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ON THE FRACTIONAL COVERING NUMBER OF HYPERGRAPHS*

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Abstract. The *fractional covering number* τ^* of a hypergraph $H = (V, E)$ is defined to be the minimum possible value of $\sum_{x \in V} t(x)$ where t ranges over all functions $t: V \to \mathbb{R}$ which satisfy $\sum_{x \in e} t(x) \geq 1$ for all edges $e \in E$. In the case of ordinary graphs G, it is known that $2\tau^*(G)$ is always an integer. By contrast, it is shown (among other things) that for any rational $p/q \geq 1$, there is a 3-uniform hypergraph H with $\tau^*(H) = p/q$.

Key words. hypergraphs, covering number

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1. Notation. A *hypergraph* $\mathcal{H}$ is a pair $(V(\mathcal{H}), E(\mathcal{H}))$ where $V(\mathcal{H})$ is a finite set (called *vertices*) and $E(\mathcal{H})$ is a family of subsets of $V(\mathcal{H})$ (called *edges*). The *rank* of $\mathcal{H}$ is the maximum size of an edge, $r(\mathcal{H}) := \max \{|E|: E \in E(\mathcal{H})\}$. If every edge has r elements then $\mathcal{H}$ is called an r-uniform hypergraph, or r-graph for short. The 2-uniform hypergraphs are called (simple) *graphs*. The *matching number* $\nu(\mathcal{H})$ of $\mathcal{H}$ is the maximum number of pairwise disjoint edges in $E(\mathcal{H})$, i.e.,

$$\nu(\mathcal{H}) = \max \{w: \text{there exists } E_1, \cdots, E_w \in E(\mathcal{H}), E_i \cap E_j = \emptyset \text{ for } i \neq j\}.$$

The *covering number* $\tau(\mathcal{H})$ of $\mathcal{H}$ is the minimum cardinality of a *cover* T, where $T \subset V(\mathcal{H})$ is a cover if $T \cap E \neq \emptyset$ for all $E \in E(\mathcal{H})$. If $\emptyset \in E(\mathcal{H})$ then $\nu = \tau = \infty$. The great importance of these notions is supported by the fact that virtually all combinatorial problems can be reformulated as the determination of the covering or matching number of an appropriate hypergraph. The calculation of τ and ν for an arbitrary hypergraph is an NP-hard problem. Thus, any result that gives estimates, at least for a certain class of hypergraphs, is especially valuable. One of the simplest estimates can be obtained from the *linear programming* bound, in other words, from the real relaxations of τ and ν. A *fractional matching* of $\mathcal{H}$ is a function $w: E(\mathcal{H}) \to \mathbb{R}$ (i.e., a vector in $\mathbb{R}^{|E(\mathcal{H})|}$) satisfying $w(E) \geq 0$ for every edge $E \in E(\mathcal{H})$ and

$$\sum \{w(E): x \in E \in E(\mathcal{H})\} \leq 1 \quad \text{for every } x \in V(\mathcal{H}).$$

The *value* of the fractional matching w is defined to be $|w| = \sum \{w(E): E \in E(\mathcal{H})\}$. The maximum of $|w|$ when w ranges over all fractional matchings is called the *fractional matching number* and is denoted by

$$\nu^*(\mathcal{H}) = \max \{|w|: w \text{ is a fractional matching of } \mathcal{H}\}.$$

Similarly, the *fractional covering number* is the minimum value of fractional covers of $\mathcal{H}$, i.e.,

$$\tau^*(\mathcal{H}) = \min \left\{ \sum_{x \in V(\mathcal{H})} t(x): t: V(\mathcal{H}) \to \mathbb{R}, t(x) \geq 0, \text{ and } \sum_{x \in E} t(x) \geq 1 \text{ for all } E \in E(\mathcal{H}) \right\}.$$

The determination of the fractional matching and covering number is a linear programming problem. This is a dual pair, so by the Duality Principle of linear programming we have $\tau^*(\mathcal{H}) = \nu^*(\mathcal{H})$ for every hypergraph $\mathcal{H}$. In general for every $\mathcal{H}$ and for every

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fractional cover t and matching w we have

$$|w| \leqq |t|,$$

so that if we have $|w| = |t|$ then both are optimal.

A subhypergraph $\mathscr{H}'$ is formed by a subset of edges of $\mathscr{H}$, i.e., $E(\mathscr{H}') \subset E(\mathscr{H})$, $\cup \{E \in E(\mathscr{H}')\} \subset V(\mathscr{H}') \subset V(\mathscr{H})$.

2. Fractional matchings in graphs. Edmonds [E] pointed out that an old theorem of Tutte [T] implies that $2\tau^*(\mathscr{G})$ is always an integer for a graph $\mathscr{G}$. Balinski [B], Balinski and Spielberg [BS], and Nemhauser and L. Trotter [NT] proved that even much more is true. To state their results, define the *fractional matching polytope* of the hypergraph $\mathscr{G}$, denoted by FMP ($\mathscr{H}$), as the set of all fractional matching vectors in $\mathbb{R}^{|E(\mathscr{H})|}$, i.e.,

$$\text{FMP}(\mathscr{H}) = \{\mathbf{w} \in \mathbb{R}^{|E(\mathscr{H})|}: \{w(E)\}_{E \in E(\mathscr{H})} \text{ is a fractional matching of } \mathscr{H}\}.$$

Analogously, the fractional covering polytope (FCP) of $\mathscr{H}$ is

$$\text{FCP}(\mathscr{H}) = \{\mathbf{t} \in \mathbb{R}^{|V(\mathscr{H})|}: \{t(x)\}_{x \in V(\mathscr{H})} \text{ is a fractional cover of } \mathscr{H}\}.$$

These are obviously polyhedra. If we can effectively describe all their vertices and facets, then in a certain sense we can solve any optimization problem concerning fractional matchings and covers. This description was given in [B], [BS], and in [NT] (a discussion of this and more graph theoretical background can be found in Lovász [L79]). Their results imply:

(2.1) all the vertices of the polytopes FMP ($\mathscr{G}$) and FCP ($\mathscr{G}$) have coordinate values which are 0, $\frac{1}{2}$, or 1.

3. 3-graphs with arbitrary denominator. It is obvious that for arbitrary hypergraphs, a statement similar to (2.1) is not true. For every rational number $r = p/q$ ($\geqq 1$), there exists a hypergraph $\mathscr{H}$ with $\tau^*(\mathscr{H}) = p/q$ (e.g., the complete q-graph on p elements). Lovász [L75] proved that for every choice of integers $1 \leqq \nu \leqq \tau$ and rational number $r > 1$ satisfying $\nu \leqq r \leqq \tau$ there exists a hypergraph $\mathscr{H}$ with $\nu(\mathscr{H}) = \nu$, $\tau^*(\mathscr{H}) = r$, and $\tau(\mathscr{H}) = \tau$. (If $\nu = \tau = 1$ then necessarily $\tau = 1$.) However, his hypergraphs have large ranks. In this section we prove that a similar statement holds even for hypergraphs of rank 3. For a real number x, denote by $\{x\}$ its *fractional part*, i.e., $\{x\} = x - \lfloor x \rfloor$.

THEOREM 3.1. *Let $0 \leqq r < 1$ be a rational number. Then there exists a hypergraph $\mathscr{H}$ of rank 3 with $\{\tau^*(\mathscr{H})\} = r$.*

For the proof we are going to use the following constructions.

Example 3.2. (A hypergraph of rank 3 with $4k + 2$ edges, and with $\{\tau^*\} = 2^k/(2^{k+1} - 1)$.) We define $\mathscr{H}^2(k)$ as follows.

$$V(\mathscr{H}^2(k)) = \{x_1, x_2, \cdots, x_{3k}\} \cup \{a_1, a_2, \cdots, a_k\} \cup \{e_1, e_2\}.$$

Let $A_{2i-1} = \{x_{3i-2}, a_i\}$, $A_{2i} = \{x_{3i-1}, a_i\}$, $B_{2i-1} = \{x_{3i-2}, x_{3i-1}, x_{3i}\}$ for $1 \leqq i \leqq k$, and $B_{2i} = \{x_{3i}, x_{3i+1}\}$ for $1 \leqq i \leqq k-1$, and $B_0 = \{x_1, x_{3k}, e_1\}$, $E_0 = \{x_{3k}, e_0\}$, $E_1 = \{e_0, e_1\}$ (see Fig. 1).

To find $\tau^*(\mathscr{H}^2(k))$ consider the following fractional matching λ: $E(\mathscr{H}^2(k)) \to \mathbb{R}$ and cover t.

Denote $2^{k+1} - 1$ by N. Let

$$\lambda(A_{2i-1}) = \lambda(B_{2i-1}) = 2^{k-i}/N \quad \text{for } 1 \leqq i \leqq k,$$

$$\lambda(A_{2i}) = (N - 2^{k-i})/N \quad \text{for } 1 \leqq i \leqq k,$$

$$\lambda(B_{2i}) = (N - 2^{k-i})/N \quad \text{for } 0 \leqq i \leqq k-1,$$

$$\lambda(E_1) = 2^k/N \quad \text{and} \quad \lambda(E_0) = (2^k - 1)/N.$$

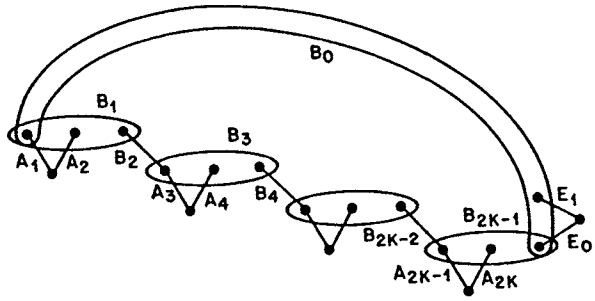

FIG. 1

Then $|\lambda| = 2k + 2^k/N$.

Define

$$t(a_i) = (N - 2^{i-1})/N \quad \text{for } 1 \leqq i \leqq k,$$

$$t(x_{3i-2}) = t(x_{3i-1}) = 2^{i-1}/N \quad \text{for } 1 \leqq i \leqq k,$$

$$t(x_{3i}) = (N - 2^i)/N \quad \text{for } 1 \leqq i \leqq k,$$

$$t(e_1) = (2^k - 1)/N, \qquad t(e_0) = 2^k/N.$$

Then t is a fractional cover with $|t| = 2k + 2^k/N$.

Example 3.3. (A hypergraph of rank 3 with $2k-1$ edges and with $\{\tau^*\}$ = (odd integer)/2^{k-1}). Define $\mathscr{H}^3(k)$ as follows. $V(\mathscr{H}^3(k)) = \{x_1, \cdots, x_k, y_1, \cdots, y_k\}$,

$$E(\mathscr{H}^3(k)) = \{\{x_k, y_k\}, A_i, B_i (1 \leqq i \leqq k-1)\}$$

where $A_i = \{x_i, y_i, x_{i+1}\}$, $B_i = \{x_i, y_i, y_{i+1}\}$. Then

$$\tau^*(\mathscr{H}^3(k)) = \frac{2k}{3} + \frac{2}{9} + \frac{(-1)^{k-1}}{9 \cdot 2^{k-1}}. \tag{3.1}$$

To prove (3.1) consider the following (optimal) fractional matching w and cover t:

$$w(A_i) = w(B_i) = \frac{1}{3} + \frac{(-1)^{i-1}}{3 \cdot 2^i}, \qquad (1 \leqq i \leqq k-1)$$

$$w(\{x_k, y_k\}) = \frac{2}{3} + \frac{(-1)^{k-1}}{3 \cdot 2^{k-1}},$$

and

$$t(x_i) = t(y_i) = \frac{1}{3} + \frac{(-1)^{k-i}}{3 \cdot 2^{k+1-i}}.$$

Proof of Theorem 3.1. For nonnegative integers $n_1, \cdots, n_s$ and hypergraphs $\mathscr{H}_1, \cdots, \mathscr{H}_s$, we denote the disjoint union of n_i copies of $\mathscr{H}_i$ by $\sum n_i \mathscr{H}_i$. Let $0 \leqq r < 1$ be rational, $r = p/q$, $(p, q) = 1$. Let $q = 2^a b$ where b is odd. Choose nonnegative integers A and B such that

$$p/q = \left\{\frac{A}{2^a} + \frac{B}{b}\right\}.$$

If a is *even*, define

$$\mathbf{H} := 9A\mathbf{H}^3(a+1) + 2B((2^{\phi(b)} - 1)/b)\mathbf{H}^2(\phi(b) - 1)$$

where ϕ denotes the Euler ϕ-function (i.e., $\phi(m)$ is the number of integers t, $1 \leqq t < m$, which are relatively prime to m).

If a is *odd*, define

$$\mathbf{H} := 18A\mathbf{H}^3(a+2) + 2B((2^{\phi(b)}-1)/b)\mathbf{H}^2(\phi(b)-1).$$

Then an easy calculation shows that $\{\tau^*(\mathscr{H})\} = r$, as required. □

4. An upper bound on the denominator. Let $N_r = \{\tau^*(\mathscr{H})$: $\mathscr{H}$ has rank at most $r\}$. Then $N_2 = \{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \cdots\}$ and $\cup_{r\geqq 2} N_r$ consists of all rationals not smaller than 1.

THEOREM 4.1. *If $u/v \in N_r$, $(u, v) = 1$, then $u/v \geqq (2 \log v)/(r \log r)$. In particular, the set N_r is a discrete sequence.*

The proof of this result is based on the following ideas. A hypergraph $\mathscr{H}$ is called τ^*-*critical* if $\tau^*(\mathscr{H}') < \tau^*(\mathscr{H})$ holds for each subhypergraph $\mathscr{H}'$ of $\mathscr{H}$, i.e., we cannot delete an edge without changing (decreasing) the value of τ^*.

LEMMA 4.2 (Füredi [Fü81]). *If $\mathscr{H}$ is τ^*-critical, then $|E(\mathscr{H})| \leqq |\cup\{E \in E(\mathscr{H})\}|$, i.e., $\mathscr{H}$ has no more edges than nonisolated vertices.*

Other (more general) versions of this lemma are well known in the theory of linear programming. This lemma just means that the number of constraints of a linear program can be reduced to the number of variables without changing the optimal value.

LEMMA 4.3. *If $\mathscr{H}$ is τ^*-critical of rank r then $|E(\mathscr{H})| \leqq r\tau^*$.*

Proof of 4.3. There exists an optimal fractional matching w_0 (i.e., $|w_0| = \tau^*$) which is a vertex of the fractional matching polytope FMP ($\mathscr{H}$). Thus, the vector $\{w_0(E)\}_{E\in E(\mathscr{H})}$ is contained in at least $|E(\mathscr{H})|$ facets of FMP ($\mathscr{H}$), say

$$w(E) = 0 \quad \text{if } E \in \mathscr{E}_0 \subset E(\mathscr{H}),$$

$$\sum_{x\in E} w(E) = 1 \quad \text{if } x \in V_0 \subset V(\mathscr{H}),$$

where $|V_0| + |\mathscr{E}_0| \geqq |E(\mathscr{H})|$. We have

$$|V_0| = \sum_{x\in V_0}\left(\sum_{x\in E} w(E)\right) = \sum_{E\in E(\mathscr{H})} w(E)|E\cap V_0| \leqq \sum w(E)r = \tau^* r.$$

Hence $|E(\mathscr{H}) - \mathscr{E}_0| \leqq r\tau^*$. Let $E(\mathscr{H}') = E(\mathscr{H}) - \mathscr{E}_0$. Then $\tau^*(\mathscr{H}') \geqq |w_0|$, since w_0 is a fractional matching of $\mathscr{H}'$. However, $\tau^*(\mathscr{H}') \leqq \tau^*(\mathscr{H})$, i.e., $\tau^*(\mathscr{H}') = \tau^*(\mathscr{H})$. $\mathscr{H}$ is τ^*-critical, so we have $\mathscr{H} = \mathscr{H}'$, $|E(\mathscr{H})| \leqq r\tau^*$. □

Applications of these lemmas can be found in [Fü86], [FF]. We now move to the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $\mathscr{H}$ be a hypergraph of rank r with $\tau^*(\mathscr{H}) = u/v$. We can assume that $\mathscr{H}$ is τ^*-critical. Hence, $|E(\mathscr{H})| \leqq (ru)/v$ by Lemma 4.3. The value of τ^* can then be obtained (by Cramer's rule) as a ratio of two 0-1 determinants of size at most $(ru)/v$. Every row contains at most r 1's by Hadamard's upper bound,

$$v = |\text{denominator}| = |\det(0\text{-}1 \text{ matrix})| \leqq r^{(1/2)ru/v}.$$

PROPOSITION 4.4. *The smallest seven elements of N_3 are* $1, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, 2$.

Proof. Figure 2 shows the incidence matrices of seven hypergraphs of rank 3 with these fractional matching numbers (optimal fractional matchings and coverings are also indicated). (By Lemmas 4.2 and 4.3, it is sufficient to consider hypergraphs having at most five edges and vertices.) A short case-by-case study shows that these are all values of N_3 which are not larger than 2. □

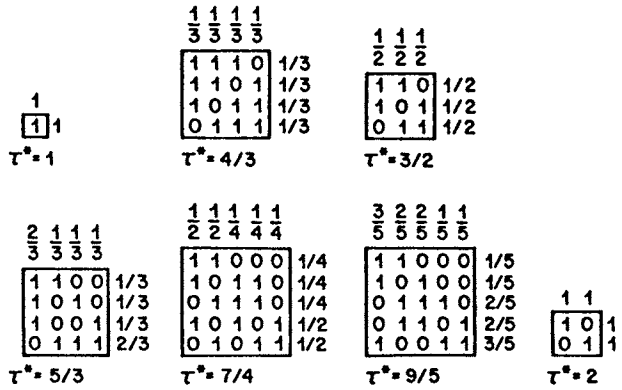

FIG. 2

5. Remarks.

Remark 5.1. One can prove in a way similar to that used for Lemma 4.3 that every vertex $\mathbf{c} \in \mathbb{R}^{|E(\mathscr{H})|}$ of FMP $(\mathscr{H})$ has at most $r\tau^*$ nonzero coordinates. This implies that there exists an integer $M = M(\mathbf{c}) \leqq r^{(1/2)r\tau^*}$ such that $M\mathbf{c}$ is an integer point.

Remark 5.2. Let $N_r = \{t_1^{(r)}, t_2^{(r)}, \cdots, t_i^{(r)}, \cdots\}$, $t_i^{(r)} < t_{i+1}^{(r)}$. Although this is a discrete sequence, Theorem 3.1 implies that for $r \geqq 3$,

$$\lim_{i \to \infty} (t_{i+1}^{(r)} - t_i^{(r)}) = 0$$

(since $\tau^*(\mathscr{H}') = 1 + \tau^*(\mathscr{H})$ if $\mathscr{H}'$ is formed from $\mathscr{H}$ by adding a single disjoint edge).

Remark 5.3. Define $d_r(n) = \max\{\text{denominator of } \tau^*(\mathscr{H})\colon \mathscr{H} \text{ is an } r\text{-graph with } |E(\mathscr{H})| \leqq n\}$. The examples in § 3 and Theorem 4.1 imply that

$$n \log \sqrt{2} + O(1) \leqq \log d_3(n) \leqq n \log \sqrt{3}.$$

It seems likely that $\lim_{n+\infty} \log d_3(n)/n$ exists. If so, is it equal to $\frac{1}{2} \log 2$?

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