\{|\bigcup (A_i \cup B_i)| : \{A_i, B_i\} \text{ an } (a,b)\text{-system}\}.$$

Clearly $n_1(a,b) \le n(a,b)$. It is easy to see that $n(a,0) = n_1(a,0)$ $(= a+1)$, $n(a,1) = n_1(a,1) = [(a+2)^2/4]$, $n_1(2,3) = 9$, $n(2,3) \ge 10$, so in general n_1 is smaller than n. But the upper bound for n_1 given in Theorem 4.12 holds for $n(a,b)$.

Theorem 4.15 (Tuza [240]). *If $a \ge b$, $a \ge 1$ then*

$$\frac{1}{4}\binom{a+b+1}{b+1} < n(a,b) \le \sum_{i=1}^{2b-2}\binom{i}{\lfloor i/2\rfloor} + \sum_{i=2b-1}^{a+b-1}\binom{i}{b} < \binom{a+b+1}{b+1}.$$

Conjecture 4.16 ([240]). (i) $n(a,b) = n_1(a,b)$ iff $a \ge b$,
(ii) if $a \ge b+2$ then

$$n_1(a,b) = \left\lceil \frac{a}{b+1} \right\rceil \binom{\left\lfloor \frac{ab}{b+1} \right\rfloor + b}{b} + b + \left\lfloor \frac{ab}{b+1} \right\rfloor$$

Tuza also proved that for $b > 4a + 3$ we have $n(a,b) > n_1(a,b)$. Moreover the somewhat surprising fact that.

Theorem 4.17 ([240]). $n_1(a,b) = n_1(b+1, a-1)$.

t-Expansive graphs and hypergraphs. For a hypergraph $\mathscr{H}$ let us denote the minimum number t for which

$$|\bigcup \mathscr{H}'| \le |\mathscr{H}'| + t \tag{4.8}$$

holds for all $\mathscr{H}' \subset \mathscr{H}$ by $t(\mathscr{H})$. If $t(\mathscr{H}) = t$, then we call $\mathscr{H}$ *t-expansive*. If $\mathscr{H}$ is a graph then its expansion number $t(\mathscr{H})$ equals to the matching number $\nu(\mathscr{H})$. Denote by $\mathscr{H} - x$ the set-system $\{H \in \mathscr{H} : x \notin H\}$. We call $\mathscr{H}$ *t-stable* (or *critically t-expansive*) if

$$t(\mathscr{H} - x) = t(\mathscr{H})$$

holds for all $x \in \bigcup \mathscr{H}$. Similarly the graph $\mathscr{G}$, (or hypergraph $\mathscr{H}$) is *ν-stable* if $\nu(\mathscr{G} - x) = \nu(\mathscr{G})$ holds for every point x. The ν-stability and t-stability coincide for graphs. The notion of ν-stability was introduced by Lovász [177], who proved that the upper bounds given in Theorem 4.8 and 4.14(i) hold for ν-stable hypergraphs as well. Every ν-critical hypergraph is ν-stable, and from any ν-stable family one can obtain a ν-critical one on the same vertex set contracting its edges. Gallai's theorem mentioned above as (4.7) is more precisely the following:

Theorem 4.18 (Gallai [141]). *If a graph $\mathscr{G}$ is ν-stable and connected then it is factor-critical, i.e., $G - x$ has a one-factor for all $x \in \bigcup \mathscr{G}$.*

Hence $|\bigcup \mathscr{G}| \leq 3\nu$ holds for all (not necessarily connected) ν-stable graphs. Here equality holds only in the case if $\mathscr{G}$ is disjoint union of ν triangles. This result plays an important role in the Edmonds-Gallai structure theorem (see, e.g., Edmonds [88], Gallai [141], ([186], Problems 7.26–32, or [188]). The following theorem generalizes Gallai's result for t-stable hypergraphs. This theorem shows that one of the natural extensions of the matching number of graphs to hypergraphs is the expansion number, and not only the usual matching number.

Theorem 4.19 [133]. *Let $\mathscr{H}$ be a family of finite sets having at least two-element members. Suppose that $\mathscr{H}$ is t-stable, i.e., $t(\mathscr{H}) = t = t(\mathscr{H} - x)$ holds for all vertices x. If $\mathscr{H}$ is connected then $|\bigcup \mathscr{H}| \leq 2t + 1$.*

Before the proof some remarks. Equality holds iff the graph $\mathscr{G} = \{E : E \subset H \in \mathscr{H}, |E| = 2\}$ is a factor-critical graph on $2t + 1$ vertices. In the case $|\bigcup \mathscr{H}| = 2t + 1$. $\mathscr{H}$ is not necessarily 2-uniform. E.g., $\mathscr{H} = \{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,3,5\}\}$ is 2-stable.

Corollary 4.20. *Suppose $\mathscr{H}$ is a t-stable hypergraph with edges having at least 2 elements. Then $|\bigcup \mathscr{H}| \leq 3t$. Here equality holds iff $\mathscr{H}$ is the disjoint union of ν triangles.* □

The crucial part of Gallai's proof is the following statement: If the graph $\mathscr{G}$ is ν-stable then $\nu(\mathscr{G} - x - y) < \nu(\mathscr{G})$, $\nu(\mathscr{G} - y - z) < \nu(\mathscr{G})$ imply $\nu(\mathscr{G} - x - z) < \nu(\mathscr{G})$. This means that the relation $x \sim y$: $\nu(\mathscr{G} - x - y) < \nu(\mathscr{G})$ is an equivalence relation on $\bigcup \mathscr{G}$. A similar statement for hypergraphs does not hold. E.g., the hypergraph $\mathscr{H}$ given on the pointset $\{1,2,3,4,5,6,7,8\}$, $\mathscr{H} = \{\{4,6,8\},\{3,5,7\},\{2,6,7\},\{1,5,8\},\{2,3,8\},\{1,4,7\}\{2,4,5\},\{1,3,6\}\}$ is critically 4-expansive and $t(\mathscr{H} - \{1,3\}) < 4$, $t(\mathscr{H} - \{3,2\}) < 4$ but $t(\mathscr{H} - \{1,2\}) = 4$. (See Fig. 4.1).

We have to mention that the concept of t-expansion is not unknown in hypergraph theory. E.g., Brace and Daykin [57] proved that $t(\mathscr{H}) = t$, $|\bigcup \mathscr{H}| = n$ implies $|\mathscr{H}| \leq (n - t + 1)2^t$, where equality holds iff $\bigcup \mathscr{H} = X = A \cup B$, $|A| = n - t$, $|B| = t$ and $\mathscr{H} = \{H \subset X : |H \cap A| \leq 1\}$. Daykin [77] proved for $m \geq 2t$, Bang, Sharp and Winkler [21] proved for $m \geq 1.3t$ and Daykin and Frankl [78] proved for $m \geq t + 25$ that if $\mathscr{H}$ is a t-expansive hypergraph on the m-element set X then $\min_{x \in X} deg_{\mathscr{H}}(x) \leq 2^t$, where $deg_{\mathscr{H}}(x) = |\{H \in \mathscr{H} : x \in H\}|$ is the *degree* of x in the hypergraph $\mathscr{H}$. It is conjectured that this holds for $m \geq t + 3$.

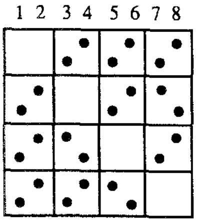

Fig. 4.1

The proof of Theorem 4.19. Let $\mathscr{H}$ be a t-stable hypergraph with at least 2-element edges. If $H \in \mathscr{H}$ and $A \subset H$, $|A| \geq 2$ then the hypergraph $\mathscr{H} \cup \{A\}$ is also t-stable on the same underlying set. Hence we can suppose that $\mathscr{H}$ is almost a downset, i.e., if $|A| \geq 2$, $A \subset H \in \mathscr{H}$ then $A \in \mathscr{H}$. Let us define $\mathscr{G} = \{A \in \mathscr{H}: |A| = 2\}$. $\mathscr{H}$ is connected, so $\mathscr{G}$ is a connected graph as well. As usual, $\Gamma(x)$ denotes the neighborhood of the point x in the graph $\mathscr{G}$, i.e., $\Gamma(x) = \{y: \{x, y\} \in \mathscr{G}\}$.

A subsystem $A \subset \mathscr{H}$ is called *maximal* if $|\bigcup \mathscr{A}| = |\mathscr{A}| + t$, the members of $\mathscr{A}$ are pairwise disjoint and $|\bigcup \mathscr{A}|$ is maximal with respect to these constraints. Let us choose a maximal subsystem and denote it by $\mathscr{B}$. Let $\bigcup \mathscr{B} = X$.

Lemma 4.21. *For each $x \in X$ there exists a t-expansive set-system $\mathscr{C}_x$ such that $\bigcup \mathscr{C}_x$ covers $X - \{x\}$, $x \notin \bigcup \mathscr{C}_x$ and it consists of pairwise disjoint edges of $\mathscr{H}$.*

Proof. $\mathscr{H}$ is a stable t-expansive hypergraph, hence there exists a set-system $\mathscr{C} = \{C_1, \ldots, C_l\} \subset \mathscr{H} - x$, such that $|\bigcup \mathscr{C}| = l + t$. Let $C_i' = C_i - \bigcup \{C_j: j < i\}$. The existence of the system $\{C_i': |C_i'| \geq 2\}$ shows that the following family of subsystems is non-empty:

$$C_x = \{\mathscr{D}, \mathscr{D} \subset \mathscr{H}, \mathscr{D} \text{ contains disjoint members, } \mathscr{D} \text{ is } t\text{-expansive and } x \notin \textstyle\bigcup \mathscr{D}\}.$$

Let $\mathscr{C}_x$ denote a subsystem belonging to C_x for which $|\mathscr{B} \cap \mathscr{C}_x|$ (i.e., the number of the common members) is maximal. We are going to show that $X - \{x\} \subset \bigcup \mathscr{C}_x$. Suppose for contradiction that $y \in X - \{x\}$ but $y \notin \bigcup \mathscr{C}_x$. Let $B \in \mathscr{B}$ the edge for which $y \in B$. We distinguish two cases. If $x \in B$ then $\{x, y\} \in \mathscr{H}$ and the subsystem $\mathscr{C}_x \cup \{x, y\}$ would be $(t + 1)$-expansive. If $x \notin B$ then let $\{C_1, \ldots, C_l\} = \{C \in \mathscr{C}_x: C \cap B \neq \varnothing\}$. The set-system $\mathscr{C}_x - \{C_1, \ldots, C_l\} \cup \{C_i - B: |C_i - B| > 1\} \cup \{B\}$ belongs to C_x, too, and has more common members with $\mathscr{B}$ then $\mathscr{C}_x$. This contradiction proves, that such a y does not exists, i.e., $(\bigcup \mathscr{C}_x) \cap X = X - \{x\}$. □

Proposition 4.22. $|\bigcup \mathscr{B}| = |X| \leq 2t$. *Here equality holds if $\mathscr{B} \subset \mathscr{G}$.*

Proof. $\mathscr{B}$ consists of at least two-element disjoint sets, hence we get $2|\mathscr{B}| \leq |\bigcup \mathscr{B}| = |\mathscr{B}| + t$, i.e., $|\mathscr{B}| \leq t$, hence $|\bigcup \mathscr{B}| \leq 2t$. □

Proposition 4.23. *The sets $\bigcup \mathscr{C}_x$ $(x \in X)$ and the set $\bigcup \mathscr{B}$ cover $\bigcup \mathscr{H}$.*

Proof. Suppose for contradiction that $y \in \bigcup \mathscr{H} - \bigcup \{\bigcup \mathscr{C}_x: x \in X\} - \bigcup \mathscr{B}$. Then there exists a 2-element subset $\{x, y\} \in \mathscr{H}$. Joining the set $\{x, y\}$ to $\mathscr{C}_x$ we get a $(t + 1)$-expansive subsystem, which is a contradiction. □

Returning to the proof of Theorem 4.19 we distinguish two cases.

1) If each $(\bigcup \mathscr{C}_x) \subset X$ then Proposition 4.23 yields that $|\bigcup \mathscr{H}| = |\bigcup \mathscr{B}|$. Now $|\bigcup \mathscr{H}| \leq 2t$, by Proposition 4.22, and we are ready.
2) From now on we can suppose that there exists a $\mathscr{C}_x$ such that $(\bigcup \mathscr{C}_x) - X \neq \emptyset$. Then $|\bigcup \mathscr{C}_x| \geq |X|$. But $\mathscr{B}$ is maximal, hence $|\bigcup \mathscr{C}_x| = |X|$ and $|(\bigcup \mathscr{C}_x - X| = 1$. Let $\bigcup \mathscr{C}_x - X = \{z\}$. Obviously, $\Gamma(z) \subset X$ ($y \in \Gamma(z) - X$ implies that $\mathscr{B} \cup \{z, y\}$ is $(t+1)$-expansive, which is a contradiction).

Proposition 4.24. *Let $u \in \Gamma(z)$. Then $\mathscr{C}_u$ is maximal t-expansive and* $(\bigcup \mathscr{C}_u) \subset (X \cup \{z\})$. ($\mathscr{C}_u$ is defined by Lemma 4.21).

Proof. By definition (more exactly by Lemma 4.21) $(\bigcup \mathscr{C}_u) \cap X = X - \{u\}$. If $z \notin \bigcup \mathscr{C}_u$ then the system $\mathscr{C}_u \cup \{z, u\}$ is $(t+1)$-expansive, which is a contradiction. Hence we get $\bigcup \mathscr{C}_u \supset (X - \{u\} \cup \{z\})$. This yields that $|\bigcup \mathscr{C}_u| \geq |X|$, i.e. $\mathscr{C}_u$ is maximal, as well. So we have $\bigcup \mathscr{C}_u = X - \{u\} \cup \{z\}$. □

Change the role of $\mathscr{B}$ and z with $\mathscr{C}$ and u. We get $\Gamma(u) \subset X \cup \{z\}$. This yields that $\Gamma(\Gamma(z)) \subset X \cup \{z\}$. Continuing procedure we get that the component of $\mathscr{G}$ which contains z is contained in $X \cup \{z\}$. Hence $\bigcup \mathscr{H} = \bigcup \mathscr{G} \subset X \cup \{z\}$. Finally, $|X \cup \{z\}| \leq 2t + 1$, by Proposition 4.22.

The case of equality is clear. □

ν-stable hypergraphs. A hypergraph $\mathscr{H}$ is *l-wise ν-stable* if $\nu(\mathscr{H} - L) = \nu(\mathscr{H})$ for every l-element set L, i.e. for every l-element set L one can find $E_1, \ldots, E_\nu \in \mathscr{H}$ such that $L, E_1, \ldots, E_\nu$ are pairwise disjoint.

Theorem 4.25 (Lovász [177]). *If $\mathscr{H}$ is l-wise ν-stable of rank r then*

$$|\bigcup \mathscr{H}| \leq \binom{r\nu + r}{r} \tag{4.9}$$

$$|\mathscr{H}| \leq (r\nu)^r \tag{4.10}$$

Proof. (4.9) follows from Theorem 4.8. To prove (4.9) consider the same hypergraph as in the proof of Theorem 4.14 (i) and apply Theorem 3.23. □

If for a hypergraph $\mathscr{H}$ of rank r the equation $\tau = r\nu$ holds then it is $(r-1)$-wise ν-stable, hence

Corollary 4.26 ([177]). *If $\mathscr{H}$ is hypergraph of rank r and $\tau = r\nu$ holds then* (4.9) *and* (4.10) *hold.*

Problem 4.27 (Lovász). Give a better upper bound for $|\mathscr{H}|$ and $|\bigcup \mathscr{H}|$ for the hypergraphs with property $\tau = r\nu$. Give better upper bounds for the size of the l-wise ν-stable hypergraphs.

Conjecture 4.28 (Lovász). If $\mathscr{H}$ is $(r-1)$-wise ν-stable of rank r then it could not be r-partitie.

Conjecture 4.28 is a stronger version of the following.

Conjecture 4.29 (Lovász, Ryser). If $\mathscr{H}$ is r-partite then $\tau \leq (r-1)\nu$.

The case $r = 2$ is the famous König theorem (see [188]). Another special cases were proved by Tuza [237, 238] (whenever $r = 3$ and $\nu \leq 4$, $r = 4$ and $\nu \leq 2$, $r = 5$ and $\nu = 1$). Tuza and Szemerédi [232] have a tricky proof for tripartite hypergraphs $\tau \leq \frac{25}{9}\nu$, which was recently improved to $\tau \leq \frac{8}{3}\nu$ by Tuza [273].

Conjecture 4.30 (Tuza [270, 272]). Let $\mathscr{G}$ be a simple graph (there are no multiple edges). Define a 3-uniform hypergraph $\mathscr{H} = \mathscr{H}(\mathscr{G})$ over the edge-set of $\mathscr{G}\mathscr{H} =:$ $\{\{E_1, E_2, E_3\} \subset \mathscr{G}$: E_1, E_2 and E_3 form a triangle$\}$. Then $\tau(\mathscr{H}) \leq 2\nu(\mathscr{H})$. If it is true there are many extremal cases, namely the (disjoint) union of $i\mathbf{K}_2^4$ and $\frac{1}{2}(\nu - i)\mathbf{K}_2^5$.

Conjecture 4.31 (Erdös and Füredi [133]). Let $\mathscr{G}$ be a graph and $\mathscr{F}$ be the hypergraph consisting of the vertices of the triangles of $\mathscr{G}$. If $\mathscr{F}$ is ν-stable then $|\bigcup \mathscr{F}| \leq 5\nu$, and here equality holds if $\mathscr{G}$ consists of ν disjoint $\mathbf{K}_2^5$.

If these conjectures are true there are several further generalizations.

5. The Linear Programming Bound for τ and ν

The calculation of the covering and matching number of an arbitrary hypergraph is an NP-complete problem [260], hence every result which gives estimates is especially valuable. One of the simplest and good estimate can be obtained from the *linear programming bound*, in other words from the real relaxation of τ and ν.

Let H be a hypergraph. A real function w over $E(\mathbf{H})$ is called a *fractional matching of* $\mathbf{H}$ if

$$w(E) \geq 0 \quad \text{for all } E \in E(\mathbf{H}), \text{ and}$$

$$\sum_{p \in E} w(E) \leq 1 \quad \text{for all } p \in V(\mathbf{H}).$$

(Note that $\{w(E)\}$ is a vector in the $|E(\mathbf{H})|$-dimensional Euclidean space, $\mathbb{R}^{|E(H)|}$). The *value* of the fractional matching w is

$$|w| = \sum_E w(E).$$

The maximum of $|w|$ when w ranges over all fractional matchings is called the *fractional matching number* and is denoted by

$$\nu^*(\mathbf{H}) = \max\{|w|: w \text{ is a fr. matching of } \mathbf{H}\}.$$

Similarly, a *fractional cover of* $\mathbf{H}$ is a real function over $V(\mathbf{H})$ satisfying

$$t(p) \geq 0 \quad \text{for all } p \in V(\mathbf{H}), \text{ an}$$

$$\sum_{p \in E} t(p) \geq 1 \quad \text{for all } E \in E(\mathbf{H}),$$

and $\min|t| = \tau^*(\mathbf{H})$ is called the fractional covering number of $\mathbf{H}$. If $\varnothing \in E(\mathbf{H})$, then let $\nu = \nu^* = \tau^* = \tau = \infty$. For every $\mathbf{H}$ we have

$$\nu \leq \nu^*, \quad \tau^* \leq \tau \tag{5.1}$$

because a matching $\mathcal{M} \subset E(\mathbf{H})$ defines a fractional matching w with $w(E) = 0$ or 1 according to $E \notin \mathcal{M}$ or $E \in \mathcal{M}$. Similarly the characteristic vector of a cover T (i.e., $t(p) = 1$ if $p \in T$, otherwise 0) is a fractional cover, too. Moreover, for every w and t we have

$$|w| \le |t|. \tag{5.2}$$

The proof is only one line:

$$|w| = \sum_E w(E) \le \sum_E w(E)\Big(\sum_{p\in E} t(p)\Big) = \sum_p t(p)\Big(\sum_{p\in E} w(E)\Big) \le \sum_p t(p) = |t|. \tag{5.3}$$

(5.1) and (5.2) implies that

$$\nu \le \nu^* \le \tau^* \le \tau,$$

and if we find a w and a t with $|w| = |t|$, then $\nu^* = \tau^* = |w| = |t|$, i.e., both are optimal. (Later, we will see that $\nu^* = \tau^*$ holds for every $\mathbf{H}$).

Examples 5.0. (i) For every bipartite graph $\mathbf{G}$ we have $\nu = \tau = \nu^*(\mathbf{G})$, by König's theorem.
(ii) If C_{2k+1} is an odd cycle then $w(E) \equiv \frac{1}{2}$, (and $t(p) \equiv \frac{1}{2}$) is a fractional matching (cover) so we have $\nu = k$, $\nu^* = \tau^* = k + \frac{1}{2}$, $\tau = k + 1$.
(iii) $\tau^*(\mathbf{K}_r^u) = u/r$, because $w(E) \equiv 1\Big/\binom{u-1}{r-1}$ is a fractional matching, and $t(p) \equiv 1/r$ is a fractional cover,
(iv) $\tau^*(PG(2,q)) = q + 1/(q+1)$, and in more general if $\mathbf{H}$ is a symmetric (r, λ)-design with $|V(\mathbf{H})| = |E(\mathbf{H})| = (r^2 - r + \lambda)/\lambda$ then $\tau^*(\mathbf{H}) = (r-1)/\lambda + 1/r$. (See the beginning of Section 3).

We can generalize the above examples in the following way. Denote by $D(\mathbf{H})$ the maximum *degree* of the hypergraph $\mathbf{H}$, i.e., $D(\mathbf{H}) = \max_x |\{E: x \in E \in E(\mathbf{H})\}|$. If all the vertices have degree D then we call $\mathbf{H}$ *D-regular*. The function $w(E) \equiv 1/D$ $(E \in E(\mathbf{H}))$ is a fractional matching, so

$$\frac{|E(\mathbf{H})|}{D(\mathbf{H})} \le \nu^*. \tag{5.4}$$

If $m(\mathbf{H})$ denotes the cardinality the smallest edge of $\mathbf{H}$, then $t(p) \equiv 1/m(\mathbf{H})$ is a fractional cover, hence

$$\tau^* \le \frac{|V(\mathbf{H})|}{m(\mathbf{H})}. \tag{5.5}$$

The above two trivial inequalities imply

Proposition 5.1. *If* $\mathbf{H}$ *is a D-regular r-graph then*

$$\nu^* = \tau^* = \frac{|V(\mathbf{H})|}{r} = \frac{|E(\mathbf{H})|}{D}. \qquad \square$$

The next proposition is easy, too (see e.g., Lovász [177]).

Proposition 5.2. *If the automorphism-group of* **H**, Aut(**H**), *is transitive over the vertices then*

$$\tau^*(\mathbf{H}) = \frac{|V(\mathbf{H})|}{m(H)};$$

if Aut(**H**) *is transitive over the edges then*

$$\tau^*(\mathbf{H}) = \frac{|E(\mathbf{H})|}{D(\mathbf{H})}. \qquad \square$$

Example 5.3. (i) (*Truncated* (r, λ) – *design*). Let **H** be an r-graph obtained from a symmetric (r, λ)-design deleting a point p and all the r edges through this point. Then $|V(\mathbf{H})| = (r^2 - r)/\lambda$, **H** is $(r - \lambda)$-regular, $\tau^*(\mathbf{H}) = (r - 1)/\lambda$.
(ii) (Edmonds). Let **M** be a *matroid* with vertex-set V, and denote by $\mathscr{B}$ the family of bases. Then

$$\tau^*(\mathscr{B}) = \min_{A \subseteq V} \left\{ |A| \Big/ \min_{B \in \mathscr{B}} |B \cap A| \right\}, \text{ and}$$

$$\nu(\mathscr{B}) = \lfloor \tau^*(\mathscr{B}) \rfloor.$$

A brief survey of linear programming. The basic problem of linear programming is the following: Find $\min \mathbf{c}^T\mathbf{x}$ where $\mathbf{c} \in \mathbb{R}^n$ (a column vector) and $\mathbf{x}$ satisfies the constraints

$$A\mathbf{x} \geq \mathbf{b}$$
$$\mathbf{x} \geq 0 \tag{5.6}$$

where **A** is an $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$. A point **x** satisfying (5.6) is called a *feasible* solution. The set of feasible solutions forms a convex set in $\mathbb{R}^n$ bounded by hyperplanes, i.e., a polyhedron. If a polyhedron is not infinite we all it a *polytope.* A hyperplane H is a *supporting* hyperplane of the (closed) polyhedron $P \subset \mathbb{R}^n$ if $P \cap H \neq \varnothing$ but all the points of P are lying in the same halfspace bounded by H. In this case $P \cap H$ is a *face.* If $P \cap H$ is a single point we call it a *vertex* of P. If a face is maximal then it is called a *facet* (or proper face).

The *dual* of the problem (5.6) is the following m-dimensional linear programming problem:

$$\begin{array}{c} \mathbf{y}^T A \leq \mathbf{c} \\ \mathbf{y} \geq 0 \\ \hline \max \mathbf{b}^T\mathbf{y} = ? \end{array} \tag{5.7}$$

Theorem 5.4 (Duality Theorem of Linear Programming). *There are* 3 *possibilities.*
(i) *The constraints of* (5.6) *lead to contradiction, no feasible solution exists. Then in the problem* (5.7) $\max b^T y = +\infty$.
(ii) *The problem* (5.6) *has a solution and a finite optimum* M. *Then* $\max \mathbf{b}^T\mathbf{y}$ *exists and its value is* M, *too.*
(iii) *In the problem* (5.6) $\min \mathbf{c}^T\mathbf{x} = -\infty$. *Then the dual problem does not have any feasible solution.*

Define the incidence-matrix $A(\mathbf{H})$ of a hypergraph $\mathbf{H}$ by $A = (a_{E,p})$ where $E \in E(\mathbf{H})$, $p \in V(\mathbf{H})$ and

$$a_{E,p} = \begin{cases} 1 & \text{if } p \in E \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{b} \in \mathbb{R}^{|E(H)|}$, $\mathbf{c} \in \mathbb{R}^{|V(H)|}$ have all components equal to 1. Then the inequalities (5.6) define a non-empty polyhedron, the fractional covering polyhedron of $\mathbf{H}$ (briefly $FCP(\mathbf{H})$) and the inequalities (5.7) define a non-empty bounded polytope the fractional matching polytope, $FMP(\mathbf{H})$. (We supposed that $\varnothing \notin E(\mathbf{H})$). Moreover the optimum of the problems (5.6) and (5.7) give $\tau^*(\mathbf{H})$ and $\nu^*(\mathbf{H})$, resp. So the Duality Theorem implies

Theorem 5.5. *For every hypergraph $\nu^* = \tau^*$ holds.* □

τ^*-critical hypergraphs. A *partial* hypergraph $\mathbf{H}'$ is formed by a subset of $E(\mathbf{H})$, i.e., $E(\mathbf{H}') \subset E(\mathbf{H})$ and $\bigcup \{E: E \in E(\mathbf{H}')\} \subset V(\mathbf{H}') \subset V(\mathbf{H})$. A *subhypergraph* $\mathbf{H}_0$ is a partial hypergraph of the *induced* subhypergraph $\mathbf{H}|V_0$ where $V_0 = V(\mathbf{H}_0) \subset V(\mathbf{H})$ and $E(\mathbf{H}|V_0) = \{E \cap V_0: E \in E(\mathbf{H})\}$.

A family $\mathbf{H}$ is called *τ^*-critical* if $\tau^*(\mathbf{H}') < \tau^*(\mathbf{H})$ holds for every partial hypergraph $\mathbf{H}' \neq \mathbf{H}$.

Lemma 5.6 (Füredi [130]). *If $\mathbf{H}$ is τ^*-critical then* $|E(\mathbf{H})| \leq |\bigcup \{E: E \in E(\mathbf{H})\}|$ $(\leq |V(\mathbf{H})|)$.

To prove Lemma 5.6 we need a simple Lemma from linear programming. First we recall Helly's theorem.

Theorem 5.7 (Helly, see in [76]). *Let $C_1, \dots, C_m$ be convex sets in $\mathbb{R}^d$, $m > d$. Suppose that every $d + 1$ of them has a non-empty intersection. Then $\bigcap_{i=1}^{m} C_i \neq \varnothing$.* □

Denote by $M(\{\mathbf{a}_i, b_i\}, \mathbf{c}, I\}$ the minimum of the following linear program

$$\begin{aligned} \mathbf{a}_i^T \mathbf{x} &\geq b_i \quad \text{for } i \in I, \mathbf{x} \in \mathbb{R}^n \\ \mathbf{x} &\geq 0 \\ \min \mathbf{c}^T \mathbf{x} &= ? \end{aligned} \tag{5.8}$$

Lemma 5.8. *If a linear program* (5.8) *with n variables has a finite optimum (and $|I|$ is finite), then there exists a $J \subset I$ such that $|J| \leq n$ and $M(\{\mathbf{a}_i, b_i\}, \mathbf{c}, I) = M(\{\mathbf{a}_i, b_i\}, \mathbf{c}, J)$.*

In other words this lemma states that the number of constraints can be reduced to n without changing the optimum value.

Proof of Lemma 5.8. Dropping some of the inequalities of (5.8) the minimum value of the program can only decrease. Hence we have to prove that there is a $J \subset I$, $|J| = n$ such that $M(\{\mathbf{a}_i, b_i\}, \mathbf{c}, J) \geq M =: M(\{\mathbf{a}_i, b_i\}, \mathbf{c}, I)$. Suppose on the contrary, that for every $J \subset I$, $|J| = n$, we have $M(\{\mathbf{a}_i, b_i\}, \mathbf{c}, J) < M$. This means that any n

of the halfspaces $\{\mathbf{y}: \mathbf{a}_i^T\mathbf{y} \geq b_i\}$ have a common point with the convex polyhedron $\{\mathbf{x} \in \mathbb{R}^n: \mathbf{c}^T\mathbf{x} < M, \mathbf{x} \geq 0\}$. The system (5.8) has a feasible solution, hence any $n+1$ of the halfspaces $\{\mathbf{y}: \mathbf{a}_i^T\mathbf{y} \geq b_i\}$ have a point in common. Now, Helly's theorem (Theorem 5.7) implies that the intersection of all the $|I|+1$ convex sets $\{\mathbf{y}: \mathbf{a}_i^T\mathbf{y} \geq b_i\}$ and $\{\mathbf{x}: \mathbf{c}^T\mathbf{x} < M, \mathbf{x} \geq 0\}$ is non-empty, i.e., it contains a point $\mathbf{x}_0$. This $\mathbf{x}_0$ is a feasible solution of (5.8) and $\mathbf{c}^T\mathbf{x}_0 < M = M(\{\mathbf{a}_i, b_i\}, \mathbf{c}, I)$. This contradiction proves the existence of an appropriate J. □

Proof of Lemma 5.6. To determine $\tau^*(\mathbf{H})$ one has to solve a linear program of dimension $|\bigcup E(\mathbf{H})|$, with the index set I, where $|I| = |E(\mathbf{H})|$. Of course, this program always has a finite optimum. So by Lemma 5.8 we obtain a partial hypergraph $\mathbf{H}_1$, $E(\mathbf{H}_1) \subset E(\mathbf{H})$ satisfying $|E(\mathbf{H}_1)| \leq |\bigcup E(\mathbf{H})|$, and $\tau^*(\mathbf{H}_1) = \tau^*(\mathbf{H})$. As $\mathbf{H}$ is τ^*-critical we have $\mathbf{H} = \mathbf{H}_1$. □

Lemma 5.9 ([70]). *If* $\mathbf{H}$ *is* τ^**-critical then* $|E(\mathbf{H})| \leq r(\mathbf{H})\tau^*(\mathbf{H})$.

Proof. There exists an optimal fractional matching w_0 which is a vertex of the fractional matching polytope $FMP(\mathbf{H})$. Then the vector $\{w_0(E)\}_{E \in E(H)}$ is contained in at least $|E(\mathbf{H})|$ facets of $FMP(\mathbf{H})$. The equation of a facet of $FMP(\mathbf{H})$ is either $y_E = 0$ or $\sum_{p \in E} y_E = 1$ for some $p \in V(\mathbf{H})$. Denote by V_0 and $\mathscr{E}_0$ the set of facets through $\mathbf{w}_0$,

$$w(E) = 0 \quad \text{if } E \in \mathscr{E}_0 \subset E(\mathbf{H}), \text{ and}$$
$$\sum_{p \in E} w(E) = 1 \quad \text{if } p \in V_0 \subset V(\mathbf{H}),$$

where $|V_0| + |\mathscr{E}_0| \geq |E(\mathbf{H})|$. We have

$$|V_0| = \sum_{p \in V_0}\left(\sum_{p \in E} w(E)\right) = \sum_{E \in E(\mathbf{H})} w(E)|E \cap V_0| \leq \left(\sum w(E)\right)r = r\tau^*. \quad (5.9)$$

Hence $|E(\mathscr{H}) - \mathscr{E}_0| \leq r\tau^*$. Let $E(\mathbf{H}') = E(\mathbf{H}) - \mathscr{E}_0$. Then

$$\nu^*(\mathbf{H}') \geq |w_0| = \nu^*(\mathbf{H}),$$

since w_0 is a fractional matching of $\mathbf{H}'$ as well. However, $\nu^*(\mathbf{H}') \leq \nu^*(\mathbf{H})$, so $\nu^*(\mathbf{H}') = \nu^*(\mathbf{H})$. $\mathbf{H}$ is ν^*-critical, so we have $\mathbf{H} = \mathbf{H}'$, $|E(\mathbf{H})| \leq r\tau^*$. □

In the same way as we did in 5.9 one can prove that.

Proposition 5.10. *Every vertex* $\mathbf{v} \in \mathbb{R}^{|E(H)|}$ *of* $FMP(\mathbf{H})$ *has at most* $r\tau^*$ *non-zero components* (*for every hypergraph* $\mathbf{H}$).

For a hypergraph $\mathbf{H}$ call the set $S \subset V(\mathbf{H})$ a *supporting set* (or support) if S is a support of some optimal fractional cover t, i.e., $S = \{x \in V(\mathbf{H}): t(x) > 0\}$.

Proposition 5.11. (i) The family of supporting sets is closed under union,
(ii) If S is a supporting set and w an optimal fractional matching then every point $x \in S$ is saturated by w $\left(\text{i.e., } \sum_{x \in E} w(E) = 1.\right)$
(iii) The union of all supporting sets is not larger than $\sum_E w(E)|E| \leq \tau^* r$.

Proof. (i) If t_1 and t_2: $V(\mathbf{H}) \to \mathbb{R}$ are optimal fractional covers with support S_1 and S_2, then $S_1 \cup S_2$ is the support of $(t_1 + t_2)/2$.
(ii) This statement is a special case of the so-called "complementary slackness condition" in linear programming. We have as in (5.3)

$$\tau^* = \sum_E w(E) \le \sum_E \sum_{p \in E} w(E)t(p) = \sum_{p \in S} t(p)\left(\sum_{p \in E} w(E)\right) \le \sum_p t(p) = \tau^*$$

Thus equalities are forced, so

$$\sum_{p \in E} w(E) = 1$$

holds for every $p \in S$.
(iii) By (ii) we have

$$|S| = \sum_{p \in S} 1 = \sum_{p \in S} \sum_{p \in E} w(E) \le \sum w(E)|E|. \qquad \square$$

k-covers and k-matchings. A collection $\{E_1, \ldots, E_s\}$ with $E_i \in E(\mathbf{H})$ (and $E_i = E_j$ is possible) is called a *k-matching* of $\mathbf{H}$ if every $p \in V(\mathbf{H})$ is contained in at most k of E_i. The maximum cardinality of a k-matching is the *k-matching number* of $\mathbf{H}$ and is denoted by $\nu_k(\mathbf{H})$. Clearly,

$$\nu \le \frac{\nu_k}{k} \le \nu^*.$$

Similarly, the function t: $V(\mathbf{H}) \to \{0, 1, \ldots, k\}$ is a *k-cover* if $\sum_{p \in E} t(p) \ge k$ holds for every $E \in E(\mathbf{H})$. The *k-covering number* is $\tau_k(\mathbf{H}) = \min\left\{\sum_{p \in V(H)} t(p)\text{: } t \text{ is a } k\text{-cover of } \mathbf{H}\right\}$. Then $\tau = \tau_1$ and

$$\tau^* \le \frac{\tau_k}{k} \le \tau.$$

Proposition 5.12.
(i) $\nu_k + \nu_l \le \nu_{k+l}$
(ii) $\tau_{k+l} \le \tau_k + \tau_l$
(iii) $\lim_{k \to \infty} \tau_k/k = \tau^* = \lim_{k \to \infty} \nu_k/k$
(iv) there exists an integer $K = K(\mathbf{H})$ such that

$$\tau^*(\mathbf{H}) = \tau_K(\mathbf{H})/K = \nu_K(\mathbf{H})/K.$$

(v) $$\nu \le \max_{E(H') \subset E(H)} \frac{|E(H')|}{D(H')} \le \tau^* \le \min_{T \text{ is a cover}} \frac{|T|}{\min_{E \in E(H)} |T \cap E|} \le \tau$$

Proof. (i) and (ii) are trivial. They imply that the limits in (iii) exist. (iv) implies that the common value of these limits is τ^*. (iv) is implied by the fact that τ^* is a rational number because it is an optimum of a linear program with integer coefficients. (v) is trivial. $\square$

Proposition 5.11 (ii) implies the following observation of Berge.

Proposition 5.13. *If for the hypergraph* **H** *we have* $\tau_k(\mathbf{H}) = \tau(\mathbf{H})k$ *then* $\tau_p = \tau p$ *holds for all* $1 \le p \le k$. □

A simple algorithm to find a k-matching. The following process was defined by Lovász [177]. Let **H** be a hypergraph. Define the sequence of edges $E_1, E_2, \ldots$ (repetitions are allowed) in the following way. Choose E_1 arbitrarily. If $E_1, \ldots, E_i$ are chosen determine to every vertex of **H** its frequency, and order the vertices such that these numbers are decreasing. Then let E_{i+1} that edge which starts with the highest index. This method gives the following theorem

Theorem 5.14 ([177, 180]). *For every hypergraph* **H** *of rank r we have*

$$r\nu_k \ge k\tau + (k-1)(r-1). \qquad \square$$

Corollary 5.15. *If* **H** *is a t-wise intersecting hypergraph of rank r then*

$$\tau \le \frac{r-1}{t-1} + 1. \qquad \square$$

Using 5.11 (iii) we have

Corollary 5.16. $r\tau^* \ge \tau + r - 1$.

A direct proof for Corollary 5.16. Let $t\colon V(\mathbf{H}) \to \mathbb{R}$ be an optimal fractional cover, $t(x_1) \ge t(x_2) \ge \cdots \ge t(x_s) > 0$ and $t(x) = 0$ for $x \notin \{x_1, \ldots, x_s\}$. Define ℓ as follows:

$$\sum_{0 \le i \le \ell} t(x_{s-i}) < 1 \le \sum_{0 \le i \le \ell+1} t(x_{s-i}).$$

Then $\{x_1, \ldots, x_{s-\ell-1}\}$ is a cover, and $s - \ell - 1 \le r\tau^* - r + 1$. □

Fractional matchings in graphs. Edmonds pointed out, that an old theorem of Tutte [236] implies that $2\tau^*(\mathscr{G})$ is always an integer for a graph $\mathscr{G}$.

Theorem 5.17 (Edmonds [89]). *For a graph* $\mathscr{G}$ *we have* $\nu_2(\mathscr{G}) = 2\tau^*(\mathscr{G}) = \tau_2(\mathscr{G})$.

Balinski [19], Balinski and Spielberg [20] and Nemhauser and L. Trotter [205] proved that even much more is true. To state their results define the *fractional point packing polytope* of the hypergraph **H**, as the set of all vectors **v** in $\mathbb{R}^{|V(\mathbf{H})|}$, such that **v** is a fractional point-packing, i.e.,

$$FPP(\mathbf{H}) = \left\{\mathbf{v} \in \mathbb{R}^{|V(\mathbf{H})|}\colon \mathbf{v} \ge 0 \text{ and for all } E \in E(\mathbf{H}) \text{ we have } \sum_{x \in E} \mathbf{v}(x) \le 1\right\}.$$

If we can effectively describe all the vertices and facets of a polytope then in a certain sense we can solve any optimization problem on it. This description was given in [19], [20] and [205]. (A discussion of this and more graphtheoretical background can be found in Lovász [184]).

Theorem 5.18 ([19], [20]). *Let* $C_1, \ldots, C_p$ *be vertex-disjoint odd circuits and* $\mathscr{M} \subset E(\mathscr{G})$ *be a matching independent from these circuits. (I.e.,* $V(C_i) \cap V(\mathscr{M}) = \varnothing$ *for all* $1 \le i \le p$*). Let*

$$w(E) = \begin{cases} 1/2 & \text{if } E \in \bigcup E(C_i) \\ 1 & \text{if } E \in \mathcal{M} \\ 0 & \text{otherwise.} \end{cases}$$

Then **w** *is a vertex of* $FMP(\mathcal{G})$, *and all vertices of this polytope have the above form.*

Theorem 5.19 ([205]). *Let* $\mathcal{G}$ *be a graph,* $\{A, B, C\}$ *a partition of* $V(\mathcal{G})$ *in such a way that*
(i) A *is an independent set,*
(ii) B *contains all neighbors of* A,
(iii) *there is no bipartite component in the induced subgraph on* C.
Then define

$$v(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in B, \\ 1/2 & \text{if } x \in C. \end{cases}$$

Then **v** *is a vertex of* $FPP(\mathcal{G})$ *and all vertices of* $FPP(\mathcal{G})$ *can be obtained in the above way.*

R. Aharoni [8] has a generalization for *infinite* graphs.

Corollary 5.20. *All coordinates of vertices of the polytopes* $FMP(\mathcal{G})$, $FPP(\mathcal{G})$, *and* $FCP(\mathcal{G})$ *equal to* $0, \frac{1}{2}$ *or* 1. □

Of course, it is much more important to investigate the *matching* and *covering* polytopes instead of their real relaxations. Define the *matching polytope* of **H** as follows

$$MP(\mathbf{H}) = \operatorname{conv}\left\{\mathbf{w} \in \mathbb{R}^{|E(\mathbf{H})|}: w(E) = 0 \text{ or } 1 \text{ and } \sum_{p \in E} w(E) \le 1 \text{ for all } p \in V(\mathbf{H})\right\}.$$

I.e., $MP(\mathbf{H})$ is the convex hull of the characteristic vectors of the matchings in **H**. For graphs, this polytope was described by Edmonds [89] (see also Edmonds and Pulleyblank [91]).

The following inequality is due to Lovász [178].

Theorem 5.21 ([178]). *Let* $\mathcal{G}$ *be a graph. Then* $\tau^* \le \frac{1}{2}(\tau + \nu)$.

Proof. We have to define a fractional cover t: $V(\mathcal{G}) \to \mathbb{R}$ such that $|t| \le (\tau + \nu)/2$. Let $T \subset V(\mathcal{G})$ be a minimal cover, $|T| = \tau$. Put 1/2 weight into each vertex $x \in T$. The graph $\mathcal{G}_0 = \{E \in E(\mathcal{G}): |E \cap T| \le 1\}$ is bipartite. Let T_0 be a (minimum) cover of $\mathcal{G}_0$ and put another 1/2 weight into each vertex $x \in T_0$. Then we obtain a fractional
cover t of $\mathcal{G}$ with weight

$$|t| = \tfrac{1}{2}|T| + \tfrac{1}{2}|T_0| = \tfrac{1}{2}\tau + \tfrac{1}{2}\tau(\mathcal{G}_0).$$

By the König-Hall theorem we have $\tau(\mathcal{G}_0) = \nu(\mathcal{G}_0) \le \nu(\mathcal{G})$. □

3-Graphs with arbitrary denominators. It is obvious, that for arbitrary hypergraphs a statement similar to Corollary 5.20 is not true. For every rational number p/q

(≥ 1) there exists a hypergraph $\mathbf{H}$ with $\tau^*(\mathbf{H}) = p/q$. (E.g., the complete q-graph over p elements). Even more is true:

Proposition 5.22 (Lovász [177]). *Let $a, b, c, d \geq 1$ be integers, $a \leq b/c \leq d$, $b/c > 1$. Then there exists a hypergraph* $\mathbf{H}$ *with*

$$\nu(\mathbf{H}) = a, \quad \tau^*(\mathbf{H}) = b/c \quad \textit{and} \quad \tau(\mathbf{H}) = d.$$

Proof. Let B and D be two disjoint sets with $|B| = bd$, $|D| = d$. Let $\mathscr{F} = \left\{F \cup \{p\}: F \in \binom{B}{cd}, p \in D\right\}$. Then $\tau(\mathscr{F}) = d$, $\tau^*(\mathscr{F}) = b/c$. If $\nu(\mathscr{F}) > a$ then consider two disjoint members of $\mathscr{F}$, $F, F' \in \mathscr{F}$ and a new element x_1 outside $\bigcup \mathscr{F}$. Let $\mathscr{F}_1 = \mathscr{F} - \{F, F'\} \cup \{F \cup \{x_1\}, F' \cup \{x_1\}\}$. Continue this process until we get an $\mathscr{F}_m$ with $\nu(\mathscr{F}_m) = a$. □

It is easy to see that for the case $b/c = 1$ Proposition 5.22 does not hold, because $\tau^*(\mathbf{H}) = 1$ implies $\tau(\mathbf{H}) = 1$ (see (5.1)). The hypergraphs used in the above proof have large ranks. A similar statement is true for hypergraphs of rank 3. For a real number x denote by $\{x\}^*$ its *fractional part*, i.e., $\{x\}^* = x - [x]$.

Theorem 5.23 (Chung, Füredi, Garey and Graham [70]). *Let $0 \leq r < 1$ be a rational number. Then there exists a hypergraph* $\mathbf{H}$ *of rank* 3 *with* $\{\tau^*(\mathbf{H})\}^* = r$.

Proof. It follows immediately from the following two constructions:

Example 5.24. (A hypergraph $\mathbf{H}$ of rank 3 with $4k + 2$ edges, and with $\{\tau^*\}^* = 2^k/(2^{k+1} - 1)$). Let $V(\mathbf{H}) = \{x_1, \ldots, x_{3k}\} \cup \{a_1, \ldots, a_k\} \cup \{l_1, l_2\}$, and $A_{2i-1} = \{x_{3i-2}, a_i\}$, $A_{2i} = \{x_{3i-1}, a_i\}$, $B_{2i-1} = \{x_{3i-2}, x_{3i-1}, x_{3i}\}$ for $1 \leq i \leq k$, and $B_{2i} = \{x_{3i}, x_{3i+1}\}$ for $1 \leq i \leq k - 1$, and $B_0 = \{x_1, x_{3k}, l_1\}$, $E_0 = \{x_{3k}, l_0\}$, $E_1 = \{l_0, l_1\}$. (See Fig. 5.1).

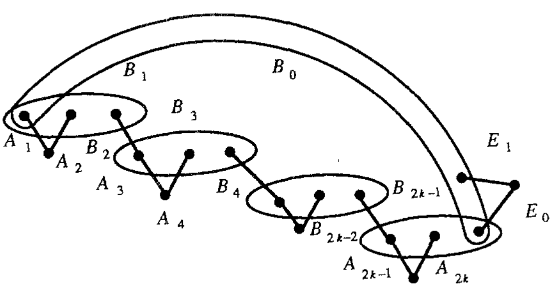

Fig. 5.1

To find $\tau^*(\mathbf{H})$ consider the following fractional matching λ:

$$\lambda(A_{2i-1}) = \lambda(B_{2i-1}) = 2^{k-i}/N \quad \text{for } 1 \leq i \leq k,$$

$$\lambda(A_{2i}) = \lambda(B_{2i}) = (N - 2^{k-i})/N \quad \text{for } 1 \leq i \leq k - 1,$$

$$\lambda(BA_0) = \lambda(E_0) = (2^k - 1)/N, \lambda(A_{2k}) = (N - 1)/N \quad \text{and}$$

$$\lambda(E_1) = 2^k/N,$$

where N stands for $2^{k+1}-1$. Then $|\lambda| = 2k + 2^k/N$. Define $t(a_i) = (N - 2^{i-1})/N$, $t(x_{3i-2}) = t(x_{3i-1}) = 2^{i-1}/N$, and $t(x_{3i}) = (N - 2^i)/N$ for $1 \le i \le k$, and $t(l_1) = (2^k - 1)/N$, $t(l_0) = 2^k/N$. Then t is fractional cover with $|t| = 2k + 2^k/N$. So we have $\tau^* = 2k + 2^k/N$. □

Example 5.25. (A hypergraph **H** of rank 3 with $2k-1$ edges and with $\{\tau^*(\mathbf{H})\}^* =$ (odd integer)$/2^{k-1}$). Let $V(\mathbf{H}) = \{x_1, \ldots, x_k, y_1, \ldots, y_k\}$, $E(\mathbf{H}) = \{\{x_k, y_k\}, A_i, B_i$ for $1 \le i \le k-1\}$ where $A_i = \{x_i, y_i, x_{i+1}\}$, $B_i = \{x_i, y_i, y_{i+1}\}$. Then

$$\tau^*(\mathbf{H}) = \frac{2k}{3} + \frac{2}{9} + \frac{(-1)^{k-1}}{9 \cdot 2^{k-1}}.$$

An upper bound on the denominator. Let $N_r = \{\tau^*(\mathbf{H})\colon \mathbf{H}$ has rank at most $r\}$. Then by Theorem 5.17 we have $N_2 = \{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots\}$ and $\bigcup_{r \ge 2} N_r$ consists of all rational numbers not smaller than 1.

Theorem 5.26 (Chung, Füredi, Garey and Graham [70]). *If* $\frac{u}{v} \in N_r$, $(u, v) = 1$, *then* $\frac{u}{v} \ge 2 \log v / r \log r$. *Hence the set* N_r *is a discrete sequence.*

For example (see in [70]) the initial segment of N_3 is the following: $N_3 = \{1, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, 2, \ldots\}$.

Proof of Theorem 5.26. Let **H** be a hypergraph of rank r with $\tau^* = u/v$. We can suppose that **H** is τ^*-critical, hence $|E(\mathbf{H})| \le ru/v$ by Lemma 5.9. So the value of $\nu^*(\mathbf{H})$ can be obtained (by Cramer's rule) as a ratio of two $0-1$ determinants of size at most ru/v. Every row contains at most r entries equal to 1 so by Hadamard's upper bound on the determinant we have

$$v = |\text{denominator}| = |\det(0-1 \text{ matrix})| \le \sqrt{r}^{\,ru/v}. \qquad \square$$

Let $N_r = \{t_1^{(r)}, t_2^{(r)}, \ldots, t_i^{(r)}, \ldots\}$ with $t_i^{(r)} < t_{i+1}^{(r)}$. This is a discrete sequence, but Theorem 5.23 implies that for $r \ge 3$

$$\lim_{i \to \infty} t_{i+1}^{(r)} - t_i^{(r)} = 0.$$

Define $d_r(n) = \max\{\text{denominator of } \tau^*(\mathbf{H})\colon \mathbf{H} \text{ is an } r\text{-graph with } E(\mathbf{H}) \le n\}$. Examples 5.25 and 5.26 imply

$$\tfrac{1}{2} n \log 2 + 0(1) \le \log d_3(n) \le \tfrac{1}{2} n \log 3. \tag{5.9}$$

Problem 5.27. It seems to be very likely that $\lim_{n \to \infty} \log d_3(n)/n$ exists (and it is $\frac{1}{2}\log 2$). Give bounds for $\log d_r(n)$.

Normal hypergraphs and generalizations. A hypergraph **H** is *normal* if for every partial hypergraph **H**′ we have $\nu(\mathbf{H}') = \tau(\mathbf{H}')$. This notion has an important role in the proof of the Weak Perfect Graph Conjecture (see Lovász [174]). A structure theorem for normal hypergraphs is due to Berge [25] and Berge and LasVergnas [30]. (This says that **H** is normal iff every odd cycle has an edge containing at least 3 vertices of the cycle.) (See in [26]). It is easy to prove the following propositions:

Proposition 5.28. *A hypergraph is normal iff it has no* ν*-stable partial hypergraph.*

Proposition 5.29. *A hypergraph is normal iff each of its τ-critical partial hypergraph consists of disjoint edges.*

Theorem 5.30 (Lovász [175]). *A hypergraph* **H** *is normal iff for every partial hypergraph* **H**′ $\tau^*(\mathbf{H}')$ *is an integer.*

Berge proved the following sharpening of Theorem 5.30.

Theorem 5.31 ([27]). *If* $\tau_2(\mathbf{H}') = 2\tau(\mathbf{H}')$ *holds for each hypergraph* **H**′ *obtained from* **H** by removing edges, *then it is normal.*

Lovász proved the following generalizations of 5.30 for $k = 2$ ([178]) and $k = 3$ ([182]).

Theorem 5.32. *Let k be given,* $k = 2$ *or* 3. *If* $k\tau^*(\mathbf{H}')$ *is an integer for each partial hypergraph* **H**′ *of* **H** *then* $\nu_k(\mathbf{H}) = \tau_k(\mathbf{H})$.

Proof of the case $k = 3$ (sketch). We use induction on $|E(\mathbf{H})|$ to prove that $\nu_3 = 3\nu^*$. Let w: $E(\mathbf{H}) \to \mathbb{R}$ an optimal fractional matching with maximum number of edges with $w(E) \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}$. If there exists an edge E with $w(E) = 0$ or 1 then we can use the inductional hypothesis. Then follows that $\frac{1}{3} \le w(E) \le \frac{2}{3}$ holds for all edge E. Consider $\mathscr{H}^- = \{E: \frac{1}{3} < w(E) < 1/2\}$, $\mathscr{H}^+ = \{E: 1/2 < w(E) < 2/3\}$. Suppose $|\mathscr{H}^+| \ge |\mathscr{H}^-|$. Then we can increase $|w|$ by adding ϵ to the edges of $\mathscr{H}^+$ and decreasing $w(E)$ by ϵ for $E \in \mathscr{H}^-$. So we can suppose that for all E we have $w(E) \in \{1/3, 1/2, 2/3\}$. □

The following statements are weaker than Theorem 5.32.

If $k\tau^*(\mathbf{H}')$ is an integer for each partial hypergraph **H**′ of **H** then $\nu_k(\mathbf{H}) = k\tau^*(\mathbf{H})$. (5.10)

If $k\tau^*(\mathbf{H}')$ is an integer for each partial hypergraph **H**′ of **H** then $\tau_k(\mathbf{H}) = k\tau^*(\mathbf{H})$. (5.11)

But the following example of Seymour ([224]) shows that Theorem 5.32 and (5.10) do not hold for $k = 60$ and (5.11) for $k = 20$.

Example 5.33. (For all partial hypergraph $\tau_{60}(\mathbf{H}') = 60\tau^*(\mathbf{H}')$ but $\tau_{60}(\mathbf{H}) \ne \nu_{60}(\mathbf{H})$.) Let $X = \{1,2,3,4,5,6,7,8,9\}$ and $\mathscr{E} = \{E_1, E_2, \ldots, E_7\}$ where $E_1 = X - \{1,3,5\}$, $E_2 = X - \{1,4,6\}$, $E_3 = X - \{2,3,6\}$, $E_4 = X - \{2,4,5\}$, $E_5 = X - \{7\}$, $E_6 = X - \{8\}$, $E_7 = X - \{9\}$. (See Fig. 5.2).

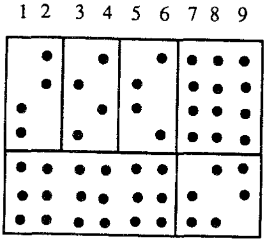

Fig. 5.2

Multiplication of vertices. Let $\mathbf{H} = (X, \mathscr{E})$ be a hypergraph. We recall that *multiplying a point* x by $k \geq 0$ (an integer) means that we replace x by k new points $x_1, \ldots, x_k$, and at the same time replace each edge E containing x by the new edges $(E - \{x\}) \cup \{x_1\}, \ldots, (E - \{x\}) \cup \{x_k\}$. So multiplying x by 0 coincides with *removing* x. Lovász proved the following.

Theorem 5.34. [179]. *If* $\nu_2(\mathbf{H}') = 2\nu(\mathbf{H}')$ *holds for each hypergraph* $\mathbf{H}'$ *obtained from* $\mathbf{H}$ *by multiplication of points, then* $\tau(\mathbf{H}) = \nu(\mathbf{H})$ *(i.e.,* $\mathbf{H}$ *is normal).*

He conjectured and Schrijver [224] proved the following generalizations:

Theorem 5.35 ([224]). *If* $k\tau^*(\mathbf{H}')$ *is an integer for each hypergraph* $\mathbf{H}'$ *arising from* $\mathbf{H}$ *by multiplication of points, then* $\tau_k(\mathbf{H}) = k\tau^*(\mathbf{H})$ $(= \nu_k(\mathbf{H}).)$

Theorem 5.36 ([224]). *If* $\nu_{2k}(\mathbf{H}') = 2\nu_k(\mathbf{H}')$ *for each hypergraph* $\mathbf{H}'$ *arising from* H *by multiplication of points, then* $\tau_k(\mathbf{H}) = k\tau^*(\mathbf{H})$ $(= \nu_k(\mathbf{H}).)$

The main tool of the proof of Theorems 5.35 and 5.36 is the following important lemma which was proved (in different forms) by Gomory [143], Chvátal [71], Hoffman [158], Fulkerson [127], Edmonds and Giles [90], Lovász [182], Schrijver and Seymour [224].

Lemma 5.37. *Let* P *be a convex polyhedron in* $\mathbb{R}^n$. *If for each vector* $w \in Z^n$ *the number* $\min\{wx\colon x \in P\}$ *is an integer, or* $\pm\infty$, *then each vertex of* P *has integer coordinates.*

Proof of 5.37. Let x_0 be a vertex of P, and consider its ith coordinate. There exists an integer vector $w \in Z^n$ such that both $\min\{wx\colon x \in P\}$ and $\min\{w'x\colon x \in P\}$ are attained at x_0, where w' arises from w by adding 1 to the ith coordinate of w and leaving the remaining coordinates unchanged. So wx_0 and $w'x_0$ are integers, hence also $w'x_0 - wx_0$, the ith coordinate of x_0. □

Proof of 5.35 (hint). Consider the fractional covering polytope $FCP(\mathbf{H})$. Then $kFCP(\mathbf{H})$ has integer vertices. □

Product of hypergraphs. The direct product of two hypergraphs $\mathbf{H}_1$ and $\mathbf{H}_2$ is defined by

$$V(\mathbf{H}_1 \times \mathbf{H}_2) = V(\mathbf{H}_1) \times V(\mathbf{H}_2),$$
$$E(\mathbf{H}_1 \times \mathbf{H}_2) = \{E_1 \times E_2\colon E_1 \in E(\mathbf{H}_1), E_2 \in E(\mathbf{H}_2)\}.$$

It is easy to prove the following properties

$$\tau^*(\mathbf{G} \times \mathbf{H}) = \tau^*(\mathbf{G}) \times \tau^*(\mathbf{H}) \tag{5.12}$$

$$\nu(\mathbf{G})\nu(\mathbf{H}) \leq \nu(\mathbf{G} \times \mathbf{H}) \leq \nu(\mathbf{G})\tau^*(\mathbf{H}) \tag{5.13}$$

$$\tau(\mathbf{G})\tau^*(\mathbf{H}) \leq \tau(\mathbf{G} \times \mathbf{H}) \leq \tau(\mathbf{G})\tau(\mathbf{H}) \tag{5.14}$$

E.g., to prove the first inequality in (5.14) observe that if S is a cover of $\mathbf{G} \times \mathbf{H}$ then $S \cap (V(\mathbf{G}) \times H)$ has at least $\tau(\mathbf{G})$ elements for every $H \in E(\mathbf{H})$. Hence $t(y) := \#\{x\colon (x, y) \in S\}$ is a $\tau(\mathbf{G})$ cover of $\mathbf{H}$.

One cannot improve (5.13) and (5.14) in general as it is shown by the next two theorems. The first one is attributed to Berge and Simonovits [31] by Lovász [177].

Theorem 5.38 ([177]). *Let* $\mathbf{H}$ *be a hypergraph, then*

$$\tau^*(\mathbf{H}) = \min_{\mathbf{G}} \frac{\tau(\mathbf{G} \times \mathbf{H})}{\tau(\mathbf{G})}$$

where $\mathbf{G}$ *runs over all hypergraphs.*

Proof. Let k be an integer $v \geq \tau_k(\mathbf{H})$. If $\mathbf{G} = \mathbf{K}^v_{v-k+1}$, the complete hypergraph over v elements then $\tau(\mathbf{G} \times \mathbf{H}) = \tau_k(\mathbf{H})$. Now we can use Proposition 5.12 (iv). □

The next theorem first appeared in [218] in an implicit form.

Theorem 5.39 (Rosenfeld [218]). *Let* $\mathbf{H}$ *be a hypergraph, then*

$$\tau^*(\mathbf{H}) = \max_{\mathbf{G}} \frac{\nu(\mathbf{G} \times \mathbf{H})}{\nu(\mathbf{G})},$$

where $\mathbf{G}$ *runs over all hypergraphs.*

Proof. If $\mathbf{G} = \mathbf{K}^k_1$ (i.e., k 1-element edges) then $\nu(\mathbf{G} \times \mathbf{H}) = \nu_k(\mathbf{H})$. □

The following results have an information-theoretical background (see [187] about Shannon-capacity). Use the brief notation $\mathbf{H}^t = \mathbf{H} \times \mathbf{H}^{t-1}$ $(:= \mathbf{H} \times \mathbf{H} \times \cdots \times \mathbf{H})$.

Theorem 5.40 (McEliece and Posner [191]).

$$\lim_{t\to\infty} \sqrt[t]{\tau(\mathbf{H}^t)} = \tau^*(\mathbf{H}).$$

Proof. Using (5.12) we have

$$(\tau^*(\mathbf{H}))^t = \tau^*(\mathbf{H}^t) \leq \tau(\mathbf{H}^t). \tag{5.15}$$

To prove an upper bound we are going to use Theorem 6.29 from the next chapter. Denote the maximum degree of a hypergraph $\mathbf{G}$ by $D(\mathbf{G})$. Clearly,

$$D(\mathbf{H}^t) = (D(\mathbf{H}))^t.$$

Hence by Corollary 6.29 we have

$$\begin{aligned}\tau(\mathbf{H}^t) &< (1 + \log D(\mathbf{H}^t))\tau^*(\mathbf{H}^t)\\ &= (1 + t\log D(\mathbf{H}))\tau(\mathbf{H})^t\end{aligned}$$

Then by (5.15) we have for all $t \geq 1$

$$\tau^*(\mathbf{H}) \leq \sqrt[t]{\tau(\mathbf{H}^t)} \leq \sqrt[t]{1 + t\log D(\mathbf{H})}\,\tau^*(\mathbf{H}).$$ □

This short proof is due to Lovász [176]. Surprisingly, the corresponding statement about $\nu(\mathbf{H}^t)$ is not true. Using (5.13) it is easy to see that

$$Q(\mathbf{H}) = \lim_{t\to\infty} \sqrt[t]{\nu(\mathbf{H}^t)}$$

there exists. This number is called the *Shannon capacity* of $\mathbf{H}$. The conjecture $Q(\mathbf{H}) = \tau^*(\mathbf{H})$ is not true. E.g., $Q(C_3) = 1 < \tau^*(C_3) = 3/2$, and $Q(C_5) = \sqrt{5} < \tau^*(C_5) = 5/2$, (see in [187]).

Problem 5.41. Give bounds for the Shannon capacity of a hypergraph. An excellent survey about the Shannon capacity of graphs is [149].

τ- and ν-critical hypergraphs. Now we sharpen Propositions 5.28 and 5.29.

Proposition 5.42 ([175]). *If a hypergraph* $\mathbf{H}$ *is τ-critical, then either* $\mathbf{H}$ *consists of disjoint edges, or*

$$\tau(\mathbf{H}) > \tau^*(\mathbf{H}).$$

Proof. Let $C(E)$ be a cover of $\mathbf{H} - \{E\}$ with cardinality $\tau - 1$ for all $E \in E(\mathbf{H})$. Then $t(x) =: |\{C: x \in C(E), E \in E(\mathbf{H})\}|$ is an $|E(\mathbf{H})| - 1$ cover of $\mathbf{H}$, so

$$\tau^* \leq (\tau - 1)|E(\mathbf{H})|/(|E(\mathbf{H})| - 1). \qquad \square$$

Note that essentially the same proof gives

$$\tau^* \leq \tau - \frac{D-1}{D}.$$

Proposition 5.43 ([175]). *If a hypergraph* $\mathbf{H}$ *is t-wise ν-stable then*

$$\nu^*(\mathbf{H}) \geq \nu(\mathbf{H}) + t \Big/ \min_{E \in E(H)} |E|.$$

Proof. We define a fractional matching $w: E(\mathbf{H}) \to \mathbb{R}$ with value $|w| = \nu + t/\min|E|$. Let $E_0 \in E(\mathbf{H})$ be an edge of minimum cardinality. If $S \in \binom{E_0}{t}$ then we have a matching $E_S^1, E_S^2, \ldots, E_S^\nu$ with $E_S^i \cap S = \varnothing$. Let $w(E_0) = t/|E_0|$ and for $E \neq E_0$ $w(E) = \#\{S: E \in \{E_S^i\}_{1\leq i\leq \nu}\} \Big/ \binom{|E_0|}{t}$. $\square$

Regularisable hypergraphs. A hypergraph $\mathbf{H}$ is said to be *regularisable* if by replacing each edge E by an appropriate non-empty set of edges equal to E, we get a new hypergraph with all vertices having the same degree.

For a graph $\mathbf{G}$, the weight $t(x) := 1/2$ on each vertex $x \in V(\mathbf{G})$ is always a fractional cover. It is optimal one if and only if $V(\mathbf{G})$ can be covered by disjoint edges and odd cycles.

Theorem 5.44 (Berge [28]). *The fractional cover $t(x) \equiv 1/2$ is the unique optimal if and only if* $\mathbf{G}$ *is regularisable and has no bipartite component.*

Proof. Easily follows from Corollary 5.20, and 5.1. $\square$

J. Csima [75] found a characterization of regularisable hypergraphs. His result was used by Berge [28] to investigate quasi-regularisable hypergraphs.

Problem 5.45. Find further properties of the regularisable r-graphs.

6. Further Bounds on τ and τ^*

r-Graphs with maximal τ^*. Several examples (e.g., Example 4.0) show that the trivial inequality $\tau \leq r\nu$ cannot be improved in general. Nevertheless, as Lovász [177] observed the inequality $\tau^* \leq r\nu$ is not sharp. He showed that $\tau^* < r\nu$ holds for any hypergraph, furthermore

$$\tau^*(r, \nu) = \sup\{\tau^*(\mathbf{H}): r(\mathbf{H}) \leq r, \nu(\mathbf{H}) \leq \nu\} < r\nu.$$

Indeed, either $\tau(\mathbf{H}) \leq r\nu - 1$ or $\tau(\mathbf{H}) = r\nu$. In the second case there exist only finitely many $\mathbf{H}$ for every fixed r and ν by Corollary 4.26. For this case we can make a proof similar to Proposition 5.42. For $\nu = 1$ he proved $\tau^*(r, 1) \leq r - 1 + 2/(r + 1)$ and conjectured $\tau^*(r, 1) \leq r - 1 + 1/r$. This conjecture was proved in the following stronger form:

Theorem 6.1 ([130]). *Let* $\mathbf{H}$ *be a hypergraph of rank* r, $r \geq 3$, $\nu(\mathbf{H}) = \nu$. *Suppose that* $\mathbf{H}$ *does not contain as a partial hypergraph* $(p + 1)$ *vertex disjoint copies of finite projective planes of order* $r - 1$. *Then*

$$\tau^*(\mathbf{H}) \leq (r - 1)\nu + p/r. \tag{6.1}$$

If such a finite plane exists then Theorem 6.1 is sharp, consider p copies of a finite projective plane and $(\nu - p)$ copies of the truncated plane (see Example 5.3). Then for the obtained hypergraph equality holds in (6.1). Another extremal family can be obtained from the *tuisted* projective planes. See [266] and [135].

Corollary 6.2. *If a* $PG(2, r - 1)$ *exists then* $\tau^*(r, \nu) = (r - 1 + 1/r)\nu$. *Otherwise* $\tau^*(r, \nu) \leq (r - 1)\nu$.

Moreover in the first case the only optimal hypergraph is the union of ν $PG(2, r - 1)$. For every other hypergraph $\mathbf{H}$ we have

$$\tau^*(\mathbf{H}) \leq \left(r - 1 + \frac{1}{r}\right)\nu - \frac{1}{r}. \tag{6.2}$$

Proof of 6.1. It is sufficient to give a suitable fractional cover t of $\mathbf{H}$. We shall give it by induction on ν, while r is fixed. The proof of the case $\nu = 1$ is similar to the case $\nu > 1$ so we do not separate it. For $E(\mathbf{H}) = \varnothing$ we put $\tau^*(\mathbf{H}) = 0$. We can also suppose that $\mathbf{H}$ is a τ^*-critical hypergraph without isolated vertices. Hence $|E(\mathbf{H})| \leq |V(\mathbf{H})|$ by Lemma 5.6. Consequently, for the minimum degree $d(x_0)$ we have

$$0 < \min_{x \in V(\mathbf{H})} d(x) \leq \frac{1}{|V(\mathbf{H})|} \sum_{x \in V(\mathbf{H})} d(x) \leq \frac{r|E(\mathbf{H})|}{|V(\mathbf{H})|} \leq r. \tag{6.3}$$

Case 1. There exists a point $x_0 \in V(\mathbf{H})$ with $0 < d_{\mathbf{H}}(x_0) = k < r$. Put $\mathscr{H}_0 = \{E \in E(\mathbf{H}): x_0 \in E\} = \{E_1, E_2, \ldots, E_k\}$, and $\mathscr{H}_i = \{E \in E(\mathbf{H}): E \cap E_i = \varnothing\}$ for $1 \le i \le k$. For the hypergraph $\mathscr{H}_i$ the induction hypothesis can be applied, so there exists a fractional cover t_i: $V(\mathbf{H}) \to \mathbb{R}$ of $\mathscr{H}_i$ such that $|t_i| \le (r-1)(v-1) + p/r$. (If here $\mathscr{H}_i = \varnothing$ then $t_i \equiv 0$). Let

$$t(x) = \begin{cases} 0 & \text{if } x = x_0 \\ \dfrac{1}{k}\left(d_0(x) + \sum_{i=1}^{k} t_i(x)\right) & \text{if } x \neq x_0, \end{cases}$$

where $d_0(x)$ is the degree of x in the hypergraph $\mathscr{H}_0$. It is easy to check that t is a fractional cover of $\mathbf{H}$ with $|t| = (r-1) + \dfrac{1}{k}\sum |t_i| \le (r-1)v + p/r$.

Case 2 (sketch). $\min_{x \in V(\mathbf{H})} d(x) \ge r$. Then by (6.3) $\mathbf{H}$ is r-regular, r-uniform, so $|V(\mathbf{H})| = |E(\mathbf{H})|$ and $\tau^*(\mathbf{H}) = \dfrac{1}{r}|E(\mathbf{H})|$ by Proposition 5.1. We have to show that $|E(\mathbf{H})| \le (r^2 - r)v + p$. Let E_1 be an arbitrary edge of $\mathbf{H}$, and put $\mathscr{H}_1 = \{E \in E(\mathbf{H}): E \cap E_i = \varnothing\}$. Applying the inductional hypothesis to $\mathscr{H}_1$ and using (5.3) we have

$$\begin{aligned} |E(\mathbf{H})| &= |\{E: E \cap E_i \neq \varnothing\}| + |\mathscr{H}_1| \le 1 + r(r-1) + \tau^*(\mathscr{H}_1)D(\mathscr{H}_1) \\ &\le (r^2 - r)v + p + 1. \end{aligned}$$

Here the right-hand side is at most $(r^2 - r)v + p$ if there is an edge E with $|E_1 \cap E| \ge 2$, or if there are only $p - 1$ disjoint $PG(2, r-1)$ in $\mathscr{H}_1$. So we can suppose that $\mathscr{H}_1$ contains p copies of $PG(2, r-1)$. These edges in $PG(2, r-1)$ are disjoint from the rest, because $\mathscr{H}$ is r-regular, i.e., $\mathscr{H}$ is not connected if $p \ge 1$. So we can suppose that $p = 0$, and

$$|E \cap E_1| = 0 \text{ or } 1 \quad \text{for all distinct edges } E, E_1 \in E(\mathbf{H}), \text{ and} \tag{6.4}$$

$$|E(\mathbf{H})| = |V(\mathbf{H})| = (r^2 - r)v + 1. \tag{6.5}$$

To finish the proof we need the following.

Proposition 6.3. *Suppose* $\mathbf{K}$ *is an r-regular r-graph,* $r \ge 3$, *satisfying* (6.4), (6.5) *and* $v(\mathbf{K}) = v$. *Then a finite projective plane of order* $r - 1$ *is a partial hypergraph of* $\mathbf{K}$.

For the proof of Proposition 6.3 we refer to [130]. □

Corollary 6.4 (Gyárfás [147]). *If* $\mathbf{H}$ *is r-partite then* $\tau^*(\mathbf{H}) \le (r-1)v$.

Proof. It follows from Theorem 6.1 because $PG(2, r-1)$ is not r-partite. □

This result gives (a slight) support to Conjecture 4.29. It is sharp, if $PG(2, r-1)$ exists, as it is shown by the example of v copies of the truncated projective plane (see Example 5.3). Using his result ($\tau^*(r, 1) \le r - 1 + 2/(r+1)$) mentioned above, Lovász [177] proved the following conjecture of Bollobás [49] and Erdös [96]:

$$\text{If } \mathbf{H} \text{ is an intersecting, regular } r\text{-graph, then } |V(\mathbf{H})| \le r^2 - r + 1. \tag{6.6}$$

Theorem 6.1 and Proposition 5.1 imply

Corollary 6.5. *If* **H** *is regular r-graph then* $|V(\mathbf{H})| \le \nu(r^2 - r + 1)$. *Moreover equality holds if and only if* **H** *is the disjoint union of* ν $PG(2, r-1)$. *Furthermore, if such a plane does not exists then* $|V(\mathbf{H})| \le \nu(r^2 - r)$. □

The $r = 2$ case was proved by Bollobás and Eldridge [54].

Proposition 6.6. $\tau^*(r+1, 1) > \tau^*(r, 1)$.

Proof. Suppose **H** is an intersecting r-graph with $\tau^*(\mathbf{H}) = \tau^*(r, 1)$. (Such an r-graph exists, in view of Proposition 6.8 below). Let $|F| = r + 1$, $F \cap V(\mathbf{H}) = \varnothing$ and define $\mathbf{H}^{r+1}$ as follows:

$$V(\mathbf{H}^{r+1}) = V(\mathbf{H}) \cup F,\ E(\mathbf{H}^{r+1}) = \{F\} \cup \{E \cup \{x\}: E \in E(\mathbf{H}), x \in F\}.$$

Then $\nu^*(\mathbf{H}^{r+1}) = 1 + \frac{r}{r+1}\tau^*(\mathbf{H}) \ge \tau^*(\mathbf{H}) + 1/r$. □

Conjecture 6.7.
(i) $\tau^*(r, \nu) = \nu\tau^*(r, 1)$.
(ii) $\tau^*(r+1, 1) - \tau^*(r, 1) = 1 - o(1)$
(iii) $\tau^*(r+1, \nu) > \tau^*(r, \nu)$
(iv) It would be interesting to determine $\tau^*(r, 1)$ for the values when no projective plane of order $r - 1$ exists.
For example $\tau^*(7, 1) = ?$.

Intersecting r-graphs with maximal τ^*. Let $r \ge t, s$ positive integers, $t \ge 2$ and define

$$\tau^*(r, t, s) = \sup\{\tau^*(\mathbf{H}): \mathbf{H} \text{ is } t\text{-wise } s\text{-intersecting of rank } r\}.$$

So with this notation $\tau^*(r, 2, 1) = \tau^*(r, 1)$ investigated in the previous section. To determine $\tau^*(r, t, s)$ it is enough to consider τ^*-critical hypergraphs **H**. Then by Lemma 5.9 $|E(\mathbf{H})| \le r\tau^*(r, t, s) \le r^2$, hence we have finitely many possibilities for every fixed r. So we have

Proposition 6.8. *There exists a t-wise s-intersecting* **H** *of rank r with* $\tau^*(\mathbf{H}) = \tau^*(r, t, s)$. □

(Of course, the same holds for $\tau^*(r, v)$ considered in the previous section, as well).

Theorem 6.9 (Frankl and Füredi [121]). *Let* **H** *be a (2-wise) s-intersecting family of rank r. Then either*
(i) $\tau^*(\mathbf{H}) = (r-1)/s + (1/r)$ *and then* **H** *is a symmetric* (r, s)*-design, or*
(ii) $\tau^*(\mathbf{H}) \le (r-1)/s + (1/r) - (r-s)/r(r-1)s$.

Proof. We can suppose that $\mathscr{F} \subset E(\mathbf{H})$ is τ^*-critical, and $\tau^*(\mathscr{F}) = \tau^*(\mathbf{H})$. Then, similarly as we have done in (6.3), we obtain

$$\frac{1}{|V(\mathscr{F})|} \sum_{p \in V(\mathscr{F})} deg_{\mathscr{F}}(p) \le r\frac{|\mathscr{F}|}{|V(\mathscr{F})|} \le r. \tag{6.7}$$

Hence $\min deg_{\mathscr{F}}(p) = d \le r$. Let ν be a vertex with minimum degree $d (\ge 1)$ and $F_1, \ldots, F_d$ be the edges of $\mathscr{F}$ through ν. Let $w: \mathscr{F} \to \mathbb{R}$ be a fractional matching and let $q =: \sum_{1 \le i \le d} w(F_i)$, $q \le 1$. Then for every fixed i, $1 \le i \le d$ we have

$$q + r - 1 \geq \sum_{x \in F_i} \left(\sum_{x \in F \in \mathscr{F}} w(F) \right) = \sum_{F \in \mathscr{F}} w(F)|F \cap F_i|$$
$$\geq s \sum_{F \in \mathscr{F}} w(F) + (r - s)w(F_i). \tag{6.8}$$

Summing up (6.8) for all i we get

$$dq + d(r-1) \geq sd|w| + (r-s)q,$$

which yields

$$\frac{r-1}{s} + \frac{d-r+s}{sd} q \geq |w|. \tag{6.9}$$

The second term of the left hand side is at most $(1/r) - (r-s)/r(r-1)s$ if $d < r$, verifying the case (ii).

If $d = r$, then (6.7) implies that $\mathscr{F}$ is an r-regular r-graph. Then by Proposition 5.1 we have

$$\tau^*(\mathscr{F}) = \frac{|\mathscr{F}|}{r} = \frac{|V(\mathscr{F})|}{r}. \tag{6.10}$$

Consider an arbitrary edge $F \in \mathscr{F}_0$. Then

$$r^2 = \sum_{p \in F_0} deg_{\mathscr{F}}(p) = \sum_{F \in \mathscr{F}} |F \cap F_0| \geq r - s + s|\mathscr{F}|, \tag{6.11}$$

i.e., $|\mathscr{F}| \leq (r^2 - r + s)/s$. If $|\mathscr{F}| \leq (r^2 - r + s - 1)/s$, then (ii) holds again by (6.10). If $|\mathscr{F}| = (r^2 - r + s)/s$ then (6.11) implies that $|F \cap F_0| = s$ holds for every two F, $F_0 \in \mathscr{F}$, i.e., $\mathscr{F}$ is an (r, s)-design. Finally, by Corollary 3.5, we have that $\mathscr{F} = E(\mathbf{H})$. □

Theorem 6.9 has the following Corollary, analogous to Corollary 6.5.

Corollary 6.10. *If* **H** *is a regular, s-intersecting hypergraph of rank r then* $|V(\mathbf{H})| \leq (r^2 - r + s)/s$. *Here equality holds if and only if* **H** *is a symmetric* (r, s)*-design.*

The case $s = 1$ was conjectured by Bollobás [49] and Erdös [96]. In the case $s = 1$ the first part of Corollary 6.10 was proved by Lovász [177] using his above mentioned upper bound on $\tau^*(r, 1)$ $(\leq r - 1 + 2/(r + 1))$. Recently, Calderbank [68] found a new short proof using associations schemes.

Conjecture 6.11. If **H** is an s-intersecting r-graph, and **H** is not a symmetric (r, s)-design, then $\tau^*(\mathbf{H}) \leq (r-1)/s$.

Denote by $q^{(i)}$ the sum $q^i + q^{i-1} + \cdots + q + 1$, so $q^{(0)} = 1$. A consequence of Theorem 6.9 is the following.

Corollary 6.12. *Suppose* **H** *is a t-wise* $q^{(l)}$*-intersecting family of rank* $q^{(t+l-1)}$. *Then either*

(a) **H** *is isomorphic to* $PG(t + l, q)$ *and then* $\tau^*(\mathbf{H}) = q^{(t+l)}/q^{(t+l-1)}$, *or*
(b) $\tau^*(\mathbf{H}) < q^{(t+l)}/q^{(t+l-1)} - 1/q^{2t+2l}$.

Conjecture 6.13. Suppose **H** is a t-wise intersecting family of rank $q^{(t-1)}$, and $\mathbf{H} \neq PG(t, q)$. Then $\tau^*(\mathbf{H}) \leq q$.

Proof of 6.12. We are going to use Theorem 6.9. If $\mathbf{H}$ is $q^{(t+l-2)}$-intersecting then we are done. Suppose that there exist $E_1, E_2 \in E(\mathbf{H})$ with $|E_1 \cap E_2| < q^{(t+l-2)}$. Let $\mathbf{a}$ be the maximal integer such that one can find $H_1, H_2, \ldots, H_a \in E(\mathbf{H})$ with $|H_1 \cap \cdots \cap H_a| < q^{(t+l-a)}$. We have $2 \le a < t$ as $\mathbf{H}$ is t-wise $q^{(l)}$-intersecting. Hence every $H \in E(\mathbf{H})$ intersects $Y =: H_1 \cap \cdots \cap H_a$ in at least $q^{(t+l-a-1)}$ elements. Thus the function $t: Y \to \mathbb{R}$, $t(y) \equiv 1/q^{(t+l-a-1)}$ is a fractional cover of $\mathbf{H}$. This means

$$\tau^*(\mathbf{H}) \le |Y|/q^{(t+l-a-1)} \le (q^{t+l-a} + \cdots + q^2 + q)/(q^{t+l-a-1} + \cdots + 1)$$
$$= q < q^{(t+l)}/q^{(t+l-1)} - 1/q^{2t+2l}.$$

□

Proposition 6.14. (i) *Suppose* $2 \le t \le r \le 3t/2 - 1$. *Then*

$$\tau^*(r, t, 1) = 1 + 2/(3t - r).$$

(ii) *Suppose* $r = (3t - 1)/2$. Then $\tau^*(r, t, 1) = 1 + 2/r$.

One of the extremal hypergraph in 6.14(ii) is the complement of $(t + 1)/2$ disjoint union of $\mathbf{K}_2^3$. For the proof we efer to [133], it is based on Theorem 4.20. Using the following proposition we can determine $\tau^*(r, t, s)$ whenever $s > r - \sqrt{r(t-1)}$.

Proposition 6.15. *Suppose* $\mathbf{H}$ *is a t-wise s-intersecting family of rank* $lt + s - l$ *where* $t \ge 2, l \ge 1$. *Suppose* $s > (t-1)l(l-1)$, *then either*
(a) $\mathbf{H}$ *is* $(lt + s - 2l)$*-intersecting* (e.g., $\mathbf{H} = K_{lt+s-l}^{lt+s}$), *or*
(b) $\tau^*(\mathbf{H}) \le (s + l - 1)/s \; (< (lt + s)/(lt + s - l))$.

Proof. (The case $t = 2$ was proved in [114], in general see in [121]). We proceed as in the proof of Corollary 6.12. Suppose on the contrary that there exist H_1, $H_2 \in E(\mathbf{H})$, $|H_1 \cap H_2| < lt + s - 2l$. Let k be the maximal such that there exist $H_1, \ldots, H_k \in E(\mathbf{H})$ with $|H_1 \cap \cdots \cap H_k| < lt + s - kl$, we have $2 \le k < l$. Then for each $H \in E(\mathbf{H})$ we have $|H \cap (H_1 \cap \cdots \cap H_k)| \ge lt + s - (k+1)l$, yielding $\tau^*(\mathbf{H}) \le (lt + s - kl - 1)/(lt + s - kl - k) \le (s + l - 1)/s$. □

The ratio of the matching polytope and the fractional matching polytope. Füredi, Kahn and Seymour [137] found the following sharpening of results Theorems 6.1 and 6.2.

Theorem 6.16 ([137]). *Let* $\mathbf{H}$ *be an r-graph. Blow up the matching polytope of* $\mathbf{H}$ *from the origin with ratio* $(r - 1 + 1/r)$. *Then we have*

$$FMP(\mathbf{H}) \subseteq \left(r - 1 + \frac{1}{r}\right) MP(\mathbf{H}).$$

Conjecture 6.17. Applying the following linear operation for $MP(\mathbf{H})$. For $\{x(E)\}_{E \in E(\mathbf{H})}$ let $L(\mathbf{x}) = \left\{\left(|E| - 1 + \frac{1}{|E|}\right)x(E): E \in E(\mathbf{H})\right\}$, and $L(MP(\mathbf{H})) = \{L(\mathbf{x}): \mathbf{x} \in MP(\mathbf{H})\}$. Then

$$FMP(\mathbf{H}) \subseteq L(MP(\mathbf{H})).$$

Another generalization of Theorem 6.2 is the following

Theorem 6.18 ([137]). *Let* **H** *be an arbitrary hypergraph. Then there exists a matching*
$\mathscr{M} \subset E(\mathbf{H})$ *with*

$$\tau^*(\mathbf{H}) \leq \sum_{E \in \mathscr{M}} |E| - 1 + \frac{1}{|E|}.$$

Proof. We will use induction on $|E(\mathbf{H})|$. We can suppose that **H** is τ^*-critical, i.e.,

$$|E(\mathbf{H})| \leq |\bigcup \{E: E \in E(\mathbf{H})\}|.$$

by Lemma 5.6. Let $w: E(\mathbf{H}) \to \mathbb{R}$ be an optimal fractional matching. Then 5.11(iii) implies the following sharper version of Proposition 5.6

$$|E(\mathbf{H})| \leq \sum_{E} w(E)|E|. \tag{6.12}$$

This implies that

$$\frac{\sum w(E)|E|}{|E(\mathbf{H})|} \geq 1,$$

so there exists an $E_0 \in E(\mathbf{H})$ with $w(E_0)|E_0| \geq 1$. Let $\mathbf{H}_0$ be the hypergraph consisting of the edges of **H** disjoint to E_0. Then, by induction, there exists an $\mathscr{M}_0$ matching with

$$\tau^*(\mathbf{H}_0) \leq \sum_{E \in \mathscr{M}_0} |E| - 1 + \frac{1}{|E|}.$$

Moreover

$$\tau^*(\mathbf{H}) - \tau^*(\mathbf{H}_0) \leq \sum_{E \cap E_0 \neq \varnothing} w(E) \leq |E_0| - (|E_0| - 1)w(E_0) \leq |E_0| - 1 + \frac{1}{|E_0|},$$

so $\mathscr{M} = \mathscr{M}_0 \cup \{E_0\}$ satisfies Theorem 6.18. □

r-Partite hypergraphs. Conjecture 4.29 would imply $\tau \leq (r-1)\tau^*$. Much more is true.

Theorem 6.19 (Lovász [175]). *If* **H** *is an r-partite hypergraph, then*

$$\tau(\mathbf{H}) \leq \frac{r}{2}\tau^*(\mathbf{H}).$$

For the proof we need the following slightly stronger statement:

Theorem 6.20 ([175]). *If* **H** *is an r-partite hypergraph then every k-cover decomposable into* $\lceil 2(k-1)/r \rceil + 1$ *covers.*

Proof of Theorem 6.20. Let $t(x)$: $V(\mathbf{H}) \to \{0, 1, 2, \ldots, k\}$ be a k-cover $V(\mathbf{H}) = V_1 \cup \cdots \cup V_r$ is the r-partition. It is easy to see that for every $r \geq 2, m \geq 0$ there exists an $r \times (m+1)$ matrix $((a_{ij}))$ such that

(a) every row is a permutation of $\{0, 1, 2, \ldots, m\}$,
(b) the sum of every column is at most $\lceil rm/2 \rceil$.

Let $m = [2(k-1)/r]$ and define T_j as follows $(1 \le j \le m+1)$:

$$T_j = \{x \in V(\mathbf{H}): t(x) > a_{ij}, x \in V_i (1 \le i \le r)\}.$$

Then every T_j is a cover. Indeed, suppose on the contrary that $E_0 = \{e_1, \dots, e_r\} \in E(\mathbf{H})$ with $e_i \in V_i$ and $t(e_i) \le a_{ij}$. Then

$$\sum_{x \in E_0} t(x) \le \sum_i a_{ij} \le \left\lceil \frac{rm}{2} \right\rceil = \left\lceil \frac{r[2(k-1)/r]}{2} \right\rceil \le k-1,$$

a contradiction. □

Proof of Theorem 6.19. By Proposition 5.12 (iv) there exists infinitely many k with $\tau_k = k\tau^*$. Using Theorem 6.20 we have

$$k\tau^* = \tau_k \ge \left(1 + \left\lfloor \frac{2(k-1)}{r} \right\rfloor\right)\tau \ge \frac{2k-1}{r}\tau.$$

If $k \to \infty$ we obtain Theorem 6.19. □

Of course, the case $r = 2$ in Theorem 6.18 follows from the König theorem as well.

Problem 6.21. Can we improve on Theorem 6.19?

The speed of convergence of the sequence τ_k/k. For every hypergraph $\mathbf{H}$ the sequence $\{\tau_k/k\tau^*\}_{k=1,2\dots}$ tends to 1 and infinitely many times equals to 1 (by Proposition 5.12). For $r = 2$ (i.e., in the case of graphs) it reaches the value 1 already for $k = 2$ (and for every even k), by Theorem 5.17. Theorem 5.23 shows that there is no upper bound on k for 3-graphs for the smallest value where $\tau_k = k\tau^*$.

Proposition 6.22 (Lovász [177]). *Let* $\mathbf{H}$ *be a hypergraph of rank r, then*
(a) $\tau_2/2\tau^* \le r/2$,
(b) $\tau_r/r\tau^* \le 2 - (1/(r-1))$,
(c) $\tau_k/k\tau^* \le 1 + (r-1)/k$.

Proof. (a) is an easy consequence of 5.11 (iii). Let $t\colon V(\mathbf{H}) \to \mathbb{R}$ be an optimal fractional cover and let $V_0 = \{x: 0 < t(x)\}$, $V_1 = \{x: t(x) = 1\}$. Then

$$t_2(x) = \begin{cases} 2 & \text{for } x \in V_1 \\ 1 & \text{for } x \in V_0 - V_1 \\ 0 & \text{otherwise} \end{cases}$$

is a 2-cover with value at most $r\tau^*$.

(b) is a special case of (c).

For the proof of (c) define $t_k(x)$ as follows

$$t_k(x) = [kt(x)]$$

Clearly, t_k is a k-cover, and

$$\sum_x t_k(x) = k\tau^* + \sum_{x \in V_0} \{kt(x)\}^* \tag{6.13}$$

Let $\mathbf{H}_0$ be the partial subhypergraph of $\mathbf{H}$ whose edges has value exactly 1, i.e.,

$E(\mathbf{H}_0) = \left\{E \in E(\mathbf{H}): \sum_{x \in E} t(x) = 1\right\}$. Then $\tau^*(\mathbf{H}_0) = \tau^*(\mathbf{H})$. Moreover

$$\sum_{x \in E_0} \{kt(x)\}^* \leq r - 1 \tag{6.14}$$

holds for all $E_0 \in E(\mathbf{H}_0)$. Let w be an optimal fractional cover of $\mathbf{H}_0$ then (6.14) implies that

$$\sum_{x \in E_0} \{kt(x)\}^* w(E_0) \leq (r - 1)w(E_0),$$

hence

$$(r - 1)\tau^*(\mathbf{H}_0) \geq \sum_{E_0 \in E(\mathbf{H})} \sum_{x \in E_0} \{kt(x)\}^* w(E_0) \geq \sum \{kt(x)\}^*.$$

This and (6.13) imply (c). □

Problem 6.23. Improve on Proposition 6.22. What can we say about r-partite r-graphs?

R. Aharoni [252] proved further relations between ν and τ for r-partite hypergraphs. *A lower bound for the matching number.* (Aharoni, Erdös and Linial [9]). Let $\mathbf{H}$ be a hypergraph, denote $|V(\mathbf{H})|$ and $|E(\mathbf{H})|$ by V and e, $m := \min\{|E|: E \in E(\mathbf{H})\}$.

Theorem 6.24 ([AhErLi]). *With the above notations*

$$\nu \geq \frac{(\tau^*)^2}{V - \dfrac{m-1}{e}(\tau^*)^2} \geq \frac{(\tau^*)^2}{V}.$$

The case of $\nu = 1$, i.e., the inequality

$$\sqrt{\frac{Ve}{e - m + 1}} \geq \tau^*$$

for intersecting hypergraphs was proved by Pach and Surányi [207]. For the proof of 6.24 we need the following result of Motzkin and Straus who using that idea gave a new proof for Turán theorem. Let $\mathbf{G} = (V, E)$ be a graph $|V| = n$, define the polynomial

$$g(x_1, \ldots, x_n) = \sum_{\{i,j\} \in E} x_i x_j.$$

We are looking for the maximum of g where all $x_i \geq 0$ and $\sum x_i = 1$. Denote by $\omega(\mathbf{G})$ the largest clique in $\mathbf{G}$, i.e., the largest complete subgraph. Setting $1/\omega$ weights to the vertices of a largest clique and 0 elsewhere we get $\max g \geq \frac{1}{2}\left(1 - \frac{1}{\omega}\right)$.

Theorem 6.25 ([201]). *For every graph* $\mathbf{G}$, *if* $x_i \geq 0$, *then*

$$g(x_1, \ldots, x_n) \leq \frac{1}{2}\left(1 - \frac{1}{\omega(\mathbf{G})}\right)\left(\sum x_i\right)^2.$$ □

Proof of 6.24. Let $w: E(\mathbf{H}) \to \mathbb{R}$ be an optimal fractional matching. Consider the V by e 0–1 matrix A where $A_{ij} = 1$ iff $i \in E_j$ (i.e., the incidency matrix of $\mathbf{H}$). Then

$$\mathbf{w}^T A^T A \mathbf{w} = (A\mathbf{w})^T (A\mathbf{w}) \le 1_v^T 1_v = V \tag{6.15}$$

The matrix $A^T A$ is an $e \times e$ matrix indexed the edges E_i $(1 \le i \le e)$ with the (i,j) entry being $|E_i \cap E_j|$. To make a lower bound we can estimate it as follows:

$$A^T A \ge diag(|E_i| - 1) + J - B$$

where J is the $e \times e$ matrix with all entries equal one, and B is a 0–1 matrix having 1 at (i,j) iff E_i and E_j are disjoint. So (6.15) implies

$$V \ge \mathbf{w}^T A^T A \mathbf{w} \ge \mathbf{w}^T diag(|E_i| - 1)\mathbf{w} + \mathbf{w}^T J \mathbf{w} - \mathbf{w}^T B \mathbf{w}. \tag{6.16}$$

We give a separate lower bound for each of the three terms.

$$\begin{aligned} \mathbf{w}^T diag(|E_i| - 1)\mathbf{w} &= \sum_{i=1}^{e} (|E_i| - 1)(w(E_i))^2 \ge (m-1) \sum_{E \in E(H)} w(E)^2 \\ &\ge (m-1)\frac{(\sum w(E))^2}{e} = \frac{m-1}{e}(\tau^*)^2. \end{aligned} \tag{6.17}$$

$$\mathbf{w}^T J \mathbf{w} = (\textstyle\sum w(E))^2 = (\tau^*)^2. \tag{6.18}$$

The matrix B can be considered as an adjacency matrix of a graph $\mathbf{G}$ whose largest clique has v vertices, so Theorem 6.25 yields

$$-\mathbf{w}^T B \mathbf{w} \ge -\left(1 - \frac{1}{v}\right)(\textstyle\sum w(E))^2 \tag{6.19}$$

Summing up (6.17)–(6.19) we conclude Theorem 6.24. □

Our next aim is to find the hypergraphs $\mathbf{H}$ for which equality holds in (the first inequality of) Theorem 6.24. All $w(E)$ $(E \in E(\mathbf{H}))$ must equal by (6.16), so their common value is $1/D$ (where D denotes the maximum degree of $\mathbf{H}$). Again (6.16) implies that $\mathbf{H}$ is an m-graph. Then (6.15) implies that every point is saturated by $\mathbf{w}$, so $\mathbf{H}$ is D-regular. We get

$$V = \frac{me}{D}, \qquad \tau^* = \frac{e}{D}.$$

Substituting these values to Theorem 6.24 we get

$$V = \frac{e}{mD - m + 1} \tag{6.20}$$

Consider the *linegraph* $L(\mathbf{H})$ of $\mathbf{H}$, i.e., $V(L(\mathbf{H})) = E(\mathbf{H})$ and two vertices of $L(\mathbf{H})$ are connected if the corresponding edges in $E(\mathbf{H})$ are intersecting. $D(L(\mathbf{H})) \le m(D-1)$ and $L(\mathbf{H})$ does not contain $v+1$ independent vertices. These two properties (and the Turán-theorem) imply that $L(\mathbf{H})$ is the union of v complete graphs $\mathbf{K}_2^{1+m(D-1)}$. So $\mathbf{H}$ can be decomposed into v disjoint D-regular, intersecting m-graphs. Moreover every edge intersects exactly $m(D-1)$ others, so $|E \cap E'| \le 1$ for all $E, E' \in E(\mathbf{H})$. Then easily follows that $D \le m$ and

Proposition 6.26. *Equality holds in Theorem 6.24 if and only if* $\mathbf{H}$ *is the disjoint union of hypergraphs* $\mathbf{H}_1, \dots, \mathbf{H}_\nu$ *where each* $\mathbf{H}_i$ *is the dual of a* $S(Dm - m + 1, D, 2)$ *Steiner system.*

E.g., $\mathbf{H} = \nu$ disjoint m-sets or, $H = \nu$ copies of $PG(2, m-1)$ or the truncated projective planes of order $m - 1$, etc.

Proposition 6.27 ([9]). *Let* $\mathbf{H}$ *be a hypergraph on* V *vertices with* b *edges. Then*

$$\tau \le \begin{cases} \min\left\{V, \sqrt[3]{\nu V \log \frac{b}{\sqrt{\nu V}}}\right\} & \text{if } b > e\sqrt{\nu V}, \\ b & \text{if } b \le e\sqrt{\nu V}, \end{cases}$$

where $e = 2.71828\dots$ *as usual.*

The bounds in Proposition 6.27 are rather tight as they showed by random constructions. Moreover they gave simple algorithms to find as large matching or cover as it was given in Theorem 6.25 and Proposition 6.27, which algorithm have polynomial running time. Their algorithm simply a greedy one, what we deal with in the next section.

Covering by greedy algorithm. One of the most natural methods to produce a small cover of a given hypergraph $\mathbf{H}$ is the so-called "Greedy Algorithm". We proceed as follows: Let x_1 be a point of maximum degree. Suppose that $x_1, \dots, x_i$ are already selected and they still do not cover all edges, then let x_{i+1} be a point which covers as many as possible from the uncovered edges. The process stops when all edges are covered.

Generally, the Greedy Algorithm is not the best, but we can hope that it gives a rather good estimate. The following theorem was proved by Lovász [176]. A slightly different result was given by Stein [229].

Theorem 6.28 ([176, 180]). *Let* $\mathbf{H}$ *be a hypergraph and denote its maximum degree by* D. *Let* $\{x_1, x_2, \dots, x_T\}$ *be a cover obtained by the greedy covering algorithm. Then*

$$T \le \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{D}\right) \max_{\mathscr{H} \subset E(\mathbf{H})} \frac{|\mathscr{H}|}{D(\mathscr{H})}.$$

Corollary 6.29. $\tau \le \left(1 + \frac{1}{2} + \cdots + \frac{1}{D}\right)\tau^* < (1 + \log D)\tau^*.$

Using the same method Stein proved

$$\tau \le \frac{|E(\mathscr{H})|}{D} + \frac{|V(\mathscr{H})|}{\min|E|}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{D}\right). \tag{6.21}$$

Proof of Theorem 6.28. Let $M = \max E(\mathbf{H}')/D(\mathbf{H}')$, where $\mathbf{H}'$ is an arbitrary partial hypergraph of $\mathbf{H}$. Let t_i denote the number of steps in the greedy algorithm in which the chosen point covers i new edges. After $t_D + t_{D-1} + \cdots + t_{s+1}$ steps the hypergraph $\mathbf{H}_s$ formed by the uncovered edges has maximum degree $\le s$ and cardinality

$|E(\mathbf{H}_s)| = \sum_{i \le s} it_i$. Hence we have for all $1 \le s \le D$

$$\sum_{i=1}^{s} it_i \le Ms. \tag{6.22}$$

Multiply these inequalities by $1/s(s-1)$ for $2 \le s \le D$ and by $1/D$ for $s = 1$ we get

$$T = \sum_{i=1}^{D} t_i \le M \sum_{i=1}^{D} 1/i. \qquad \square$$

Proof of Corollary 6.29. Clearly $\tau \le T$ and $|E(\mathbf{H}')|/D(\mathbf{H}') \le \tau^*(\mathbf{H}')$ by (5.4). The obvious inequality $\tau^*(\mathbf{H}') \le \tau^*(\mathbf{H})$ completes the proof. $\square$

Remark 6.30. Let $\mathbf{H}^i$ the hypergraph consisting of the uncovered edges after i steps. Then

$$|E(\mathbf{H}^i)| = |E(\mathbf{H}^{i-1})| - D(\mathbf{H}^{i-1}) \le \left(1 - \frac{1}{M}\right)|E(\mathbf{H}^{i-1})| \le \left(1 - \frac{1}{M}\right)^i |E(\mathbf{H})|.$$

Hence

$$|E(H^{[M \log D]})| \le \left(1 - \frac{1}{M}\right)^{M \log D} |E(\mathscr{H})| < \frac{|E(\mathscr{H})|}{D} \le \tau^*,$$

so in this way we again obtain that

$$T \le [\tau^* \log D] + [\tau^*]. \tag{6.23}$$

Another proof can be obtained for (6.23) by probabilistic method. Let $t(x)$: $V(\mathbf{H}) \to \mathbb{R}$ be an optimal fractional cover, and in the i-th round of our process choose the point $x \in V(\mathbf{H})$ into our cover T with probability $t(x)$. Then after r rounds for all $E \in E(\mathbf{H})$ we have

$$Prob(E \text{ is not covered by } T) = \left(\prod_{x \in E} (1 - t(x))\right)^s < \frac{1}{e^s},$$

and the expected value of $|T|$ is $E(T) = s\tau^*$. So there exists a cover T' of size at most $s\tau^* + |E(\mathbf{H})|e^{-s}$.

Actually, Lovász [176] proved a slightly more, namely that for any cover obtained by the greedy algorithm the following holds:

$$T \le \frac{\tilde{v}_1}{1 \cdot 2} + \frac{\tilde{v}_2}{2 \cdot 3} + \cdots + \frac{\tilde{v}_{D-1}}{(D-1)D} + \frac{\tilde{v}_D}{D},$$

where $\tilde{v}_i = \max\{|\mathscr{M}|: \mathscr{M} \subset E(\mathbf{H}), deg_{\mathscr{M}}(x) \le i \text{ for all } x\}$, the maximum size of a simple i-matching in $\mathbf{H}$.

Problem 6.31. Is it true that one can improve Corollary 6.29 by a constant factor? What about if we restrict our attention to a smaller class of hypergraphs? (E.g., r-graphs, or τ^* is bounded).

For the complete r-graph $\mathbf{K}_r^{2r}$ we have $\tau/\tau^* = (r+1)/2$ and $\log D = \log \binom{2r-1}{r-1} \sim r \log 4$. (See [262]).

Minimal complementing subsets in a finite group. The following theorem is a slight generalization of a theorem of Lorentz [173]. This version (in almost this form) is due to Halberstam and Roth ([153], pp. 14–16).

Corollary 6.32. *Let G be a finite group and $A \subset G$ be a subset of G. Then there exists a set $B \subset G$ such that $AB = G$ and*

$$|B| \le (1 + \log|A|)\frac{|G|}{|A|}.$$

Proof (Lovász [176], Stein [229]). Construct a hypergraph $\mathbf{H}$ as follows, $V(\mathbf{H}) = G$, $E(\mathbf{H}) := \{A^{-1}g\colon g \in G\}$. Let B be a cover of $\mathbf{H}$, then $AB = G$. Moreover $\mathbf{H}$ is $|A|$-regular so by Proposition 5.1 (or 5.2) we have $\tau^* = |G|/|A|$, and $D(\mathbf{H}) = |A|$. Then Corollary 6.29 completes the proof. □

If there are more choices, then one can prove more:

Theorem 6.33 (Finkelstein, Kleitman, Leighton [259]). *For any finite group G and integer a, $1 \le a \le |G|$, one can find sets $A \subset G$, $B \subset G$ such that $AB = G$ and $|A| = a$, $|B| \le 2\dfrac{|G|}{|A|}$.*

Chromatic number of graphs. If $\alpha(\mathbf{G})$ denotes the maximum number of *independent points* in the graph $\mathbf{G}$, and $\chi(\mathbf{G})$ is its chromatic number, then clearly $\chi(\mathbf{G}) \ge |V(\mathbf{G})|/\alpha(\mathbf{G})$. Even more

$$\chi(\mathbf{G}) \ge \max_{\mathbf{H}} \frac{|V(\mathbf{H})|}{\alpha(\mathbf{H})}, \tag{6.24}$$

where $\mathbf{H}$ ranges over all induced subgraphs of $\mathbf{G}$. The following theorem shows that (6.24) is not so far from the true order of magnitude of $\chi(\mathbf{G})$.

Corollary 6.34 (Lovász [176]). *For any graph G*

$$\chi(\mathbf{G}) \le (1 + \log\alpha(\mathbf{G})) \max_{\mathbf{H}} \frac{|V(\mathbf{H})|}{\alpha(\mathbf{H})}.$$

An example of Erdös and Hajnal [103] (the shift graph) shows that the inequality in Corollary 6.34 is sharp (appart from a constant factor).

Proof of 6.34 (sketch). Apply Corollary 6.29 to the following hypergraph $\mathbf{H}$: $V(\mathbf{H})$ is the family of independent sets in $\mathbf{G}$, $E(\mathbf{H})$ has $V(\mathbf{G})$ members $\{E_p\colon p \in V(\mathbf{G})\}$ defined by $E_p = \{I \in V(\mathbf{H})\colon p \in I\}$. □

3-Chromatic hypergraphs. A hypergraph is said to be 2-*chromatic* if its vertices can be 2-colored in such a way that no edge is monochromatic. Let $m(r)$ denote the minimum cardinality of a non-2-chromatic r-graph. Erdös and Hajnal [102] proved that $m(r) \ge 2^{r-1}$ and more than 20 years later Beck [24] proved $m(r) \ge r^{(1/3)-\varepsilon}2^r$ (for a short account of this, see [227]). On the other hand using probabilistic method Erdös proved

Theorem 6.35 ([93]). *There is a non-2-chromatic r-graph with not more than*

$\left(\frac{e\log 2}{4}+o(1)\right)r^2 2^r$ *edges.*

Proof (sketch). (Lovász [176]). Given a set X of $2N$ points (where $N=[r^2/4]$ will be the best choice). Form a hypergraph $\mathbf{H}$ as follows. Let $V(\mathbf{H})$ consists of all r-tuples of X. For every partition $P=\{P_1,P_2\}$ of X we form an edge of $\mathbf{H}$ consisting of all r-subsets of X which entirely belong to P_1 or P_2. Now, if T is a cover of $\mathbf{H}$ then the r-tuples of X corresponding to elements of T form a non-2-chromatic r-graph, so $m(r)\le\tau(\mathbf{H})$. It is easy to see that $\tau^*(\mathbf{H})=\binom{2N}{r}\Big/2\binom{N}{r}$ and $D(\mathbf{H})=2^{2N-r}$. Now applying 6.29 to $\mathbf{H}$ we obtain

$$\tau(\mathbf{H})\le(1+(2N-r)\log 2)\frac{\binom{2N}{r}}{2\binom{N}{r}}=\left(\frac{e\log 2}{4}+o(1)\right)r^2 2^r. \qquad \square$$

7. Some Applications of Linear Programming for Extremal Problems

The following useful lemma is a generalization of (5.4).

Lemma 7.1. *Let $a\colon E(\mathbf{H})\to\mathbb{R}^+$ be a real-valued, non-negative function on the edge-set of the hypergraph $\mathbf{H}$. Then we have*

$$\max_{p\in V(\mathbf{H})}\left(\sum_{p\in H}a(H)\right)\ge\frac{1}{\tau^*}\left(\sum_{H\in E(\mathbf{H})}a(H)\right).$$

Proof. Let $M=\max_{p\in V(\mathbf{H})}\{\sum a(H)\colon p\in H\in E(\mathbf{H})\}$. Then the function $a/M\colon E(\mathbf{H})\to\mathbb{R}^+$ is a fractional matching of $\mathbf{H}$. Hence $(\sum a(H))/M\le\nu^*(\mathbf{H})$. $\square$

The largest (r,ν,D) multihypergraph. Abbott, Hanson, Katchalski and Liu investigated the following problem in a series of papers [1, 2, 4, 5]. Let r, ν, D be positive integers and denote $N=N(r,\nu,D)$ the largest integer for which there exists an r-uniform hypergraph with N (not necessarily distinct) edges and having no independent set of edges of size greater than ν (i.e., the matching number is at most ν) and no vertex of degree exceeding D. Such a hypergraph will be called an *(r,ν,D)-hypergraph.*

The problem of evaluating $N(r,\nu,D)$ for all values of the parameters seems to be very difficult. Nevertheless, they established a couple of upper and lower bounds and obtained exact values of $N(r,\nu,D)$ for various infinite classes of values of r, ν and D. They proved

Theorem 7.1 ([1]). $N(2,\nu,D)=\nu\left\lfloor\frac{3}{2}D\right\rfloor$.

Theorem 7.2 ([1]). *$N(r,\nu,D)\le\nu(r(D-1)+1)$, and here equality holds if and only if there exists an $S(\nu,D,2)$ Steiner-system over $\nu=r(D-1)+1$ vertices.*

Theorem 7.3.
(i) ([1]). $N(r, v, 2) = (r + 1)v$
(ii) ([2]).

$$N(r, v, 3) = \begin{cases} (2r + 1)v & \text{if } r \equiv 0, 1 \pmod 3 \\ 2rv & \text{if } r \equiv 2 \pmod 3. \end{cases}$$

Although the proof of Theorem 7.2 is easy, it gives the exact values for $N(r, v, D)$ for several large classes of parameters. (It is well-known, that if $S(v, D, 2)$ exists then $(v - 1)/(D - 1)$ and $\binom{v}{2}\Big/\binom{D}{2}$ are integers, and these two constraints are sufficient for $v > v_0(D)$. See Wilson [245].) All in these cases $r \geq D$. Here we concentrate to the case when D is large.

Example 7.4. Suppose that there exists a projective plane of order $r - 1$, $PG(2, r - 1)$. Let L_0 be a line and $A_0 \subset L_0$ a set of $D - r[D/r]$ elements. Let $\mathbf{H}$ be the multi-hypergraph obtained from $PG(2, r - 1)$ such that the multiplicity of a line L is

$$\begin{array}{ll} \lfloor D/r \rfloor & \text{if } L \cap A_0 = \varnothing \\ \lceil D/r \rceil & \text{if } L \cap A_0 \neq \varnothing, L \neq L_0 \\ D - (r - 1)[D/r] & \text{if } L = L_0. \end{array}$$

Then $\mathbf{H}$ is intersecting of rank r, maximum degree D and $E(\mathbf{H}) = rD - (r - 1)[D/r]$.

The above example is due to Bermond, Bond and Saclé [36]. If we take v disjoint copies of $\mathbf{H}$ then we get

$$v(rD - (r - 1)[D/r]) \leq N(r, v, D), \tag{7.1}$$

whenever a $PG(2, r - 1)$ exists. Recall the definition of $\tau^*(r, v) = \max\{\tau^*(\mathbf{G}): r(\mathbf{G}) \leq r, v(\mathbf{G}) \leq v\}$, see Theorem 6.1. Here we will prove (see [134]).

Theorem 7.5. *For any r, v and D we have*

$$\tau^*(r, v)D - r\tau^*(r, v) < N(r, v, D) \leq \tau^*(r, v)D.$$

Theorem 7.6. *If* $D \geq (r - 1)^2 v$ *and a* $PG(2, r - 1)$ *exists then* $N(r, v, D) = v(rD - (r - 1)[D/r])$ *holds.*

Theorem 7.5 and 6.1 imply that

$$\lim_{D \to \infty} N(r, v, D)/D = \tau^*(r, v) \leq v(r - 1 + (1/r)). \tag{7.2}$$

In [2] it was proved that

$$\lim_{D \to \infty} N(r, 1, D)/D \leq r - 1 + \max_n \frac{n(r^2 - r) - r^4 + 4r^3 - 6r^2 + 4r}{n^2 - n(2r + 1) + r^3 - 2r^2 + 3r}, \tag{7.3}$$

where the maximum is taken over all $n \geq r^2 - r + 1$. The right-hand-side of (7.3) is $(r-1)+0.25+0(1/r)$ always larger than the bound in (7.2). In the case $\nu=1$ Theorem 7.6 was conjectured by Bermond, Bond and Saclé [36] and in a slightly weaker form in [35]. In the case $\nu=1$ they proved that equality holds in (7.1) for $r\leq 4$ (for all D). Moreover they determined also $N(r,1,3)$ (see Theorem 7.3) and $N(r,1,4)$ for $r\not\equiv 3\ (mod\,4)$. This case was completed by Bermond and Bond [33]:

$$N(r,1,4)=\begin{cases}3r+1 & \text{if } r\equiv 0 \text{ or } 1\ (mod\,4),\\ 3r & \text{if } r\equiv 2 \text{ or } 3\ (mod\,4) \text{ but } r\neq 3,\\ 8 & \text{if } r=3.\end{cases}$$

Proof of Theorem 7.5. The upper bound is a trivial consequence of (5.4). To prove the lower bound consider a hypergraph $\mathbf{H}$ of rank at most r and matching number ν such that $\tau^*(\mathbf{H})=\tau^*(r,\nu)$. (Such a hypergraph exists by Lemma 5.9.) We may suppose that it is τ^*-critical, so $|E(\mathbf{H})|\leq r\tau^*(\mathbf{H})$ by Lemma 5.9. Let $w\colon E(\mathbf{H})\to\mathbb{R}^+$ be an optimal fractional matching. Then multiplying every edge E of $\mathbf{H}$ $[w(E)D]$ times we obtain a multihypergraph giving the lower bound. □

Note that we obtained (considering a rational w) that equality holds in Theorem 7.5 for infinitely many values of D for any given r and ν.

Proof of Theorem 7.6. The lower bound for $N(r,\nu,D)$ is given by Example 7.4. To prove the upper bound consider a multihypergraph $\mathbf{H}$ with rank r, $\nu(\mathbf{H})\leq\nu$, $D(\mathscr{H})\leq D$. The case $r=2$ is covered by Theorem 7.1 so we may suppose that $r\geq 3$. Then Theorem 6.1 implies that either $\tau^*(\mathbf{H})\leq\nu(r-1+1/r)-1/r$ or $\mathbf{H}$ is obtained from ν disjoint copies of $PG(2,r-1)$ by multiplication of the lines. In the first case $\mathbf{H}$ has at most $(\tau^*(r,\nu)-1/r)D$ edges (by (5.4)), which is less than $N(r,\nu,D)$ for $D>\nu(r-1)^2$. In the latter case, if a line L in a component of $\mathbf{H}$ has multiplicity at least $[D/r]$, then that component consists of at most

$$\lceil D/r\rceil+\sum_{x\in L}(deg_{\mathbf{H}}(x)-\lceil D/r\rceil)$$

edges. Otherwise, if each line has multiplicity at most $[D/r]$, clearly each component of $\mathbf{H}$ has only $[D/r](r^2-r+1)$ edges. □

Problem. Abbott, Hanson, Katchalski and Liu also investigated $N(r,\nu,D,s)=\max\{|E(\mathbf{H})|\colon \mathbf{H}$ is a multihypergraph of rank r, with matching number ν, maximum degree D, and for all $E_1,\ldots,E_{\nu+1}\in E(\mathbf{H})$ one has $1\leq i_1<\cdots<i_s\leq\nu+1$ such that $\bigcap E_{i_j}\neq\varnothing\}$. Of course, $N(r,\nu,D)=N(r,\nu,D,2)$. There are only a few results on this function.

The largest (r,ν,D) *hypergraph* (without multiple edges). Denote by $f(r,\nu,D)$ the maximum number of r-tuples contained in an r-graph $\mathbf{F}$ with $\nu(\mathbf{F})\leq\nu$ and $D(\mathbf{F})\leq D$. Now multiple edges are not allowed. The function $f(2,\nu,D)$ was investigated by several authors [3, 109, 222].

The determination of $f(2,\nu,D)$ was completed by Chvátal and Hanson [72]. In particular they proved that if $D>2\nu$, then $f(2,\nu,D)=\nu D$. Bollobás [48] conjectured that this result has the following extension:

Suppose r is such that there is a finite projective plane of order $r-2$, or $r=2,3$. If D is sufficiently large and divisible by $r-1$, then

$$f(r, v, D) = \frac{r^2 - 3r + 3}{r - 1} vD. \tag{7.4}$$

Furthermore, the extremal r-graphs can be obtained as follows. Take v pairwise disjoint projective planes (or triangles, or points if $r = 3$ or 2) each with $(r-2)^2 + (r-2) + 1 = r^2 - 3r + 3$ points and with $r - 1$ points on each line. For each line of each plane take $D/(r-1)$ r-tuples in such a way that each of these r-tuples intersects these projective planes exactly in this line.

Bollobás [49] proved his conjecture for $r = 3$ whenever $D > 72v^3$. In general (7.4) was proved in [120]. Here we give a more exact version.

Theorem 7.7 ([134]). *For any given r and v there exists a function $c(r, v)$ such that*

$$|\tau^*(r-1, v)D - f(r, v, D)| \le c(r, v) \tag{7.5}$$

holds. Moreover if D is sufficiently large compared to r and v $(D > c(r, v))$, and there exists a finte projective plane $PG(2, r-2)$ (or $r = 2, 3$), then

$$f(r, v, D) = N(r-1, v, D). \tag{7.6}$$

Here $N(r-1, v, D)$ is defined above, and by Theorem 7.6 it is $v((r-1)D - (r-2)\cdot [D/(r-1)])$ in the case of (7.6).

Proof (sketch). The lower bounds follow from the trivial inequality $f(r, v, D) \ge N(r-1, v, D)$. To prove the upper bounds consider an r-graph $\mathbf{F}$ with $D(\mathbf{F}) \le D$, $v(\mathbf{F}) \le v$. Consider its v-critical nucleus (see Examples 4.7, Theorem 4.8 and Conjecture 4.9) S, and apply Theorems 6.1, 7.5 and 7.6 to the multihypergraph $(\mathbf{F}|S) - \mathbf{F}$. □

Large hypergraphs with given degree and diameter. A *path* in a hypergraph $\mathbf{H}$ connecting x, $y \in V(\mathbf{H})$ is a sequence $x = x_1, E_1, x_2, E_2, \ldots, x_p, E_p, x_{p+1} = y$ with $\{x_i, x_{i+1}\} \subset E_i \in E(\mathbf{H})$ for $1 \le i \le p$. Its length is p, the number of its edges. The *distance* between x and y is the length of a shortest path between them. The *diameter* of $\mathbf{H}$ is the maximum of the distances over all pair of vertices. Therefore, a hypergraph is of diameter 1 if any pair of vertices belongs to at least one edge. Call a hypergraph of maximum degree at most D, diameter at most d and rank at most r, a $\{D, d, r\}$-hypergraph. Let $n(D, d, r)$ denote the maximum number of vertices of a $\{D, d, r\}$-hypergraph. Here we are going to determine $n(D, d, r)$, at least in some very special cases.

The case of graphs $(r = 2)$ is a very old problem, an extensive literature can be found in the survey papers [32, 35] or see Bollobás' book [50]. If we restrict our attention to the case diam $= 1$, then considering the dual hypergraph (i.e., the role of vertices and edges is interchanged but the incidences are saved) we get

$n(D, 1, r)$ is the maximum number of edges of an intersecting hypergraph of rank at most D and with maximum degree at most r. (7.7)

So we have

$$n(D, 1, r) = N(r, 1, D).$$

Note that without the restriction on the size of edges, but with a fixed number of vertices, this problem has been extensively studied. (See later our next section, Theorems 7.8 and 7.9).

Bermond, Delorme and Fahri have a couple of lower and upper bounds in the case of graphs ($r = 2$), and see [34] for the case $D = 2$. The following exact result is due to Kleitman et al. [167]:

$$n(2, 2, r) = \tfrac{5}{4}r^2, \quad \text{if } r \text{ is even.}$$

A nice generalization of this can be found in [255]. Although the investigation of $n(D, d, r)$ leads to interesting combinatorial structures, and this topic is very important in modelling interconnection networks (also see [69]) we hardly need linear programming, and fractional matchings. However, if we are looking for the smallest $\{D, d, r\}$-hypergraph over n vertices, then we have to use the machinery explained in the previous chapters. We will return these questions (see Theorem 7.19).

Covering of pairs by a small number of subsets. $C(n, k, 2)$ is used to denote the minimal number of k-sets required to cover all pairs from an n-set. For fixed k, and $n \to \infty$ Erdös and Hanani [105] proved that

$$\binom{n}{2}\Big/\binom{k}{2} \le C(n, k, 2) = (1 + o(1))\binom{n}{2}\Big/\binom{k}{2}. \tag{7.8}$$

This limit theorem obviously follows from the existence theorem of $S(n, k, 2)$ Steiner-systems for all $n > n_0(k)$ if $\binom{n}{2}\Big/\binom{k}{2}$ and $(n - 1)/(k - 1)$ are integers, due to R. Wilson [245]. But the lower bound is very poor if n is not much bigger than k. Mills [197] determined the solution of $C(n, k, 2) = t$ for all t up to and including 12. For $t = 13$ he [198] and Todorov [235] determine all (n, k) pairs with $C(2, k, n) = 13$ except the pairs $(28, 9)$ and $(41, 13)$ are undecided.

Suppose t is given and let $n(k, t) = \max\{n\colon C(n, k, 2) \le t\}$, i.e., $n(k, t)$ is the largest cardinality of a set whose pairs can be covered by t k-sets. To state our theorem define

$$\tau_i^*(t) = \max\{\tau^*(\mathbf{H})\colon \mathbf{H} \text{ is intersecting, } |V(\mathbf{H})| \le t\}.$$

Theorem 7.8 ([135]). *For all t and k we have*

$$\tau_i^*(t)k - t < n(k, t) \le \tau_i^*(t)k.$$

For any given t equality holds for infinitely many k. Mills [197] also proved that $\lim_{k\to\infty} n(k, t)/k$ exists and equals to its maximum. He has determined this limit for $t \le 13$. With our notations his result is the following:

t	1	2	3	4	5	6	7	8	9	10	11	12	13
$\tau_i^*(t)$	1	1	3/2	5/3	9/5	2	7/3	12/5	5/2	8/3	14/5	3	13/4

Some of his result ($t \le 6$) was rediscovered in [225]. Using fractional matchings we can determine $n(k, t)$ if k is large (compared to t) in the case $t = q^2 + q + 1$

whenever a $PG(2,q)$ exists. First a definition. Let $0 \le b < q+1$ be an integer and $\mathbf{P}$ a $PG(2,q)$. The pair of (multi-)hypergraphs $(\mathbf{H},\mathbf{L})$ over $V(\mathbf{P})$ is a *generalized b-cover* of $\mathbf{P}$ if the following hold

(i) $\mathbf{L}$ is a subset of lines of $\mathbf{P}$ (with multiplicities)
(ii) $\mathbf{P} \cup \mathbf{H}$ is intersecting.
(iii) $deg_{\mathbf{H}}(x) \le deg_{\mathbf{L}}(x) + b$ for all $x \in V(\mathbf{P})$,
(iv) $|E(\mathbf{H})| \ge |E(\mathbf{L})|$,
(v) $E(\mathbf{H}) \cap E(\mathbf{L}) = \varnothing$.

The *value* of the b-cover is $v_b(\mathbf{H},\mathbf{L}) = |E(\mathbf{H})| - |E(\mathbf{L})|$.

Finally $v_b(\mathbf{P}) = \max v_b(\mathbf{H},\mathbf{L})$, $v_b(q) = \max\{v_b(\mathbf{P})\colon \mathbf{P}$ is a $PG(2,q)\}$. Note that $E(\mathbf{H}) \cap E(\mathbf{P}) = \varnothing$ is not forbidden. So we can always have the following lower bound for $v_b(q)$ (if $b \ge 1$).

$$v_b(q) \ge bq - q + b. \tag{7.9}$$

The construction giving the lower bound in (7.9) is analogous to Example 7.4. Let L_0 be a line of $\mathbf{P}$, $A_0 \subset L_0$, $|A_0| = b$ and then $E(\mathbf{H}) =\colon \{L \in E(\mathbf{H}),\ L \cap A_0 \ne \varnothing,\ L \ne L_0\}$, and $\mathbf{L}$ consists of $q-b$ copies of L_0. It is easy to see that $v_0(q) = 0$.

Theorem 7.9 ([135]). *Suppose that a $PG(2,q)$ exists and $t = q^2+q+1$. Let $k = (q+1)a + b$ where $0 \le b \le q$. Then for $a \ge q^2+q$ we have*

$$n(k,t) = (q^2+q+1)a + v_b(q).$$

This is an improvement (for large k) on a result of Todorov [234].

Proof of Theorem 7.8. First consider the dual of the problem, then the proof is analogous to the proof of Theorem 7.5. □

Proof of Theorem 7.9. The main tool of our proof is the following

Lemma 7.10. *Let $\mathbf{H}$ be an intersecting hypergraph over q^2+q+1 elements. If $\mathbf{H}$ does not contain a $PG(2,q)$ as a partial hypergraph then $\tau^*(\mathbf{H}) \le q + 1/(q+2)$.*

Proof. We may assume that every edge in $\mathbf{H}$ has at least $q+1$ elements. Let $\mathbf{H}^{q+1}$ consist of the $(q+1)$-element edges of $\mathbf{H}$. Then $\tau^*(\mathbf{H}^{q+1}) \le q$ by Theorem 6.1. Put a weight $1/(q+2)$ into every point of $\mathbf{H}$. So we have covered all the large ($=$ at least $(q+2)$-element) edges, and $(q+1)/(q+2)$ part of the edges of $\mathbf{H}^{q+1}$. Hence we have

$$\tau^*(\mathbf{H}) \le (q^2+q+1)/(q+2) + \frac{1}{q+2}\tau^*(\mathbf{H}^{q+1}) \le \frac{(q+1)^2}{q+2}. \qquad \square$$

Returning to the proof of Theorem 7.9 we can proceed as in the proof of Theorem 7.6 replacing Theorem 6.1 by the above Lemma 7.10. We omit the details. □

Theorem 7.11. *If $b \ge \sqrt{q}$, then $v_b(q) = bq - q + b$.*

Proof. Let $(\mathbf{H},\mathbf{L})$ be a b-cover of $\mathbf{P}$. If $\mathbf{H}$ contains a line, L_0, of $\mathbf{P}$ then by (iii) we have

$$\begin{aligned}\nu_b(\mathbf{H},\mathbf{L}) &= |E(\mathbf{H})| - |E(\mathbf{L})| \\ &\leq 1 + \sum_{x\in L_0} \max\{0, deg_{\mathbf{H}}(x) - 1 - deg_{\mathbf{L}}(x)\} \\ &\leq 1 + (b-1)(q+1).\end{aligned}$$

If $\mathbf{H}$ does not contain any line then $|H| \geq q + \sqrt{q} + 1$ holds for all $H \in E(\mathbf{H})$ by Theorem 3.1. Hence we have

$$\begin{aligned}(q + \sqrt{q} + 1)\nu_b(\mathbf{H},\mathbf{L}) &\leq (q+1)(|E(\mathbf{H})| - |E(\mathbf{L})|) + \sqrt{q}|E(\mathbf{H})| \\ &\leq \sum_{H\in E(\mathbf{H})} |H| - \sum_{H\in E(\mathbf{L})} |H| \\ &= \sum deg_{\mathbf{H}}(x) - deg_{\mathbf{L}}(x) \leq b(q^2 + q + 1).\end{aligned}$$

This yields

$$\nu_b(\mathbf{H},\mathbf{L}) \leq bq + b - b\sqrt{q}. \tag{7.10}$$

The right-hand-side of (7.10) is less than $bq + b - q$ for $b > \sqrt{q}$. □

The determination of $\nu_b(q)$ for all b seems to be very difficult. The following example shows that $\nu_b(q)$ can be much larger than the lower bound in (7.9).

Example 7.12 [135]. Let $\mathbf{P}$ be a Desarguesian projective plane of order q, where $\sqrt{q}$ is an integer. Let $B_1, B_2, \ldots, B_{q-\sqrt{q}+1}$ be a decomposition of $V(\mathbf{P})$ into Baer subplanes, L_0 a line. Let $\mathscr{A} = \{A_1, A_2, \ldots, A_{q-\sqrt{q}+1}\}$ be an intersecting family on L_0 such that $D(\mathscr{A}) = (1 + o(1))\sqrt{q}$. Define

$$E(\mathbf{H}) = \{A_i \cup B_i : 1 \leq i \leq q - \sqrt{q} + 1\}$$

$$E(\mathbf{L}) = D(\mathscr{A}) - 1 \text{ copies of } L_0.$$

Then $(\mathbf{H},\mathbf{L})$ is a generalized 1-cover of $\mathbf{P}$ yielding

$$\nu_1(q) \geq q - 2\sqrt{q} + o(\sqrt{q}).$$

Problem 7.13. For $n \geq k \geq t \geq 1$, we let $C(n,k,t)$ denote the smallest integer m such that there exist m k-tuples of an n-set S with the property that every t-tuple of S is contained in at least one of them.

We will return to the determination of $C(n,k,t)$ in the next chapter. Here we would like to mention only that if t is fixed and k tends to infinity, then we can apply the fractional matching technique as we used it above to obtain asymptotic (and a couple of exact) results. See [135] (and the case $k \leq (3t+1)/2$ see [197] and Theorems 4.20, 6.14 above).

(D,c)-colorings of complete graphs. An (D,c)-coloring of a complete graph $\mathbf{K}$ means a coloring of the edges with c colors so that all monochromatic *connected* subgraphs have at most D vertices. A (D,c)-coloring can be viewed as c partitions of a ground set into sets of cardinality at most D such that all pairs of elements appear together in some of the sets. Resolvable block designs with c parallel classes and with blocks of size D are natural examples of (D,c)-colorings. However, (D,c)-colorings are more relaxed structures since the blocks may have any sizes up to D, and the pairs of the ground set may appear together in many blocks.

Let $f(D,c)$ denote the largest integer m such that $\mathbf{K}^m$ has a (D,c) coloring. Obviously,

$$f(D,c) \leq 1 + c(D-1). \tag{7.11}$$

The function $f(D,c)$ was introduced by Gerencsér and Gyárfás [142] in 1967. $f(D,2) = D$ and $f(D,3)$ was determined in [17] and [142]. In [147] there are further results on $f(D,c)$. The problem of determining $f(D,c)$ was rediscovered by Bierbrauer and Brandis [41]. In [42] the value of $f(D,c)$ is given for all $c \leq 5$ or $D \leq 3$, see Theorem 7.15.

For a (D,c)-coloring of $\mathbf{K}^n$ we can associate a hypergraph $\mathbf{H}$ with $V(\mathbf{H}) = V(\mathbf{K}^n)$ and the edges of $\mathbf{H}$ are the vertex sets of the connected monochromatic components of $\mathbf{K}^n$. The dual hypergraph $\mathbf{H}^*$ of $\mathbf{H}$ is a c-partite intersecting hypergraph (where multiple edges are allowed). So we have

Proposition 7.14. $f(D,c) = \max\{|E(\mathbf{G})|: \mathbf{G}$ *is a c-partite multihypergraph with* $D(\mathbf{G}) \leq D\}$.

There is another interpretation of $f(D,c)$ from the point of view of Ramsey theory as well (see, e.g., Erdös and Graham [101]).

Theorem 7.15 ([42]).

$$f(D,3) = \begin{cases} 4p & \text{if } D = 2p \\ 4p+1 & \text{if } D = 2p+1 \end{cases}$$

$$f(D,4) = \begin{cases} 9p & \text{if } D = 3p \\ 9p+1 & \text{if } D = 3p+1 \\ 9p+4 & \text{if } D = 3p+2 \end{cases}$$

$$f(D,5) = \begin{cases} 16p & \text{if } D = 4p \\ 16p+1 & \text{if } D = 4p+1 \\ 16p+6 & \text{if } D = 4p+2 \\ 16p+9 & \text{if } D = 4p+3 \end{cases}$$

$$f(2,c) = \begin{cases} c+1 & \text{if } c \text{ is odd} \\ c & \text{if } c \text{ is even} \end{cases}$$

$$f(3,c) = \begin{cases} 5 & \text{if } c = 3 \\ 2c & \text{if } c \equiv 0 \,(mod\, 3), \quad c \geq 6, \\ 2c+1 & \text{if } c \equiv 1 \,(mod\, 3), \\ 2c-1 & \text{if } c \equiv 2 \,(mod\, 3). \end{cases}$$

In [40] and [41] there are further results in the case $D \leq c$. Then they use strong results from the theory of resolvable designs. Here we give a theorem which asymptotically determines $f(D,c)$ whenever D is large, c is fixed and $c - 1 = q$ is a primepower.

We recall the definition of *i-cover* of a hypergraph $\mathbf{H}$, (see Proposition 5.12). We have

$$\tau_i(\mathbf{H}) = \min_t \left\{ \sum_{x \in V(\mathbf{H})} t(x) | t\colon V(\mathbf{H}) \to \{0, 1, \ldots, i\} \text{ such that} \right.$$

$$\left. \sum_{x \in E} t(x) \geq i \text{ holds for all } E \in E(\mathbf{H}) \right\}.$$

For an integer q, when a finite projective plane exists, define $\tau_i(q) = \min\{\tau_i(\mathbf{A})\colon \mathbf{A}$ is an affine plane of order $q\}$. By definition $\tau_0(q) = 0$. We need one more definition

$$\tau_c^* = \max\{\tau^*(\mathbf{H})\colon \mathbf{H} \text{ is } c\text{-partite and intersecting}\}.$$

Theorem 7.16 ([136]).

$$D\tau_c^* - c\tau_c^* < f(D, c) \leq D\tau_c^*,$$

and for any fixed c there are infinitely many D for which equality holds.

The proof is analogous to Theorem 7.5. Then Corollary 6.4 implies that (see [147])

$$f(D, c) \leq D(c - 1). \tag{7.12}$$

Theorem 7.17 ([136]). *Suppose that there exists an affine plane of order q, and* $D = q\lceil D/q \rceil - i$ *where* $0 \leq i \leq q$. *Then for* $D \geq q^2 - q$ *we have*

$$f(D, q + 1) = \lceil D/q \rceil q^2 - \tau_i(q),$$

and the only optimal multihypergraph is obtained from a truncated projective plane (see Example 5.3) by multiplying its lines.

The case $D \equiv 0 \ (mod\, q)$ was proved in [42]. Their lower bound for $f(D, q + 1)$ for general i is a slightly smaller than the one given in Theorem 7.17. For the proof of Theorem 7.17 the following lemma was used:

Lemma 7.18 ([136]). *Suppose that* **H** *is an r-partite intersecting hypergraph (without multiple edges). Then either*
(i) **H** *is a truncated projective plane, and then* $\tau^*(\mathbf{H}) = r - 1$, *or*
(ii) $\tau^*(\mathbf{H}) \leq r - 1 - (r - 1)^{-1}$.

Minimum number of edges of a graph of diameter 2. Suppose that **G** is a (simple) graph of diameter 2 (i.e., every two vertices connected by an edge or a path of length 2), over n vertices. Then **G** has at least $n - 1$ edges, and the only extremal graph is the star. The star has a vertex of degree $n - 1$. It is natural to ask that at least how many edges must a graph on n vertices have, if its diameter is 2 and its maximum degree is at most D. Let us denote this minimum number by $e_2(n, D)$.

This question was posed by Erdös and Rényi [110] in 1962, and later Erdös, Rényi and T. Sós [111] determined the exact value of $e_2(n, D)$ for $D > n/2$. (Some of their statements, especially those without proofs, were recently corrected by Vrto and Znám [242].) They proved, e.g., that for $\frac{2}{3}n < D \leq n - 5$ one has

$$e_2(n, D) = 2n - 4,$$

significantly larger than $n - 1$! The extremal graph is shown in Fig. 7.1.

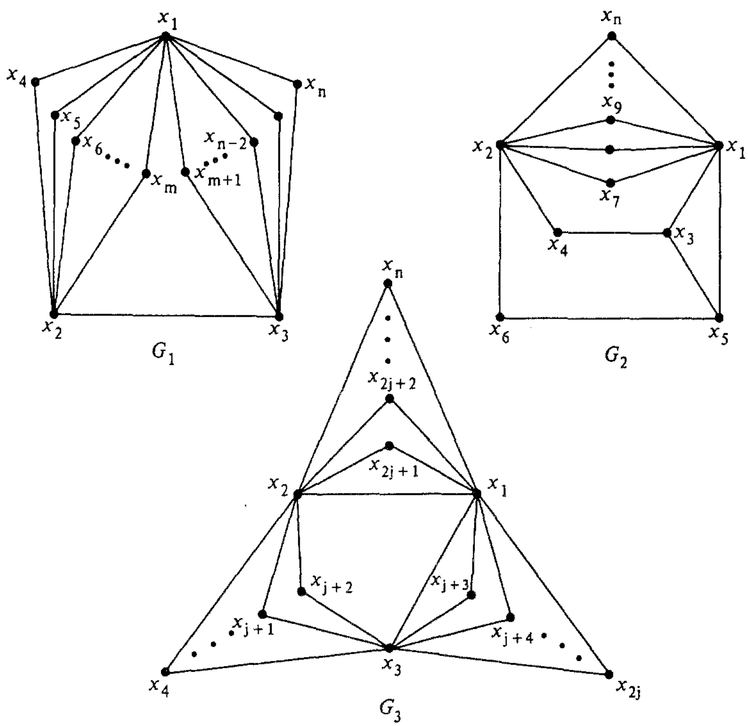

Fig. 7.1. (Redrawn from Page 174 [50])

Bollobás [47] proved the following lower bound for $D < n/2$. Let $0 < c < 1/2$, $v = [1/c]$, $u = \left[\frac{v}{v-1}\left(\frac{1}{c^2} - v\right)\right]$, then define

$$g(c) = \frac{1}{2}\left(v + cu + cv + \frac{v}{v-1}(1 - vc^2) - c^2\sqrt{u}\right).$$

Then we have

$$e_2(n, cn) \geq g(c)n + o(n).$$

This complicated lower bound and an example obtained from finite projective planes imply [47] that

$$\frac{1}{c}n < e_2(n, cn) < \left(\frac{1}{c} + \left(\frac{1}{c}\right)^{5/8}\right)n \tag{7.13}$$

that is nc^{-1} is in fact the correct order of magnitude of $e_2(n, cn)$.

Theorem 7.19 ([206]). *Suppose that* $\frac{q+1}{q^2+q+1} < c < \frac{1}{q}$ *and there exists a finite projective plane of order* q*. Then for* $n > n_0(c)$ *we have*

$$e_2(n, cn) = (q+1)n + O(1).$$

Example 7.20. The construction giving $(q+1)n + O(1)$ edges is the following. The vertex-set of $V(G)$ is the disjoint union of $V_0, V_1, \ldots, V_{q^2+q+1}, |V_0| = q^2 + q + 1$, $|V_i| \sim (n/q^2 + q + 1) - 1$. Consider a $PG(2, q)$ on the point set V_0 with lines $L_1, \ldots,$ L_{q^2+q+1}. Then connect each vertex of L_i to each vertex of V_i $(1 \le i \le q^2 + q + 1)$, and add all the edges in V_0.

They also proved in [207] that

$$e(c) = \lim_{n\to\infty} e_2(n, cn)/n \tag{7.14}$$

exists for every $0 < c < 1$, except for $c = c_1, c_2, \ldots$ where $\{c_k\}$ is a sequence tending to 0. To obtain $e(c)$ they developed the following method. For any hypergraph $\mathbf{H}$ and positive real x define

$$A(\mathbf{H}, x) := \min\left\{\sum_{E\in E(\mathbf{H})} w(E)|E| \colon w \text{ is a fractional matching of } \mathbf{H} \text{ with } |w| = x\right\}.$$

(If $v^*(\mathbf{H}) < x$ then $A(\mathbf{H}, x) = +\infty$.) Further, let

$$A(x) := \inf\{A(\mathbf{H}, x) \colon \mathbf{H} \text{ is intersecting}\}. \tag{7.15}$$

Theorem 7.21 ([207]). $e(c) = cA(1/c)$.

To calculate $A(x)$ is a finite process for all given x, because they proved that

$$A(x) = x^2 + O(x^{1.6}),$$

and instead of (7.15) the following holds

$$A(x) =: \min\{A(\mathbf{H}, x) \colon \mathbf{H} \text{ is intersecting}, |V(\mathbf{H})| \le x^2 + O(x^{1.6})\}.$$

An intersecting hypergraph with $A(x) = A(\mathbf{H}, x)$ is called x-extremal hypergraph. So the determination of $e_2(n, D)$ is more or less equivalent to the following

Problem 7.22. Describe the x-extremal hypergraphs.

There are plenty more of the open problems here, e.g., in [55] Bondy and Murty investigated graphs of diameter 2 with lower and upper bounds on their degrees. One can find more about this in Bollobás book [50].

The largest intersecting r-graph on n vertices. One of the best-known results in extremal set theory is the theorem of Erdös-Ko-Rado.

Theorem 7.23 ([107]). *Suppose $n \ge 2r$, and let $\mathbf{H}$ be an intersecting r-graph. Then* $|E(\mathbf{H})| \le \binom{n-1}{r-1}$.

This bound can be attained, and the extremal families are precisely the families $\mathscr{F}(a) = \left\{F \in \binom{[n]}{r} \colon a \in F\right\}$ for $n > 2r$. Clearly $\tau(\mathscr{F}(a)) = 1$ holds.

What if we do not allow the members of $E(\mathbf{H})$ to have an overall non-trivial intersection? How large $|E(\mathbf{H})|$ can be? The answer to this question has been given by Hilton and Milner [156]. A short proof can be found in [122].

Theorem 7.24 ([156]). *Suppose $n \geq 2r$, and let* **H** *be an intersecting r-graph with $\tau(\mathbf{H}) \geq 2$. Then*

$$|E(\mathbf{H})| \leq \binom{n-1}{r-1} - \binom{n-r-1}{r-1} + 1.$$

Let $g(n, r, \tau)$ denote the largest size of an intersecting r-graph **H** with $|V(\mathbf{H})| = n$, $\tau(\mathbf{H}) \geq \tau$.

Theorem 7.25 ([115]). *There exists a positive integer constant $c(r, \tau)$, depending only r and τ, such that*

$$g(n, r, \tau) = (c(r, \tau) + o(1))\binom{n}{r-\tau}$$

holds whenever n tends to infinity.

Here the value of $c(r, \tau)$ is determined by Conjecture 3.25. By Theorem 7.23 and 7.24 we have $c(r, 1) = 1$, $c(r, 2) = r$ and the only one more known value is $c(r, 3) = r^2 - r$ [115]. A weaker result was rediscovered in [7].

Proof. To give a lower bound

$$\max|E(\mathbf{H})| \geq (r(r-1)\cdots(r-\tau+2) + o(1))\binom{n}{r-\tau}$$

one can use Example 3.24. To prove the upper bound (and that the limit exists) consider an optimal **H** and let **B** its ν-critical nucleus (or in other words, its $(2, 1)$-critical kernel, see Chapter 4 between (4.3) and (4.4)). We have $|E(\mathbf{B})|$ is bounded (see 4.5) and it does not contain a member with less than τ elements. Hence

$$|E(\mathbf{H})| \leq \sum_{B \in E(\mathbf{B})} \binom{n-|B|}{r-|B|} \leq |E(\mathbf{B})| \cdot \binom{n-\tau}{r-\tau}.$$

Finally, it is trivial that from the finitely many possible **B** there exists a best one (which contains the most τ-sets). □

There are a lot of beautiful results and problems about different generalizations of the Erdös-Ko-Rado theorem. E.g., one can ask what is the largest size of an r-graph **H** over n elements with given $\nu(\mathbf{H})$. (This was solved for $n > n_0(r, \nu)$ by Erdös [94].) A recent survey is [118]. In the next section we will discuss only one of these questions, when a degree constrain is given.

The largest intersecting r-graph with a linear condition on the maximum degree. Let c be a real number, $0 < c \leq 1$. Erdös, Rothschild and Szemerédi [256] raised the following question: How large can an intersecting r-uniform hypergraph **H** can be if each vertex has degree at most $c|E(\mathbf{H})|$. The class of these systems over n elements is denoted by $\mathscr{F}(n, k, c)$, $f(n, k, c)$ is the maximum size of such a family.

Erdös, Rothschild and Szemerédi [256] solved the case $c = 2/3$, by proving

$$f(n, r, 2/3) = 3\binom{n-3}{r-2} \quad \text{for } n > n_0(r). \tag{7.17}$$

For a family $\mathscr{H}$ with $V(\mathscr{H}) \subset [n]$ define $\mathscr{F}(\mathscr{H}) := \left\{F \in \binom{[n]}{r}\right.$, there exists an $\left. H \in \mathscr{H} \text{ with } H \subset F\right\}$. Similarly, $\mathscr{F}_0(\mathscr{H}) = \left\{F \in \binom{[n]}{r} : F \cap V(\mathscr{H}) \in \mathscr{H}\right\}$. Obviously, $\mathscr{F}_0(\mathscr{H}) \subseteq \mathscr{F}(\mathscr{H})$ and $\mathscr{F}(\mathscr{H})$, $\mathscr{F}_0(\mathscr{H})$ are intersecting families whenever $\mathscr{H}$ is intersecting.

In (7.17) the only extremal family is $\mathscr{F}_0(\mathbf{K}_2^3)$. This result was extended by Frankl [114]

$$f(n,r,c) = 3\binom{n-3}{r-2} + \binom{n-3}{r-3} \quad \text{if} \quad \frac{2}{3} < c < 1, n > n_0(r,c). \tag{7.18}$$

In [114] and [131] $f(n,r,c)$ is determined for all $c \geq 3/7$ (for $n > n_0(r,c)$). In the case $\frac{3}{5} < c < \frac{2}{3}$, there are 6 non-isomorphic extremal families.

Developing the method of [114] in [129] the following general theorem is proved. First some definitions. Let $\mathscr{B}$ be a family of rank k, $0 < c < 1$, $\mathscr{B}_k := \{B \in \mathscr{B}, |B| = k\} \neq \varnothing$. The optimum value of the following linear programming problem (7.19) is called the *capacity* of $\mathscr{B}$ belonging to c.

$$\begin{cases} w\colon \mathscr{B} \to \mathbb{R}^+, \\ w(B) \leq 1 \text{ for all } B \in \mathscr{B}_k, \\ \sum\limits_{x \in B \in \mathscr{B}} w(B) \leq c\left(\sum w(B)\right) \quad \text{for all } x \in V(\mathscr{B}). \end{cases} \tag{7.19}$$

Then $Cap_{\mathscr{B}}(c) := \max\{|w| : w \text{ satisfies (7.19)}\}$. It may occur that $Cap_{\mathscr{B}}(c) = 0$ or ∞.

Theorem 7.26. *Suppose* $0 < c < 1$ *is given. Define k as* $1/\tau^*(k,1) \leq c < 1/\tau^*(k-1,1)$. *Define* $f(c) = \max\{Cap_{\mathscr{B}}(c)$: *$\mathscr{B}$ has rank k, intersecting*$\}$. *Then*

$$f(n,r,c) = f(c)\cdot\binom{n}{r-k} + O(n^{r-k-1})$$

holds, whenever r, c are fixed and n tends to infinity.

Theorem 7.26 implies, in particular, that for any given c the problem of Erdös, Rothschild and Szemerédi is a finite one, i.e., it can be solved—in theory—by investigating a finite number of cases. (We can suppose that $\mathscr{B}$ is ν-critical.) In [121] the method was further developed and applied to t-wise s-intersecting families. We mention here only 2 corollaries.

Corollary 7.27. *Let* $\mathscr{F} \subset \binom{[n]}{r}$ *be a t-wise intersecting family such that* $D(\mathscr{F}) \leq \frac{q^t - 1}{q^{t-1} - 1}|\mathscr{F}|$. *Then for* $n > n_0(k)$ *we have* $|\mathscr{F}| \leq |\mathscr{F}_0(PG(t,q))|$. *The extremal family is unique.*

This theorem was conjectured by Erdös [95] and proved for $t = 2$, $r \leq 3$ by Frankl [114], the case $t = 2$ by Füredi [129] and in general in [121]. Another exact result.

Corollary 7.28. *Let* $t < k \leq (3/2)t - 1$, *and* $1 - 2/(3t - k + 2) \leq c < 1 - 2/(3t - k + 3)$. *Then for* $n > n_0(r, c)$ *we have*

$$f^t(n,r,c) = \left\lfloor \frac{1}{1-c}\binom{n-k-1}{r-k} \right\rfloor.$$

$\left(\text{Here } f^t(n,r,c) = \max\left\{|\mathscr{F}|: \mathscr{F} \subset \binom{[n]}{r}, t\text{-wise intersecting, and } D(\mathscr{F}) \leq c|\mathscr{F}|\right\}.\right)$

The proof of 7.28 based on the generalized version of 7.26 and 6.14. A similar problem was solved by Frankl in [119]. He determined $d(n, r, D)$ exactly for all $n \geq 2r$ and $2\binom{n-3}{r-2} \leq D$, where $d(n,r,D) := \max\left\{|\mathscr{F}|: \mathscr{F} \subset \binom{[n]}{r}, \text{intersecting}, D(\mathscr{F}) \leq D\right\}$.

8. Asymptotically Good Coverings and Decompositions

(n, k, t) *Coverings and packings.* Let $n \geq k > t \geq 1$ be integers and set $[n] = \{1, 2, \ldots, n\}$, a generic n-set. Recall the definitions of (n, k, t) packings and coverings. A family $\mathscr{P} \subset \binom{[n]}{k}$ is an (n, k, t) *packing* if $|P \cap P'| < t$ holds for every two distinct members of $\mathscr{P}$. The packing function $P(n, k, t)$ is defined as the maximum cardinality of a packing. A family $\mathscr{C} \subset \binom{[n]}{k}$ is an (n, k, t) *covering* if every t-set of $[n]$ is contained in some member $C \in \mathscr{C}$. The covering function $C(n, k, t)$ is the minimum cardinality of an (n, k, t) covering. Elementary counting argument implies

$$P(n,k,t) \leq \binom{n}{t} \Big/ \binom{k}{t} \leq C(n,k,t). \tag{8.1}$$

This was improved by Schönheim [223]:

$$P(n,k,t) \leq \left\lfloor \frac{n}{k} \left\lfloor \frac{n-1}{k-1} \right\rfloor \cdots \left\lfloor \frac{n-t+1}{k-t+1} \right\rfloor \cdots \right\rfloor$$

$$C(n,k,t) \geq \left\lceil \frac{n}{k} \left\lceil \frac{n-1}{k-1} \right\rceil \cdots \left\lceil \frac{n-t+1}{k-t+1} \right\rceil \cdots \right\rceil.$$

Equality holds if and only if there exists an (n, k, t) Steiner system. The existence of Steiner systems (and other tactical configurations) is a central question of Combinatorial Analysis, which seems to be infinitely difficult. Here we will deal with only asymptotic results, where k and t are fixed and n is large. Obviously

$$P(n,k,1) = [n/k] \quad \text{and} \quad C(n,k,1) = [n/k] \tag{8.2}$$

hold. The first nontrivial case was proved by Fort and Hedlund [113], $C(n, 3, 2) =$

$\left\lceil \frac{n}{3} \left\lceil \frac{n-1}{2} \right\rceil \right\rceil$. A recent short expository paper of the determination of $C(n,3,2)$ and $P(n,3,2)$ is [268]. Mills [195], [196] determined $C(n,4,2)$ for all n. The case $C(n,5,2)$ is still not complete (see [198] where he proved $C(16,5,2)=15$, so $C(n,5,2)$
is known for all $n \leq 23$, and for the case $n \equiv 3 \, (mod\, 4)$ see [199]). Much less is known about the cases $t \geq 3$, or $k \geq 5$. (On $C(n,4,3)$ see, e.g., [154], on $C(n,5,2)$ see [265].)

As it is known (Wilson [245]), that for all k if $\frac{n-1}{k-1}$ and $\binom{n}{2} \Big/ \binom{k}{2}$ are integers and $n > n_0(k)$ then there exists an $(n,k,2)$ Steiner system, so we have

$$\lim_{n\to\infty} P(n,k,2)\binom{k}{2} \Big/ \binom{n}{2} = \lim_{n\to\infty} C(n,k,2)\binom{k}{2} \Big/ \binom{n}{2} = 1. \tag{8.3}$$

These equations were proved first by Erdös and Hanani [105] in 1963. They conjectured that the analogous of (8.3) is true for all t, i.e., for all $k > t \geq 1$

$$p(k,t) = \lim_{n\to\infty} P(n,k,t)\binom{k}{t} \Big/ \binom{n}{t} = 1, \tag{8.4}$$

$$c(k,t) = \lim_{n\to\infty} C(n,k,t)\binom{k}{t} \Big/ \binom{n}{t} = 1. \tag{8.5}$$

These equalities became known as the Erdös-Hanani Conjecture. They proved (8.4) and (8.5) for $t=3$, $k=p$ or $p+1$ where p is a prime power. They also showed that either of the equalities (8.4) and (8.5) imply the other. It is easy to prove the case $t=k-1$.

Theorem 8.1. $P(n,k,k-1) \geq \frac{n-k+1}{n}\binom{n}{k-1} \Big/ \binom{k}{k-1}$.

Proof. We give a construction. Let $\mathscr{C}_\alpha = \left\{ C \in \binom{[n]}{k} : \sum_{c\in C} c = \alpha (mod\, n) \right\}$, where $0 \leq \alpha < n$. Then each of the $\mathscr{C}_\alpha$ is an $(n,k,k-1)$ packing, hence

$$P(n,k,k-1) \geq \max_\alpha |\mathscr{C}_\alpha| \geq \frac{1}{n}\binom{n}{k}. \qquad \square$$

The case $t=k-2$ was proved by Bassalygo and Zinovèv [22]. Kusjurin [168] proved that for any fixed t one has $\lim_{k\to\infty} c(k,t) = 1$. Finally Rödl [217] found an ingenious technique which succeeded in proving the Erdös-Hanani conjecture.

Theorem 8.2 ([217]). *For all fixed k and t we have*

$$\lim_{n\to\infty} P(n,k,t)\binom{k}{t} \Big/ \binom{n}{t} = \lim_{n\to\infty} C(n,k,t)\binom{k}{t} \Big/ \binom{n}{t} = 1.$$

Rödl used probabilistic method. In [228] Spencer rephrased his argument. We postpone the proof of Theorem 8.2. First we will discuss more general results, and further consequences of the Rödl method.

Near perfect matchings in near regular hypergraphs. The next breakthrough in this field was the Frankl, Rödl [125] theorem where they generalized Rödl's method in a hypergraph setting. Define the *point covering* of **H** as a subset of edges whose union is $V(\mathbf{H})$. The *point covering number*, $cov(\mathbf{H})$, is the minimum number t so that there exist t edges of **H** whose union is the whole $V(\mathbf{H})$. For a r-graph **H** we have

$$\nu(\mathbf{H}) \leq |V(\mathbf{H})|/r \leq cov(\mathbf{H}). \tag{8.6}$$

An r-graph **H** has a *perfect* matching if $\nu(\mathbf{H}) = |V(\mathbf{H})|/r$. It has a *near perfect* matching if $\nu(\mathbf{H})$ is "close" to $|V(\mathbf{H})|/r$. (This was not a perfect definition.)

Now let **H** be a D-regular r-graph. Considering its dual, Corollary 6.29 implies

$$cov(\mathbf{H}) \leq \frac{|V(\mathbf{H})|}{r}(1 + \log D). \tag{8.7}$$

One cannot say much more in general as it is shown by the following example of Frankl (unpublished).

Example 8.3. Define the following $6k$-regular 3-graph **G**, over $9k$ vertices. $V(\mathbf{G}) = V_1 \cup V_2 \cup \cdots \cup V_{2k} \cup V_0$ where $|V_0| = 3k$, $|V_i| = 3$ for $1 \leq i \leq 2k$. $E(\mathbf{G}) = \left\{E \in \binom{V(\mathbf{G})}{3} : |E \cap V_0| = 1 \text{ and there exists an } i \text{ with } |V_i \cap E| = 2\right\}$. Then

$$\nu(\mathbf{G}) = 2k < \frac{|V(\mathbf{G})|}{3} = 3k < cov(\mathbf{G}) = 4k.$$

If **H** is a *random D-regular r-graph*, then [258] (for $D \to \infty$) it almost surely contains a near perfect matching. In other words, almost every D-regular r-graph has a near perfect matching. Frankl and Rödl showed, that the randomness in this statement can be replaced by some seemingly very weak conditions. The next version (which is even simpler and more powerful) is due to Pippenger (private communication, see in [212]). Recall that $deg_{\mathbf{H}}(x, y)$ means $|\{E \in E(\mathbf{H}): \{x, y\} \subset E\}|$.

Theorem 8.4. *Let* **H** *be an r-graph on n vertices,* $K > 0$ *a fixed real number. If for some d we have*
(i) $deg_{\mathbf{H}}(x) < Kd$ *for all* $x \in V(\mathbf{H})$,
(ii) $deg_{\mathbf{H}}(x) = d(1 \pm o(1))$ *for almost all x, and*
(iii) $deg_{\mathbf{H}}(x, y) = o(d)$ *for all distinct x and y,*
then $cov(\mathbf{H}) = (n/r)(1 + o(1))$ *holds.*

Note that $cov(\mathbf{H}) < (n/r)(1 + \varepsilon)$ implies $\nu(\mathbf{H}) > (n/r)(1 - r\varepsilon)$. To avoid ambiguities in $o - O$ notation we restate Theorem 8.4 with epsilons and deltas.

Theorem 8.4. *For all integer* $r \geq 2$, *reals* $K \geq 1$, $\alpha > 0$ *there exists a* $\gamma > 0$ *so that: If the r-graph* **H** *on n vertices has the following properties*
(i) $deg_{\mathbf{H}}(x) < Kd$ *for all x,*
(ii) $(1 - \gamma)d < deg_{\mathbf{H}}(x) < (1 + \gamma)d$ *holds for all but at most* γn *vertices,*
(iii) $deg_{\mathbf{H}}(x, y) < \gamma d$ *for all distinct x and y,*
then $cov(\mathbf{H}) < (n/r)(1 + \alpha)$ *holds.*

Corollaries of the Frankl-Rödl theorem. First we show that how easily Theorems 8.4 implies 8.2.

Proof of Theorem 8.2. For $n > k > t$ define a hypergraph **H** with vertex-set $\binom{[n]}{t}$. Edges of **H** are formed by copies of $\mathbf{K}_t^k$, i.e., an $\binom{k}{t}$-tuple of edges of $\mathbf{K}_t^n$ forms an edge of **H** if it is the edge-set of some complete graph $\mathbf{K}_t^k$. The rank of **H** is $\binom{k}{t}$, $deg_{\mathbf{H}}(x) = \binom{n-t}{k-t}$ for all vertex of **H**, and we have

$$deg_{\mathbf{H}}(x, y) \leq \binom{n-t-1}{k-t-1}$$

for all two distinct vertices of **H** (i.e., for all two distinct t sets of $[n]$). We can apply Theorem 8.4 obtaining that

$$cov(\mathbf{H}) = (1 + o(1))\binom{n}{t}\bigg/\binom{k}{t} \tag{8.8}$$

when $n \to \infty$ for k and t are fixed. A point cover of **H** corresponds to an (n, k, t) covering, so (8.8) implies (8.5). □

For two given t-graphs **G** and **A** let us denote the *packing number* of **G** with respect to **A** by $\pi(\mathbf{G}, \mathbf{A})$. That is the maximum number of pairwise edge disjoint copies of **A** in **G**. The above proof of Theorem 8.2 yields

Corollary 8.5 ([125]). *Suppose* **A** *is a given t-graph and n tends to infinity. Then*

$$\pi(\mathbf{K}_t^n, \mathbf{A}) = (1 - o(1))\binom{n}{t}\bigg/|E(\mathbf{A})|$$

holds.

In the case $t = 2$ Wilson [246] proved more, he showed that $\pi(\mathbf{K}_2^n, \mathbf{A}) = \binom{n}{2}\Big/|E(\mathbf{A})|$ provided $n > n_0(\mathbf{A})$, $\binom{n}{2}\Big/|E(\mathbf{A})|$ and $(n-1)/d$ are integers where d is the greatest common divisor of the degrees of **A**. In the hypergraph case, the decomposition of $\mathbf{K}_t^n$ into delta systems was investigated in [269, 202, 203]. Let $\mathscr{G}(n, p)$ denote the random graph with edge probability p, that is, each edge is present in $\mathscr{G}(n, p)$ with independent probability p.

Corollary 8.6 (Ajtai, Komlós, Rödl and Szemerédi [10]). *Suppose that* **A** *is a fixed graph,* $0 < p < 1$ *is given. If n tends to infinity we have*

$$\pi(\mathscr{G}(n, p), \mathbf{A}) = (1 - o(1))\binom{n}{2}p/|E(\mathbf{A})|.$$

In other words, almost all graph almost decomposable into disjoint copies of **A**. *In* [123] *the following sharpening of Corollary* 8.5 *is proved.*

Corollary 8.7 ([123]). *Let the t-graph* **A** *be given with* $E(\mathbf{A}) = \{A_1, \ldots, A_u\}$. *Suppose*

$n \to \infty$. Then we can place $(1 - o(1))\binom{n}{t}\Big/u$ copies of $\mathbf{A}$, $\mathbf{A}^1$, $\mathbf{A}^2, \dots$ into $[n]$ with the following properties
(i) $|V(\mathbf{A}^i) \cap V(\mathbf{A}^j)| \le t$ *for all* $i \ne j$,
(ii) *if* $B = V(\mathbf{A}^i) \cap V(\mathbf{A}^j)$, $|B| = t$ *then* $B \notin E(\mathbf{A}^i)$ *and* $B \notin E(\mathbf{A}^j)$.

In other words, for any given $\mathbf{A}$ there exists a hypergraph $\mathbf{H}$ over n vertices with $(1 - o(1))\binom{n}{t}$ edges such that it can be decomposed into edge disjoint *induced* copies of $\mathbf{A}$. (Moreover these copies of $\mathbf{A}$ cannot have more then t common points.) For example, if $\mathbf{A}$ is a fourcycle, $\mathbf{C}_4$, and n is even then such a graph and a decomposition are: $\mathbf{H} = \mathbf{K}_2^n - U$ where U is a perfect matching, and then $\{M \cup M' \colon M, M' \in U\}$ gives the desired decomposition. As in [123] the earlier version of Theorem 8.4 was used, the proof was somewhat technical. Here we outline a less complicated proof.

Proof of Corollary 8.7. Let $\mathscr{P}$ be an almost optimal $(n, k, t+1)$ packing, i.e., $|\mathscr{P}| \ge (1 - o(1))\binom{n}{t+1}\Big/\binom{k}{t+1}$ holds by Theorem 8.2. Choose a small, positive ε (e.g., $\varepsilon = 1/4^v$ where $v = |V(\mathbf{A})|$) and let $\mathbf{G}_t^n(1-p)$ be a random t-graph on n vertices such that

$$\operatorname{Prob}(T \in E(\mathbf{G}_t^n(1-p))) = 1 - p.$$

Define the random subhypergraph $\mathscr{P}_v^n$ as follows

$$P \in E(\mathscr{P}_v^n) \quad \text{if} \quad P \in \mathscr{P} \quad \text{and} \quad \mathbf{G}_t^n(1-p)|P \cong \mathbf{A}.$$

Then one can apply Theorem 8.4 to $\mathscr{P}_v^n$. □

Corollary 8.7 was used to give an asymptotic solution to the following problem. What is the maximum size of a k-graph $\mathbf{H}$ over n vertices if no edge is covered by the union of r others $(r \ge 2)$. Denote this maximum by $M(n, k, r)$. Clearly $M(n, 2, r) = n - 1$ for all $r \ge 2$, and in general $M(n, k, r) = n - (r - 1)$ for $r \ge k$. In [98, 99] it was proved that for fixed k and r, $t = [k/r]$ one has

$$c_{k,r} = \lim_{n \to \infty} M(n, k, r)\Big/\binom{n}{t}$$

exists and positive. In [123] it was proved that if $k = r(t-1) + l + 1, 0 \le l \le r$ then

$$c_{k,r} = 1\Big/\left(\binom{k}{t} - m(k, t, l)\right),$$

where $m(k, t, l) = \max\left\{|\mathscr{N}| \colon \mathscr{N} \subset \binom{[k]}{t}, v(\mathscr{N}) \le l\right\}$, the same function we mentioned above after the proof of Theorem 7.25.

The Johnson graph $J(n, k)$ is defined over $\binom{[n]}{k}$, such that $v_1, v_2 \in \binom{[n]}{k}$ are adjacent if $|v_1 \cap v_2| = k - 1$. A *sparse* matching $M \subset E(\mathbf{G})$ of a simple graph $\mathbf{G}$ is a collection of edges of **G** such that $E(\mathbf{G})|(\bigcup M) = M$, i.e., the distance between any two edges of M in G is at least 2. Let $M(n,k)$ be the maximum number of edges that a sparse matching in $J(n,k)$ can have. It is easy to see that

$$M(n,k) \le \left\lfloor \binom{n}{k-1} \Big/ (2k-1) \right\rfloor, \tag{8.9}$$

and Hemmeter and Hong [155] proved that in (8.9) equality holds for $k = 2$ and for $k = 3$ whenever $k \equiv 1 \ (mod\, 5)$. Here we have

Corollary 8.8. *For a given k when n tends to infinity one has $M(n,k) = (1 - o(1))\binom{n}{k-1}\Big/(2k-1)$.*

Proof. Let **G** be the union of two $\mathbf{K}^k_{k-1}$ on $k+1$ vertices. (I.e., $|V(\mathbf{G})| = k+1$, $|E(\mathbf{G})| = 2k-1$.) The determination of $M(n,k)$ is equivalent to the question of $\pi(\mathbf{K}^n_{k-1}, \mathbf{G})$. Then 8.5 can be applied. □

Corollary 8.9 (Brouwer [58] for $k = 3, 4$ and [125] for all k). *Let* **S** *be an $(n,k,2)$ Steiner family. Then*

$$\nu(\mathbf{S}) \ge (1 - o(1))n/k$$

whenever k is fixed and $n \to \infty$. □

Brouwer proved that $\nu(\mathscr{S}(n,3,2)) \ge \frac{n}{3} - 2n^{2/3}$ which is much better than Corollary 8.9. He announced that his proof also gives $\nu(\mathscr{S}(n,4,2)) \ge \frac{n}{4} - O(n^{2/3})$, as well.

Conjecture 8.10 (Brouwer [58]). $\nu(\mathscr{S}(n,3,2)) \ge \frac{n-4}{3}$ for all n and for all Steiner system $\mathscr{S}(n,3,2)$.
This conjecture is proved up to $n \le 19$ (Lo Faro [112]).

Edgecoloring of near regular hypergraphs. It was conjectured, that if **H** is a "near regular" r-graph then not only $\nu(\mathbf{H}) \ge (1 - o(1))n/r$ but its chromatic index is about $D(\mathbf{H})$. We recall that the *chromatic index* of **H**, $q(\mathbf{H})$ is the smallest integer q that one can decompose $E(\mathbf{H})$ into q matchings. Obviously, $q(\mathbf{H}) \ge D(\mathbf{H})$, and the famous Vizing theorem says (see, e.g., in [50]) that for a (simple) graph **G** one has

$$D(\mathbf{G}) \le q(G) \le D(\mathbf{G}) + 1. \tag{8.10}$$

As Example 8.3 shows (8.10) is not true in general. Spencer proved the hypergraph version of (8.10). Here we give a simpler, a more complete version of his result due to Spencer and Pippenger.

Theorem 8.11 ([212]). *Let* **H** *be an r-graph on n vertices such that*
(i) $deg_{\mathbf{H}}(x) = (1 + o(1))d$, *for all x*
(ii) $deg_{\mathbf{H}}(x,y) = o(d)$, *for all pair x, y, (r is fixed, $d = d(n) \to \infty$). Then*

[A] *There is a partition of* $E(\mathbf{H})$ *into* $d(1 + o(1))$ *matchings (i.e.,* $q(\mathbf{H}) = d(1 + o(1))$*)*
[B] *There is a partition of* $E(\mathbf{H})$ *into* $d(1 + o(1))$ *point coverings.*

A straightforward application of 8.11 is the following sharpening of Corollary 8.9.

Corollary 8.12 ([212]). *Fix* k *and let* $n \to \infty$. *If* $\mathbf{S}$ *is an* $(n, k, 2)$ *Steiner family, then*

$$q(\mathbf{S}) = \frac{n}{k-1} + o(n). \qquad \square$$

Earlier only very weak results were about $q(\mathscr{S}(n, k, 2))$. (To find more about this see, e.g., [73].)

Let $D(n)$ denote the maximum number of edge disjoint Steiner triple systems on $[n]$. It is easy to check that $D(n) \le n - 2$, and it is conjectured that here equality holds (for all $n \equiv 0$ or 1 $(mod\, 6)$). This conjecture was proved for infinitely many values of n by J.X. Lu [189]. Theorem 8.4 and 8.11 are not applicable to this problem because they cannot handle *exact* $(n, 3, 2)$ packings. However one can prove using Theorem 8.11 the following

Corollary 8.13. $\mathbf{K}_3^n$ *can be decomposed into* $n + o(n)$ $(n, 3, 2)$ *packings.* $\square$

Proof of the Frankl-Rödl theorem. The proof of Theorem 8.4 is a victory of the nonconstructive method. The key idea of the proof (as in the proof of Theorem 8.2) is to randomly select $\varepsilon n/r$ edges of $\mathbf{H}$, ε *small.* These edges will overlap but basically the overlap will be of order ε^2 (i.e., $\sim\varepsilon^2 n$). Thus these edges are appropriately efficient in covering vertices. Now delete all the vertices so covered and iterate. The problem is that the remaining r-graph $\mathbf{H}'$ is no longer so regular as $\mathbf{H}$ was. So our probabilistic space is changed in every step, we have to overcome a lot of technical problem. To do this we will introduce a variable δ which we think of as *tolerance* and think of a graph being regular within tolerance $1 \pm \delta$. The main step is to show that if we want $\mathbf{H}'$ to be regular within tolerance $1 \pm \delta'$ it is suffices if $\mathbf{H}$ is regular within tolerance $1 \pm \delta$ for appropriately small δ. This is sufficient since we will iterate only a *finite* number of times (dependent on ε only) as once there are few vertices left one can cover them one by one. We are going to prove the second version of Theorem 8.4. We are going to use the following notation: $1 \pm \delta$ means some number between $1 - \delta$ and $1 + \delta$.

Lemma 8.14. *Fix* r, $\varepsilon > 0$ *and* K. *Then for all* $\delta' > 0$ *there exists an* $\delta > 0$ *so that the following holds. Suppose that* n *and* D *are sufficiently large* ($n > n_0(r, \delta', \varepsilon, K)$, $D > D_0(r, \delta', \varepsilon, K)$). *Let* $\mathbf{H}$ *be an* r*-graph with* $|V(\mathbf{H})| = n > n_0$ *vertices, and let* $D > D_0$. *Suppose that*
(i) $deg_{\mathbf{H}}(x) < KD$ *for all* x,
(ii) $deg_{\mathbf{H}}(x) = (1 \pm \delta)D$ *for all but at most* δn *vertices* x,
(iii) $deg_{\mathbf{H}}(x, y) < \delta D$ *for all* x, y.
Then there exists $\mathscr{C} \subset E(\mathbf{H})$ *so that setting* $S = V(\mathbf{H}) - \bigcup \mathscr{C}$, *and* $\mathbf{H}^* = \mathbf{H} - \{\bigcup \mathscr{C}\}$ *(i.e.,* $V(\mathbf{H}^*) = S$, $E(\mathbf{H}^*) = \{E \in E(\mathbf{H}) \colon E \subset S\}$)
(iv) $|\mathscr{C}| = (\varepsilon n/r)(1 \pm \delta')$,
(v) $|S| = ne^{-\varepsilon}(1 \pm \delta')$,
(vi) $deg_{\mathbf{H}^*}(x) = De^{-\varepsilon(r-1)}(1 \pm \delta')$ *for all but at most* $\delta'|S|$ *vertices of* H^*.

Proof of Lemma 8.14. We will choose $\mathscr{C}$ randomly. We will follow the argument in [212], which does not even use the Chernoff inequality. In the proof our main tool is the Chebishev's Inequality which says that for a random variable ξ with mean value $E(\xi)$ and variance $D^2(\xi)$ one has

$$\text{Prob}(|\xi - E(\xi)| > \lambda D(\xi)) \leq \frac{1}{\lambda^2} \tag{8.11}$$

for all $\lambda > 0$. The variables $\delta_1, \delta_2, \ldots$ of the following proof are all explicitly computable functions of δ which can be made arbitrarily small by making δ appropriately small.

Define a random family $\mathscr{C}$ by

$$\text{Prob}(C \in \mathscr{C}) = p = \varepsilon/D \quad \text{for all } C \in E(\mathbf{H}).$$

These events mutually independent. As $E(\mathbf{H}) = (\sum deg_{\mathbf{H}}(x))/r$, we have, that for some δ_1 ($\delta_1 \leq (1 + K)\delta$)

$$||E(\mathbf{H})| - nD/r| < \delta_1 nD$$

holds. The expected value of $|\mathscr{C}|$ is $|E(\mathbf{H})|p = \varepsilon n/r \pm \delta_1 \delta n$, and the variance of $|\mathscr{C}|$ is $|E(\mathbf{H})|p(1-p) \leq 2n\varepsilon/r$. Then (8.11) implies

$$\text{Prob}(|\mathscr{C}| = (\varepsilon n/r)(1 \pm \delta_2)) \to 1$$

for some δ_2, whenever $n \to \infty$. So we have proved that (iv) holds almost surely.

Now we prove that (v) holds almost surely (or at least with probability greater than 0.99). For each $x \in V(\mathbf{H})$ let the random variable $\xi_x = 1$ if $x \notin \bigcup \mathscr{C}$, 0 otherwise. Set $\xi = \sum \xi_x$, so $\xi = |S|$. If for a vertex x we have $deg(x) = D(1 \pm \delta)$ then

$$E(\xi_x) = (1-p)^{deg(x)} = e^{-\varepsilon} \pm \delta_3.$$

For other x, $0 \leq E(\xi_x) \leq 1$ so

$$E(\xi) = (e^{-\varepsilon} \pm \delta_4)n.$$

We are going to give an upper bound on the variance

$$D^2(\xi) = \sum D^2(\xi_x) + \sum_{x \neq y}\sum Cov(\xi_x, \xi_y).$$

Then,

$$\sum D^2(\xi_x) \leq \sum E(\xi_x) = E(\xi) = o(E(\xi)^2). \tag{8.12}$$

And,

$$\begin{aligned} Cov(\xi_x, \xi_y) &= E(\xi_x \xi_y) - E(\xi_x)E(\xi_y) \\ &= (1-p)^{deg(x)+deg(y)-deg(x,y)} - (1-p)^{deg(x)+deg(y)} \\ &\leq (1-p)^{-deg(x,y)} - 1 < e^{\varepsilon\delta} - 1 < \delta_5. \end{aligned} \tag{8.13}$$

Then (8.12) and (8.13) imply that $D^2(\xi) < \delta_6 E(\xi)^2$, and again by the Chebishev's inequality we have

$$\text{Prob}(\xi = ne^{-\varepsilon}(1 \pm \delta_7)) > 0.99$$

giving (v).

The first step of the proof of (vi) is to observe that for almost all vertices x (i.e., all but $\leq \delta_8 n$) have the following properties
(a) $deg_{\mathbf{H}}(x) = (1 \pm \delta)D$
(b) For all but $\delta_9 D$ edges $E \in E(\mathbf{H})$, $x \in E$ one has that the number of

$$\text{edges } \{F \in E(\mathbf{H}) \colon x \notin F, E \cap F \neq \varnothing\} \text{ is } (1 \pm \delta_{10})(r-1)D.$$

(Here we used the fact that $d(x, y) < \delta D$.) Call an edge defined by (b) "good". We examine the distribution of $deg_{\mathbf{H}^*}(x)$ conditioning on the event $x \notin \bigcup \mathscr{C}$. (b) implies that any good edge survives with probability $e^{-\varepsilon(r-1)} \pm \delta_{11}$, hence

$$E(deg_{\mathbf{H}^*}(x)) = D(e^{-\varepsilon(r-1)} \pm \delta_{12}).$$

Using again the Chebishev inequality (after obtaining an upper bound on the variance) we get

$$\operatorname{Prob}(deg_{\mathbf{H}^*}(x) = e^{-\varepsilon(r-1)}D(1 \pm \delta_{13})) > 1 - \delta_{13}.$$

Hence the expected number of $x \in S$ with $deg_{\mathbf{H}^*}(x) \neq e^{-\varepsilon(r-1)}D(1 \pm \delta_{13})$ is at most $\delta_{14} n$. So with probability at least 0.99 one has $deg_{\mathbf{H}^*}(x) = e^{-\varepsilon(r-1)}D(1 \pm \delta_{14})$ for all but at most $\delta_{15} n$ vertices of $x \in S$, giving (vi).

Proof of Theorem 8.4 from Lemma 8.14. Fix $\varepsilon > 0$ such that $\varepsilon/(1 - e^{-\varepsilon}) + r\varepsilon < 1 + a$. Fix t integral such that $(1 - \varepsilon)^t < \varepsilon$, and fix $\delta > 0$ with $(1 + \delta)\left(\frac{\varepsilon}{1 - e^{-\varepsilon}} + r\varepsilon\right) < 1 + a$. We are going to use Lemma 8.14 t times. Define by reverse induction $\delta = \delta_t, \delta_{t-1}, \ldots, \delta_1, \delta_0$ so that $\delta_i \leq \delta_{i+1} e^{-\varepsilon(r-1)}$ and for $n > m_i$, $D > D_i$ one can use Lemma 8.14 with parameters

$$r = r, \quad \varepsilon = \varepsilon, \quad K = Ke^{\varepsilon(i+1)(r-1)}, \quad \delta' = \delta_{i+1}, \quad \delta = \delta_i.$$

Then Theorem 8.4 holds for $\gamma = \delta_0$, with $n_0 = \max n_i$, $D_0 = \max D_i$. Indeed, let $\mathbf{H}$ satisfy the conditions of 8.4. Apply 8.14 t times, we find $V_0 = V(\mathbf{H}) \supset V_1 \supset \cdots \supset V_t$ with $|V_i| = ne^{-\varepsilon i}(1 \pm \delta_i)$ and $\mathscr{C}_i$ with $V_{i+1} = V_i - \bigcup \mathscr{C}_i$, $|\mathscr{C}_i| = (\varepsilon n e^{-\varepsilon i}/r)(1 \pm \delta_i)$. For each $x \in V_t$ let C_x be an arbitrarily chosen edge containing x and let $\mathscr{C}_t$ denote the set of such edges. Then

$$\mathscr{C} = \mathscr{C}_0 \cup \cdots \cup \mathscr{C}_t$$

covers $V(\mathbf{H})$ and $|\mathscr{C}| \leq \frac{n}{r}(1 + \delta) \sum_{i=0}^{t-1} \varepsilon e^{-\varepsilon i} + n\varepsilon(1 + \gamma) < (n/r)(1 + a)$. □

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