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Colored packing of sets

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TO ALEX ROSA ON HIS FIFTIETH BIRTHDAY

ABSTRACT

Let $\mathcal{H}$ be a family of t-sets on $\{1,2, \ldots, k\}$. A family $\mathcal{F}$ of k-sets on v elements is called a $(v,k,\mathcal{H})$-packing if for all $F \in \mathcal{F}$ there is a copy of $\mathcal{H}$, $\mathcal{H}_F$ such that the t-sets of F corresponding to $\mathcal{H}_F$ are covered only by F. Clearly, $|\mathcal{F}| \leq \binom{v}{t} / |\mathcal{H}|$, and if $\mathcal{H}$ is the complete t-hypergraph then we obtain the usual definition of the (partial) Steiner-system. The main result of this paper is that for every fixed k and $\mathcal{H}$ the size of the largest $\mathcal{H}$-packing is $(1-o(1)) \binom{v}{t} / |\mathcal{H}|$, whenever $v \rightarrow \infty$

1. Preliminaries. Packings and near perfect matchings

Let X be a v-element set, $X = \{1,2, \ldots, v\}$. For an integer k, $0 \leq k \leq v$ we denote the collection of all k-subsets of X by $\binom{X}{k}$, while 2^X denotes the power set of X. A family of subsets of X is just a subset of 2^X. It is called *k-uniform* if it is a subset of $\binom{X}{k}$. A *Steiner-system* $\mathcal{S} = S(v,k,t)$ is an $\mathcal{S} \subset \binom{X}{k}$ such that for every $T \in \binom{X}{t}$ there is exactly one $B \in \mathcal{S}$ with $T \subset B$. Obviously, $|\mathcal{S}| = \binom{v}{t} / \binom{k}{t}$ holds. A $\mathcal{P} \subset \binom{X}{k}$ is called a *(v,k,t)-packing* if $|P \cap P'| < t$ holds for every pair $P,P' \in \mathcal{P}$. Rödl

[16] proved that

$$\max\left\{|\mathcal{P}|: \mathcal{P} \text{ is a } (v,k,t)\text{—packing}\right\} = (1-o(1))\binom{v}{t}\Big/\binom{k}{t} \tag{1}$$

holds for every fixed k,t whenever $v \to \infty$

This theorem was generalized by Frankl and Rödl [13]. To state it, we recall some definitions. For a family of finite sets $\mathcal{F}$ and an arbitrary set A the *degree* of A is defined by $deg_{\mathcal{F}}(A) =: |\{F \in \mathcal{F}: A \subset F\}|$. For $A = \{a\}$ we set $deg_{\mathcal{F}}(a) =: deg_{\mathcal{F}}(\{a\})$, the usual definition of the degree of an element. A *matching* $\mathcal{M}$ of $\mathcal{F}$ is a subfamily of pairwise disjoint members, $\mathcal{M} \subset \mathcal{F}$, $M \cap M' = \emptyset$ for all $M, M' \in \mathcal{M}$. The largest cardinality of a matching is denoted by $\nu(\mathcal{F})$. Clearly, for $\mathcal{F} \subset \binom{X}{k}$ we have

$$\nu(\mathcal{F}) \leq v/k. \tag{2}$$

(Frankl and Rödl [13]) For every $\epsilon > 0$ and k there exists a $\delta > 0$ and a $v_0 = v_0(k,\epsilon)$ such that if $\mathcal{F} \subset \binom{X}{k}$, and every degree of $\mathcal{F}$ is almost d (i.e., $|deg_{\mathcal{F}}(x) - d| \leq \epsilon d$ holds for every $x \in X$) and for every $x,y \in X$ we have $deg_{\mathcal{F}}(\{x,y\}) < d/(\log v)^3$ then

$$\nu(\mathcal{F}) > \frac{v}{k}(1-\delta)$$

holds for $v > v_0$.

For a family of sets $\mathcal{G} \subset 2^X$ the subset $A \subset X$ is an *own* part of $G \in \mathcal{G}$ if $A \subset G$ and $deg_{\mathcal{G}}(A) = 1$, i.e., A is contained only in G. Hence $\mathcal{G} \subset \binom{X}{k}$ is a (v,k,t)-packing if and only if every $G \in \mathcal{G}$ has $\binom{k}{t}$ own t-subsets. The aim of this paper is to construct such families $\mathcal{F}$ in which for every $F \in \mathcal{F}$ the family of own t-subsets of F is isomorphic to a given t-uniform hypergraph $\mathcal{H}$. Such a family is called an *$\mathcal{H}$-packing*. If $\mathcal{H}$ is the complete t-hypergraph on k elements, an $\mathcal{H}$-packing is just the usual (v,k,t)-packing. The existence of large $\mathcal{H}$-packings is proved in Chapter 2 and 5. The proof is probabilistic, the main tool for the construction is (2). In Chapter 3 we give an application solving (at least asymptotically) the question: what is the maximum size of a family $\mathcal{F} \subset \binom{X}{k}$ such that none of the members is contained in the union of r others.

2. $\mathcal{H}$-packings and colored $\mathcal{H}$-packings

Let $\mathcal{H}$ be a family of t-sets over k elements. Suppose that $\mathcal{F} \subset \binom{X}{k}$ where $|X| = v$ and for every $F \in \mathcal{F}$ there exists a copy of $\mathcal{H}$ on F (i.e., $\mathcal{H}_F \subset \binom{F}{t}$, $\mathcal{H}_F \approx \mathcal{H}$). If every t-set $T \in \mathcal{H}_F$ is covered only by F (i.e., $deg_{\mathcal{F}}(T) = 1$) then we call $\mathcal{F}$ a $(v,k,\mathcal{H})$-*packing* (or, briefly, $\mathcal{H}$-packing). Clearly,

$$|\mathcal{F}| \leq \binom{n}{t} / |\mathcal{H}| . \tag{3}$$

E.g., the following family $\mathcal{F}$ is a $(v,4,C_4)$-packing of size $(v^2/8) + O(v)$. $\mathcal{F} = \left\{\{2i-1,2i\} \cup \{2j-1,2j\}: 1 \leq i < j \leq v/2\right\}$.

Definition 2.1. Let $\mathcal{H} \subset \binom{K}{t}$, $|K| = k$, $c = \binom{k}{t} - |\mathcal{H}|$ and fix a partition $\binom{K}{t} = \mathcal{H} \cup \{T_1\} \cup \cdots \cup \{T_c\}$. In other words, this is a coloring χ: $\binom{K}{t} \rightarrow \{0,1, \ldots, c\}$ with $\chi(T) = 0$ for $T \in \mathcal{H}$. The family $\mathcal{F} \subset \binom{X}{k}$, $|X| = v$ is called a *colored* $(v,k,\mathcal{H})$-*packing* if

(i) $|F \cap F'| \leq t$ holds for every two $F, F' \in \mathcal{F}$, and

(ii) there exists a coloring of the t-sets of X with $c+1$ colors $\binom{X}{t} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_c$ such that for every $F \in \mathcal{F}$ the induced coloring of $\binom{F}{t}$ is isomorphic to χ, especially $\mathcal{C}_0 \cap \binom{F}{t} \approx \mathcal{H}$. (I.e., there exists an injection π_F: $F \rightarrow K$ such that for $T \in \binom{F}{t}$ we have $T \in \mathcal{C}_{\chi(\pi_F(T))}$.) E.g., the following family $\mathcal{F}$ is a colored $(v,4,C_4)$-packing of size $(v^2/8) + O(v)$: $\mathcal{F} = \left\{\{2i-1,2i\} \cup \{2j,2j+1\}: 1 \leq i < j < v/2\right\}$. We prove that the upper bound in (3) for the size of $\mathcal{F}$ is essentially the best possible.

Theorem 2.2. For every given k and $\mathcal{H}$ the size of the largest colored $(v,k,\mathcal{H})$-packing is $(1-o(1)) \binom{v}{t} / |\mathcal{H}|$ when $v \rightarrow \infty$

In the proof we will use the following

Lemma 2.3. Suppose $n > n_0(k)$, $k > t \geq 1$, $D \leq n^{2/3}$. Then there exists a family of k-sets $\mathbf{S}$ on the n-element set N such that

(i) $|S \cap S'| \leq t$ holds for every two $S, S' \in \mathbf{S}$

and

(ii) $|deg_{S}(T) - D| < 2\sqrt{kD\log n}$ holds for every $T \in \binom{N}{t}$. In other words, there exists a "near" $D-(n,k,t)$-design. Of course, for $D < 4k \log n$ (ii) is meaningless.

Let $f(v,k,\mathcal{H}) = \max\{|\mathcal{F}|: \mathcal{F}$ is a $(v,k,\mathcal{H})$-packing$\}$ and $f_c(v,k,\mathcal{H}) = \max\{|\mathcal{F}|: \mathcal{F}$ is a colored $(v,k,\mathcal{H})$-packing$\}$. Then $f_c \leq f$. If $|\mathcal{H}| < \binom{k}{t}$ then we cannot expect equality in (3). The following two results are consequences of a theorem of Bollobás [2].

(4) ([2]) If $|\mathcal{H}| = 1$, then $f(v,k,\mathcal{H}) = \binom{v-k+t}{t}$.

(5) ([7]) If $|\mathcal{H}| = \binom{s}{t}$ a complete hypergraph then for $v > v_0(k)$ we have $f(v,k,\mathcal{H}) \leq \binom{v-k+s}{t} / \binom{s}{t}$, where equality holds if and only if there exists a Steiner system $S(v-k+s,\ s,t)$.

Another special case of Theorem 2.2 (when $\mathcal{H}$ is a star) was proved in [7]. Theorem 2.2 says that $|f(v,k,\mathcal{H}) - \binom{v}{t} / |\mathcal{H}|| = o(v^t)$. In the cases (4) and (5) $\binom{v}{t} / |\mathcal{H}| - f(v,k,\mathcal{H}) > O(v^{t-1})$. In general, the gap may be much larger: Let K be the graph with vertex-set $\{a,b,c,d\}$ and edges $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$.

Proposition 2.4. Suppose that $\mathcal{H}$ is a graph, and that it has an induced subgraph isomorphic to K. Then

$$\binom{v}{2} / |\mathcal{H}| - f(v,k,\mathcal{H}) > \frac{1}{2\sqrt{2}\,|\mathcal{H}|}\, v^{3/2} .$$

3. r-cover-free families

We call the family of sets $\mathcal{F}$ *r-cover-free* if $F_0 \not\subset F_1 \cup \cdots \cup F_r$ holds for all $F_0, F_1, \ldots, F_r \in \mathcal{F}$ ($F_i \neq F_j$ for $i \neq j$). Let us denote by $f_r(n,k)$ the maximum cardinality of an r-cover-free family $\mathcal{F} \subset \binom{N}{k}$, $|N| = n$. Let us set $t = \lceil k/r \rceil$ (upper integer part). An (n,k,t)-packing is r-cover-free, hence by (1)

$$f_r(n,k) \geq (1-o(1)) \binom{n}{t} \Big/ \binom{k}{t}.$$

On the other hand every $F \in \mathcal{F}$ has an own t-subset. Indeed, F can be covered by r t-sets, $F = T_1 \cup \cdots \cup T_r$, $T_i \in \binom{F}{t}$. If for every i $deg_{\mathcal{F}}(T_i) > 1$ then F is covered by r others, which is a contradiction. This yields

$$f_r(n,k) \leq \binom{n}{t}.$$

Proposition 3.1. ([7]) For fixed k and r

$$\lim_{n \to \infty} f_r(n,k) \Big/ \binom{n}{t} = \lim_{n \to \infty} sup\ f_r(n,k) \Big/ \binom{n}{t} =: c_r(k)$$

exists and is positive.

In the next theorem we determine the value of $c_r(k)$, or at least we show that the calculation of it is a finite problem depending only on k.

Definition 3.2. Let k,t,l be positive integers $k \geq t(l+1)$, and $m(k,t,l) = \max\{|\mathcal{N}|: \mathcal{N} \subset \binom{K}{t}, |K| = k, \mathcal{N}$ does not contain $l+1$ pairwise disjoint members$\}$.

For $k < t(l+1)$ define $m(k,t,l) = \binom{k}{t}$ and $m(k,t,0) = 0$. Considering all the t-sets intersecting a given l-set we obtain that $m(k,t,l) \geq \binom{k}{t} - \binom{k-l}{t}$ holds. In several cases this is best possible:

(6) (Erdős, Ko and Rado [9]) $m(k,t,l) = \binom{k-1}{t-1}$ for $k \geq 2t$,

(7) (Erdős [5]) $m(k,t,l) = \binom{k}{t} - \binom{k-l}{t}$ for $k > k_0(t,l)$.

Later $k_0(t,l) < 2t^3 l$ was established by Bollobás, Daykin and Erdős [3]. We can prove $k_0(t,l) < 2t^2 l$ (cf. [12]). The case $t=2$ was solved completely by Erdős and Gallai [8] in 1959:

$$m(k,2,l) = \max\left\{\binom{2l+1}{2}, \binom{k}{2} - \binom{k-l}{2}\right\} \tag{8}$$

for $k \geq 2l+1$ and the only extremal graphs are either K_{2l+1} or $K_k - K_{k-l}$.

Conjecture 3.3. (Erdős [5]) $m(k,t,l) = \max\left\{\binom{tl+t-1}{t}, \binom{k}{t} - \binom{k-l}{t}\right\}$ for all $k \geq (t+1)l$.

A general upper bound was given by Frankl (see, e.g., in [10] or [11])

$$m(k,t,l) \leq l\binom{k-1}{t-1}.$$

For k and r let $k = r(t-1) + l+1$ where $0 \leq l < r$ (i.e., $t = \lceil k/r \rceil$).

Theorem 3.4. $c_r(k) = \left(\binom{k}{t} - m(k,t,l)\right)^{-1}$.

Other results and additional constructions can be found in [6] and [7] where exact results are proved for the case $r=2$ and the cases (6)-(8), and also for $k = 2r$.

4. Proof of Lemma 2.3

We are going to use the following consequence of Chernoff inequality [4] (originally, the Bernstein's improvement of Chebysheff inequality, see, e.g., Rényi [15]).

(9) Let $Y_1, \ldots, Y_m$ be independent random variables with $Prob(Y_i=1) = q$, $Prob(Y_i=0) = 1-q$, then

$$Prob(|\sum Y_i - mq| > \alpha\sqrt{mq}) < 2e^{-\alpha^2/2}.$$

For every $F \in \binom{N}{k}$ let Y_F be a random variable

$$Prob(Y_F=1) = D \Big/ \binom{n-t}{k-t}, \tag{10}$$

$$Prob(Y_F=0) = 1 - Prob(Y_F=1) .$$

Let $\mathcal{F}$ be the random family, defined by $\mathcal{F} = \left\{F \in \binom{N}{k} : Y_F = 1\right\}$. For $T \in \binom{N}{t}$ define $Y_T = \sum \left\{Y_F : T \subset F\right\}$. Then

$$E(Y_T) = D .$$

As Y_T is a sum of $\binom{n-t}{k-t}$ independent random variables (9) gives that for every fixed T

$$Prob(|Y_T - D| > (2kD\log n)^{1/2}) < \frac{2}{n^k} .$$

Hence

$$Prob(\exists T \in \binom{N}{t} \text{ with } |Y_T - D| > (2kD\log n)^{1/2}) < \frac{2\binom{n}{t}}{n^k} = o(1) . \tag{11}$$

Proposition 4.1. $Prob(\exists U \in \binom{N}{t+1}$ with $deg_{\mathcal{F}}(U) \geq 3k) < \frac{1}{t+1}$.

Proof: We can choose a set U in $\binom{n}{t+1}$ distinct ways. Then we can choose $3k$ k-sets through U in $\binom{\binom{n-t-1}{k-t-1}}{3k}$ ways. The probability of the appearance of such a configuration is $(D/\binom{n-t}{k-t})^{3k}$. Altogether, the probability in the left hand side is not larger than

$$\binom{n}{t+1}\binom{\binom{n-t-1}{k-t-1}}{3k}\left(\frac{D}{\binom{n-t}{k-t}}\right)^{3k} < \frac{n^{t+1}}{(t+1)!}\left(\frac{\binom{n-t-1}{k-t-1}}{\binom{n-t}{k-t}}\right)^{3k}\frac{D^{3k}}{(3k)!}$$

Using $\binom{n-t-1}{k-t-1}/\binom{n-t}{k-t} = (k-t)/(n-t) < k/n$ and $D \le n^{2/3}$ we obtain 4.1. □

Proposition 4.2. Let $s = 6\left\lfloor k^{-2.5}(D\log n)^{0.5}\right\rfloor$. Then the probability that there exists a $T \in \binom{N}{t}$ and $2s$ distinct members of $\mathcal{F}$ such that $F_1, \ldots, F_s \in \mathcal{F}$ with $T \subset F_i$ and $F_{s+i} \in \mathcal{F}$ with $|F_i \cap F_{s+i}| > t$ $(1 \le i \le s)$ is less than $1/k^2$.

Proof: It is analogous to the 4.1. We can choose such a configuration in at most

$$\binom{n}{t}\binom{\binom{n-t}{k-t}}{s}\left(\binom{n-t-1}{k-t-1}\binom{k}{t+1}\right)^{s}$$

distinct ways. The probability of the appearance of each of these configuration in $\mathcal{F}$ is $(D/\binom{n-t}{k-t})^{2s}$. For n sufficiently large (e.g., $n > \exp(k^{10}4^k)$) an easy calculation gives 4.2. □

The proof of 2.3. Choose a family $\mathcal{F}$ at random as in (10). Then the sum of the probabilities in (11), 4.1 and 4.2 is $o(1) + 1/(t+1) + 1/k^2 < 1$. Hence there exists a family $\mathcal{F}$ without the configurations described in 4.1 and 4.2 and for which

$$|deg_{\mathcal{F}}(T) - D| < (2kD\log n)^{1/2} \tag{12}$$

holds for every $T \in \binom{N}{t}$.

Now call a set $F \in \mathcal{F}$ *bad* if there exists an $F' \in \mathcal{F}$, $F \ne F'$ with $|F \cap F'| > t$. Let $\mathcal{B} = \{F \in \mathcal{F}\colon F \text{ is bad}\}$, and define $\mathcal{S} = \mathcal{F} - \mathcal{B}$. We claim that $\mathcal{S}$ fulfils the constraints of 2.3. Obviously (i) holds. If we prove that for every $T \in \binom{N}{t}$

$$deg_B(T) < (2-\sqrt{2})\sqrt{kD\log n}$$

holds, then we are ready by (12). Suppose on the contrary, that for some T we have $deg_B(T) > 3k^3s$. Then by 4.1 we can find $B_1, \ldots, B_{sk} \in \mathcal{B}$, $B_i \supset T$ such that $(B_i - T) \cap (B_j - T) = \emptyset$ for $1 \le i < j \le sk$. There exists $B_i' \in B$ with $|B_i' \cap B_i| > t$. Then we can choose a subsequence of B_j's such that $B_{i_1}, B_{i_2}, \ldots, B_{i_s}$ and

$B'_{i_1}, \ldots, B'_{i_s}$ are 2s distinct members. This contradicts 4.2. □

5. Proof of Theorem 2.2

To avoid trivialities suppose that $t \geq 2$. To construct a colored $(v,k,\mathcal{H})$-packing we begin with a family $\mathcal{S} \subset \binom{X}{k}$, $|X| = v$, given by Lemma 2.3 with $D = \sqrt{v}$. In the following calculations we suppose that h is a small but fixed positive real depending only on k (e.g., $h = 4^{-k}$). Furthermore, we suppose that $v > v_0(k)$. Let $p = v^{-h}$ and let Z_T be a random variable for every $T \in \binom{X}{t}$ with distribution

$$Prob(Z_T = i) = p \quad \text{for} \quad i = 1, 2, \ldots, c \text{ and}$$

$$Prob(Z_T = 0) = 1 - cp \ .$$

In other words we color randomly and independently the t-sets of X. Recall that $c = \binom{k}{t} - |\mathcal{H}|$. Let $\mathcal{C}_i =: \{T: Z_T = i\}$, then for $i \geq 1$

$$E(|\mathcal{C}_i|) = p\binom{v}{t},$$

and by (9)

$$Prob\left(\left||\mathcal{C}_i| - p\binom{v}{t}\right| > v^{t-1}\right) < 2e^{-v^t} . \tag{13}$$

Call a set $S \in \mathcal{S}$ *well-colored* (with respect to the coloration $\{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_c\}$) if the restriction of the coloration to $\binom{S}{t}$ is isomorphic to χ. Denote the set of well-colored k-sets by $\mathcal{W}$. If g is the number of non-isomorphic embeddings of χ into $\binom{S}{t}$ then

$$Prob(S \text{ is well-colored}) = g\ p^c (1-p)^{\binom{k}{t}-c} .$$

Define an $|\mathcal{H}|$-uniform hypergraph $\mathcal{A}$ with vertex-set $\mathcal{C}_0$, $\mathcal{A} = \left\{\binom{S}{t} \cap \mathcal{C}_0 : S \in \mathcal{S} \text{ well-colored}\right\}$. Let $d =: gp^c(1-p)^{\binom{k}{t}-c-1} |\mathcal{H}| D / \binom{k}{t}$. By the choice of p and D we have

$$d > v^{0.4} \tag{14}$$

Proposition 5.1. Let $T \in \mathcal{C}_0$, then

$$Prob(|deg_{\mathcal{A}}(T) - d| > v^{0.3}) < e^{-v^h}.$$

Proof: Consider the sets $S_1, S_2, \ldots, S_l \in \mathcal{S}$ with $T \subset S_i$. As $T \in \mathcal{C}_0$, $Prob(S_i \in \mathcal{W}) = gp^c(1-c)^{\binom{k}{t}-c-1} |\mathcal{H}| / \binom{k}{t}$. Moreover these events are independent, hence by (9)

$$Prob(|deg_{\mathcal{A}}(T) - l| > v^h \sqrt{l}) < 2e^{-v^{2h}} < e^{-v^h}.$$

However $|deg_{\mathcal{A}}(T) - d| > v^{0.3}$ implies $|deg_{\mathcal{A}}(T) - l| > v^h \sqrt{l}$. □

This proposition and (13) yield that there exists a choice of $\{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_c\}$ for which

$$|\mathcal{C}_i| < p\binom{v}{t} + v^{t-1} < v^{t-h}$$

holds for all $i \geq 1$, and for every $T \in \mathcal{C}_0$

$$|deg_{\mathcal{A}}(T) - d| \leq v^{0.3}.$$

But by (14) $d > v^{0.4}$ and for every $T_1 \neq T_2$, $T_1, T_2 \in \binom{X}{t}$ we have $deg_{\mathcal{X}}(\{T_1, T_2\}) \leq 1 < d/(\log v)^3$. Thus we can apply (2) to $\mathcal{A}$. This gives that whenever $v \longrightarrow \infty$

$$v(\mathcal{A}) \geq (1-o(1))\ |\mathcal{C}_0| / |\mathcal{H}| = (1-o(1)) \binom{v}{t} / |\mathcal{H}|.$$

Finally, a matching $\mathcal{M}$ in $\mathcal{A}$ gives a colored $\mathcal{H}$-cover, $\mathcal{F} = \{S \in \mathcal{S}: \binom{S}{t} \cap \mathcal{C}_0 \in \mathcal{M}\}$. □

6. Proof of 2.4

Let $\mathcal{F}$ be a $(v,k,\mathcal{H})$-packing and let $\mathcal{G}$ denote the set of "crowded" edges, i.e., $\mathcal{G} = \{\{i,j\}$: *either* $\{i,j\}$ *is uncovered or* $deg_{\mathcal{F}}(\{i,j\}) \geq 2\}$. For a vertex $x \in X$ denote by f_x the number of $F \in \mathcal{F}$ for which the vertex corresponding to a in F is x. Then

$$\sum_{x \in X} f_x = |\mathcal{F}| = \frac{1}{|\mathcal{H}|}\left(\binom{v}{2} - |\mathcal{G}|\right). \tag{15}$$

The main point is that

$$f_x \leq \binom{deg_{\mathcal{G}}(x)}{2}. \tag{16}$$

Indeed, if $x = a_F$ and $\{a_F, b_F\} \notin \mathcal{H}_F$, $\{a_F, c_F\} \notin \mathcal{H}_F$ then, clearly $\{b_F, c_F\} \neq \{b_{F'}, c_{F'}\}$. Moreover,

$$f_x \leq n - 1 \tag{17}$$

holds for all $x \in X$ because $\{a_F, d_F\}$ is an own edge. Finally (15)-(17) give $|\mathcal{G}| \geq v^{3/2}/2\sqrt{2}\,|\mathcal{H}|$.

7. Proof of Theorem 3.4

Construction. Let $\mathcal{N}$ be a family of t-sets over the k-element set K such that $\mathcal{N}$ does not contain $l+1$ pairwise disjoint members and suppose that $|\mathcal{N}|$ is maximal, i.e., $|\mathcal{N}| = m(k,t,l)$. Let $\mathcal{H} = \binom{K}{t} - \mathcal{N}$ and let $\mathcal{F}$ be a colored $(n,k,\mathcal{H})$-packing with size

$$|\mathcal{F}| = (1-o(1))\binom{n}{t}\Big/|\mathcal{H}|.$$

We claim that $\mathcal{F}$ is an r-cover-free family. Suppose on the contrary, that $F_0 \subset F_1 \cup \cdots \cup F_r$. As $|F_0 \cap F_i| \leq t$ and $|F_0| = r(t-1) + l+1$, we have at least $l+1$ F_i such that $|(F_i \setminus (F_1 \cup \cdots \cup F_{i-1})) \cap F_0| \geq t$. Then $|F_0 \cap F_i| = t$ must hold and thus we have at least $l+1$ disjoint sets $F_0 \cap F_i$ such that $F_0 \cap F_i \in \mathcal{N}$, a contradiction.

Upper bound. Let $\mathcal{F}_0$ be an r-cover-free family. Let $\mathcal{F}_0$ be the sets with small own parts, i.e., $\mathcal{F}_0 = \{F \in \mathcal{F} : \exists U \subset F,\ |U| \leq t-1 \text{ such that } deg_{\mathcal{F}}(U) = 1\}$. Clearly $|\mathcal{F}_0| \leq \binom{n}{t-1}$. Consider an $F \in \mathcal{F} - \mathcal{F}_0$ and let $\mathcal{N}_F$ be the non-own parts of F with t elements, i.e., $\mathcal{N}_F = \{T \in \binom{F}{t} : \exists F' \in \mathcal{F} \text{ such that } F \cap F' \supset T\}$.

Proposition 7.1. If $F \in \mathcal{F} - \mathcal{F}_0$ then $\mathcal{N}_F$ does not contain $l+1$ pairwise disjoint members.

Proof: Suppose for contradiction that $T_1, \ldots, T_{l+1} \in \mathcal{N}_F$ with $|\bigcup T_i| = (l+1)t$. Let $\mathcal{P} = \{T_1, \ldots, T_{l+1}, S_1, \ldots, S_{r-l-1}\}$ be a partition of F such that $|S_i| = t-1$. Then for each $P \in \mathcal{P}$ there exists an $F_P \in \mathcal{F}$, $F_P \neq F$ with $P \subset F_P$. Hence $F \subset \bigcup\{F_P : P \in \mathcal{P}\}$, a contradiction. □

Now Proposition 7.1 implies that $|\mathcal{N}_F| \leq m(k,t,l)$, i.e., every $F \in \mathcal{F} - \mathcal{F}_0$ contains at least $\binom{k}{t} - m(k,t,l)$ own t-subsets. Hence

$$|\mathcal{F}| = |\mathcal{F}_0| + |\mathcal{F}-\mathcal{F}_0| \le \binom{n}{t-1} + \binom{n}{t} \Big/ \left(\binom{k}{t} - m(k,t,l)\right). \ \square$$

A slightly more complicated argument gives that for $n > n_0(k)$

$$|\mathcal{F}| \le \binom{n}{t} \Big/ \left(\binom{k}{t} - m(k,t,l)\right).$$

8. Final Remarks

Actually, using the argument in Chapter 5 we can prove the following stronger statement.

Theorem 8.1. Let $\mathcal{H}$ be a family of t-sets on $\{1,2,\dots,k\}$. There exists a family $\mathcal{F} \subset \binom{X}{k}$, $|X| = v$ of size

$$|\mathcal{F}| = (1-o(1)) \binom{v}{t} \Big/ |\mathcal{H}|$$

(whenever $v \to \infty$) with the following properties:

(i) $|F \cap F'| \le t$ for all distinct $F, F' \in \mathcal{F}$

(ii) For every $F \in \mathcal{F}$ there is a permutation of its elements $F = (x_1, \dots, x_k)$ such that whenever $|F \cap F'| = t$ and $F' = (y_1, \dots, y_k)$ and $F \cap F' = \{x_{i_1}, \dots, x_{i_t}\}$ then $x_{i_\alpha} = y_{i_\alpha}$ for $1 \le \alpha \le t$, and $\{i_1, \dots, i_t\} \notin \mathcal{H}$. It remains open whether we can suppose in this theorem that the orderings on each $F \in \mathcal{F}$ can be obtained as a restriction of an ordering of X.

Another open problem arises from the fact that our proof is probabilistic. It would be interesting to give other ("real") constructions. It is not necessarily hopeless, e.g., N. Alon [1] pointed out that an exponentially large r-cover-free family can be obtained using a recent explicit construction of J. Friedman [14] of certain generalized Justensen codes.

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