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Exact Solution of Some Turán-Type Problems

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Fifteen years ago Chvátal conjectured that if $\mathscr{F}$ is a family of k subsets of an n-set, $|\mathscr{F}| > \binom{n-1}{k-1}$, d is an arbitrary integer with $d \leqslant k-1$ and $(d+1)k \leqslant dn$, then there exist $d+1$ sets in $\mathscr{F}$ with empty intersection such that the intersection of any d of them is non-empty. The validity of this conjecture is established for $n \geqslant n_0(k)$, in a more general framework. Another problem which is solved asymptotically is when the excluded configuration is a fixed sunflower. © 1987 Academic Press, Inc.

PREFACE

Extremal problems in combinatorics have a long history. In 1907 Mantel [M] proved that every graph on n vertices and more than $n^2/4$ edges contains a triangle.

If $\mathscr{G}$ is an arbitrary graph one can define the corresponding extremal problem. Namely, let $\mathrm{ex}(n, \mathscr{G})$ denote the maximum number of edges in a graph on n vertices and not containing $\mathscr{G}$ as a subgraph. The graph $\mathscr{G}$ is called an excluded subgraph. Let C_r be the cycle of length r, $r \geqslant 3$.

Mantel's result can be restated as $ex(n, C_3) = \lfloor n^2/4 \rfloor$.

Already in 1938 Erdös [E1] considered the function $\mathrm{ex}(n, C_4)$, although in other terms. Nevertheless, the exact value of $\mathrm{ex}(n, C_4)$ is only known for $n = q^2 + q + 1$ where q is a prime power (cf. Füredi [Fü2]).

Turán [T1] determined $\mathrm{ex}(n, K_r)$ for all $r \geqslant 3$ in 1940. (K_r denotes the complete graph on r vertices).

In his memory the determination of $\mathrm{ex}(n, \mathscr{G})$ is called a Turán-type problem.

Turán-type problems are often very difficult and very little is known even

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about such simple cases when $\mathcal{G}$ is C_r or a fixed complete bipartite graph (cf. [Bo3] or [ES]).

Similar extremal problems were widely investigated for k-uniform hypergraphs, i.e., collections of k-element sets, as well.

The first such result was obtained by Erdös *et al.* [EKR] before World War II. They solved the case when the excluded hypergraph consists of two pairwise disjoint edges—the answer is $\binom{n-1}{k-1}$ for all $n \geqslant 2k$.

In general, hypergraph extremal problems are much more difficult than ordinary graph problems.

In this paper we give the solution (at least asymptotically) of a relatively wide class of hypergraph extremal problems. This class includes sunflowers (Δ-systems) and simplices (see the definition in the next section).

1. Introduction

Let X be a finite set, $|X| = n$ and let $\mathcal{F}$ be a family of k-element subsets of X, i.e., $\mathcal{F} \subset \binom{X}{k}$. Such a family is often called a k-graph. Suppose that k and d are positive integers.

Definition 1.1. We say that $\mathcal{F}$ contains a d-dimensional simplex if there exist $d+1$ sets $F_1, ..., F_{d+1} \in \mathcal{F}$ satisfying

(i) $\bigcap_{i=1}^{d+1} F_i = \varnothing$;

(ii) $\bigcap_{i \neq j,\, 1 \leqslant i \leqslant d+1} F_i \neq \varnothing$ for every $1 \leqslant j \leqslant d+1$.

In words, $d+1$ sets form a d-dimensional simplex if the intersection of all of them is empty but the intersection of every choice of d of them is non-empty.

For $d = 1$ a simplex consists simply of two disjoint non-empty sets.

For $d = 2$ a simplex is called a triangle.

It is easy to check that if $\mathcal{F}$ contains a d-dimensional simplex then $k \geqslant d$ holds. Also, if $k = d$ then the unique possibility for a d-dimensional simplex is to take all the k-subsets of a $(k+1)$-set.

Let us introduce the function $s(n, k, d)$:

$$s(n, k, d) = \max\{|\mathcal{F}| : \mathcal{F} \text{ contains no } d\text{-dimensional simplex}\}.$$

Thus $s(n, k, k)$ is the maximum number of edges in a k-graph $\mathcal{F} \subset \binom{X}{k}$ which contains no complete graph on $k+1$ vertices. For $k = 2$ the answer is $\lfloor n^2/4 \rfloor$ and it was already known to Mantel [M].

For $k \geqslant 3$ this is a special case of Turán's problem (cf. [T2, E7]) and it is one of the outstanding open problems in combinatorics. It is easy to see that $s(n, k, k) = \Omega(n^k)$, e.g., $s(n, k, k) \geqslant \lfloor n/k \rfloor^k$. For the currently known best upper and lower bounds see [C, FR]. Looking at all sets containing a fixed element shows that $s(n, k, d) \geqslant \binom{n-1}{k-1}$ holds for all $k > d$.

The fact that $s(n, k, 1) = \binom{n-1}{k-1}$ for $n \geqslant 2k$ is a special case of the Erdös–Ko–Rado Theorem [EKR] (see Theorem 4.3).

In 1971 Erdös made the following conjecture:

Conjecture 1.2 ([E4]). Suppose that $\mathscr{F}$ contains no triangle, $n \geqslant 3k/2$, $k \geqslant 3$. Then $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$ holds.

The year after, Chvátal stated the more general conjecture:

Conjecture 1.3 ([Ch1]). Suppose that $\mathscr{F}$ contains no d-dimensional simplex, $k > d$, $n > (d+1)k/d$. Then $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$ holds.

In [Ch2] Chvátal proved his conjecture in the special case $k = d+1$.

The validity of Conjecture 1.3 for the case $n \leqslant dk/(d-1)$ follows from Lemma 1 in [F1].

Bermond and Frankl [BF] proved Conjecture 1.3 for an infinity of the values (n, k, d), however, their method does not work for $n > k^2$.

Let us mention the following result for large values of n.

THEOREM 1.4 ([F4]). (i) *Suppose that $k \geqslant 5$, $n \geqslant n_0(k)$. Then Conjecture* 1.2 *is true.*

(ii) *Suppose that $k \geqslant 3d+1$ and $n \geqslant n_0(k)$. Then Conjecture* 1.3 *is true.*

(iii) *For all fixed $k > d$ we have*

$$s(n, k, d) \leqslant \binom{n-1}{k-1} + c_k n^{k-2},$$

where c_k is a constant depending only on k.

Using linear independence techniques we improve (iii):

THEOREM 1.5. *Suppose that $k > d$. Then we have*

$$s(n, k, d) \leqslant \binom{n}{k-1}. \tag{1}$$

For $n > n_0(k)$ we shall establish the validity of Conjecture 1.3.

THEOREM 1.6. *Suppose that $\mathscr{F} \subset \binom{X}{k}$, $\mathscr{F}$ contains no simplex of dimension d, $k > d$, $n > n_0(k)$. Then $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$. Moreover, the inequality is strict unless $\mathscr{F} = \{F \in \binom{X}{k} : x \in F\}$ holds for some $x \in X$.*

2. SUNFLOWERS

The family $\{D_1, ..., D_s\}$ is called a *sunflower* of size s and with *center* C if $D_i \cap D_j = C$ holds for all $1 \leqslant i < j \leqslant s$ (we assume also $D_i \neq D_j$).

Let us define the following function.

$$\phi(t, s) = \max\{|\mathscr{D}| : \mathscr{D} \text{ is a } t\text{-graph containing no sunflower of size } s\}.$$

Erdös and Rado [ER] proved that $\phi(t, s) \leqslant t!(s-1)^t$.

This result has had many applications in combinatorics and in computer science; cf. e.g., [Ra].

Let us mention that Erdös [E7] offers \$1000 for deciding whether $\phi(t, 3) \leqslant c^t$ holds for some absolute constant c.

Obviously, $\phi(t, 2) = 1$. Abbott, *et al.* [AHS] showed that $\phi(2, s) = s(s-1)$ for s odd and $\phi(2, s) = \lfloor (2s-1)(s-1)/2 \rfloor$ for s even. For s odd the only way to obtain equality is by taking the disjoint union of two complete graphs on s vertices. For s even there are many optimal graphs.

Except for these $\phi(t, s)$ is only known in the case $t = 3$, $s = 3$ when $\phi(3, 3) = 20$ holds.

Duke and Erdös [DE] introduced the following function.

DEFINITION 2.1. $f(n, k, l, s) = \max\{|\mathscr{F}| : \mathscr{F} \subset \binom{X}{k}, \mathscr{F}$ contains no sunflower of size s and with center of size $l\}$.

An old result of Erdös can be formulated as

THEOREM 2.2 [E3]. *Suppose that $n > n_0(k, s)$, then one has*

$$f(n, k, 0, s) = \binom{n}{k} - \binom{n-s+1}{k}.$$

The case $s = 2$ of the general problem, i.e., $\mathscr{F}$ contains no two sets intersecting in exactly l vertices goes back to Erdös [E5]. It received particular attention in view of possible geometric applications. Erdös and Sós (see [S] determined the value of $f(n, k, l, 2)$ for $k = 3$, $l = 1$. Larman [L] proved $f(n, 5, 2, 2) = O(n^2)$.

The present authors [FF3] proved that $f(n, k, l, 2) = \binom{n-l-1}{k-l-1}$ holds for $k \geqslant 2l + 2$ and $n \geqslant n_0(k)$ and they showed also that $f(n, k, l, 2) = O(n^l)$ for $k \leqslant 2l + 1$, as it was conjectured by Erdös [E5].

In view of a result of Füredi [Fü1] this implies that $f(n, k, l, s) = O(n^{\max\{l, k-l-1\}})$ for fixed k, l, and s. In Theorem 2.7, we determine the asymptotical value of $f(n, k, l, s)$ for $k \geqslant 2l + 3$.

Let us give two constructions.

EXAMPLE 2.3. Let $\mathscr{G}$ be a family of $(l+1)$ sets without any sunflower of size s and satisfying $|\mathscr{G}| = \phi(l+1, s)$. Set $Y = \bigcup \mathscr{G}$ and suppose that $Y \subset X$. Define

$$\mathscr{F} = \left\{ F \in \binom{X}{k} : F \cap Y \in \mathscr{G} \right\}.$$

Clearly $|\mathscr{F}| = (\phi(l+1, s) - o(1)) \binom{n-l-1}{k-l-1}$ and $\mathscr{F}$ contains no sunflower of size s and with center of size l.

EXAMPLE 2.4. Define $b = l - 1 + s(k-l)$. Let $\mathscr{B} \subset \binom{X}{b}$ be an l-packing, i.e., $|B \cap B'| < l$ holds for all distinct B, $B' \in \mathscr{B}$. Define

$$\mathscr{F} = \Delta_k(\mathscr{B}) = \left\{ F \in \binom{X}{k} : \exists B \in \mathscr{B},\ F \subset B \right\}.$$

Claim 2.5. The family $\mathscr{F}$ in Example 2.4 contains no sunflower of size s and with center of size l.

Proof. Suppose that $F_1, \ldots, F_s$ form such a sunflower and $F_i \subset B_i \in \mathscr{B}$. Since distinct F_i's have l elements in common while distinct B's at most $l-1$, it follows that $B_1 = B_2 = \cdots = B_s$. Thus $(F_1 \cup \cdots \cup F_s) \subset B_1$ holds. However, $|F_1 \cup \cdots \cup F_s| = l + (k-l)s > |B_1|$, a contradiction. ∎

Since $k > l$, $|\mathscr{F}| = \binom{b}{k} |B|$ holds. By a result of Rödl [R], for fixed b we can choose $|\mathscr{B}|$ as large as $(1 - o(1)) \binom{n}{l} / \binom{b}{l}$.

We conjecture that Example 2.3 is nearly best possible for $k \geqslant 2l+1$ while Example 2.4 is nearly best possible for $k < 2l+1$.

Conjecture 2.6. Suppose that k, l, s are fixed. Then one of the following holds:

(i) $k \geqslant 2l+1$ and $f(n, k, l, s) = (\phi(l+1, s) + o(1)) \binom{n-l-1}{k-l-1}$

(ii) $k < 2l+1$, and

$$f(n, k, l, s) = \left(\binom{l-1+s(k-l)}{k} + o(1) \right) \binom{n}{l} \Big/ \binom{l-1+s(k-l)}{l}.$$

Note that for $k = 2l+1$ both examples give $\Omega(\binom{n}{l})$ sets. However, $\phi(l+1, s) \geqslant (s-1)^{l+1}$ which is an upper bound for the coefficient of $\binom{n}{l}$ in (ii).

For the case $s = 2$, $k - l$ a prime power, Conjecture 2.6 was proved in [FW].

In the case $k = 3$ and $l = 1$ the conjecture was recently proved in a more exact form by Chung and Frankl [CF].

THEOREM 2.7. *Suppose that k, l, s are fixed and $k \geqslant 2l+3$. Then*

$$f(n, k, l, s) = (\phi(l+1, s) + o(1)) \binom{n-l-1}{k-l-1}. \tag{2.1}$$

An interesting feature of this result, that it can be proved although very little is known about the function $\phi(l+1, s)$.

Let us also note that Example 2.4 shows that $f(n, 4, 2, s) \geqslant (1-o(1))((2s-1)(s-1)/6)\binom{n}{2}$, which solves a problem of Chung *et al.* [CEG].

3. Special Simplices and a Problem of Kalai

DEFINITION 3.1. The collection $\mathscr{H} = \{H_1, ..., H_{d+1}\}$ is called a *special d-dimensional simplex* if for some $(d+1)$-element set $C = \{x_1, ..., x_{d+1}\}$ (called the center) $H_i \cap C = C - \{x_i\}$ holds, moreover the sets $H_i - C$ are pairwise disjoint, $i = 1, ..., d+1$.

THEOREM 3.2. *Suppose that $k \geqslant d+3$, $n > n_0(k)$ and $\mathscr{F} \subset \binom{X}{k}$ contains no special d-dimensional simplex. Then $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$, moreover, equality holds if and only if $\mathscr{F} = \{F \in \binom{X}{k}: x \in F\}$ for some $x \in X$.*

We believe that the same is true for the cases $k = d+1$, $d+2$ as well. However, we could only prove it for the case $d = 2$.

THEOREM 3.3. *Suppose that $k \geqslant 3$, $n \geqslant n_0(k)$ and $\mathscr{F} \subset \binom{X}{k}$ contains no special triangle. Then $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$, moreover, equality holds if and only if $\mathscr{F} = \{F \in \binom{X}{k}: x \in F\}$ for some $x \in X$.*

Note that for $k \geqslant 5$ Theorem 3.3 is covered by Theorem 3.2. The proof of the case $k = 3$ uses a rather involved weight-function argument while for the case $k = 4$ a refinement of the principal methods of the paper is needed.

For a family $\mathscr{F}$ and an integer l let us define $\Delta_l(\mathscr{F}) = \{L: |L| = l, \exists F \in \mathscr{F}, L \in F\}$.

Let us define the function $c(k, d)$ recursively for $k > d$ by setting $c(d+1, d) = 1$, $c(d+2, d) = 2d+1$ and $c(k+1, d) = kc(k, d) - 1$ for $k \geqslant d+2$. Note that $c(k, d) < (k-1)!$ holds for all $k > d \geqslant 2$. For the proof of Theorem 3.2 we need the following result.

THEOREM 3.4. *Suppose that $\mathscr{F} \subset \binom{X}{k}$, $\mathscr{F}$ contains no special d-dimensional simplex, $k > d$. Then*

$$|\Delta_{k-1}(\mathscr{F})| \geqslant |\mathscr{F}|/c(k, d). \tag{3.1}$$

DEFINITION 3.5. Let us define k-trees in the following inductive way. Every k-graph consisting of a single edge is a k-tree. Suppose that $\mathscr{T} = \{T_1, \dots, T_u\} \subset \binom{X}{k}$ is a k-tree, $S \in \binom{T_i}{k-1}$, $x \notin \cup \mathscr{T}$. Then $\{T_1, \dots, T_u, \{S \cup \{x\}\}\}$ is a k-tree.

Note that for graphs ($k = 2$) this leads to the usual notion of a tree.

Conjecture 3.6 (Kalai [K2]). Suppose that $\mathscr{F} \subset \binom{X}{k}$, u is a positive integer and $|\mathscr{F}| > u\binom{n}{k-1}/k$. Then $\mathscr{F}$ contains every k-tree with $u+1$ edges.

Note that the case $k = 2$ of the above conjecture is a famous open problem of Erdös and Sós (cf. [E8]).

To see that -if true- this bound is nearly best possible, consider a $(k-1)$-packing $\mathscr{P} \subset \binom{X}{k+u-1}$ of maximal size. By the result of Rödl [R] which we cited earlier, $|\mathscr{P}| = (1 - o(1))\binom{n}{k-1}/\binom{k+u-1}{k-1}$ holds. It is easy to check that the family $\mathscr{F} = \Delta_k(\mathscr{P})$ contains *no* k-tree with $u+1$ edges and

$$|\mathscr{F}| = \binom{k+u-1}{k} |\mathscr{P}| = (1 - o(1))\, u \binom{n}{k-1} \Big/ k.$$

We prove this conjecture for a very restricted class of k-trees.

DEFINITION 3.7. Call a k-tree star-shaped if it contains an edge which intersects all other edges in $k-1$ vertices.

THEOREM 3.8. *Suppose that* $\mathscr{F} \subset \binom{X}{k}$, u *is a positive integer and* $|\mathscr{F}| > u\binom{n}{k-1}/k$. *Then* $\mathscr{F}$ *contains every star-shaped tree with* $u+1$ *edges.*

4. MORE ON SIMPLICES AND A STRANGE PHENOMENON

DEFINITION 4.1. A family $\{S_1, \dots, S_{d+1}\}$ is called a d-dimensional l-simplex (or for short (d, l)-simplex) if $|S_1 \cap \cdots \cap S_{d+1}| < l$ but the intersection of any d of the S_i has size at least l.

Claim 4.2. If $\{S_1, \dots, S_{d+1}\}$ is a (d, l)-simplex then $|S_1| \geqslant d + l - 1$.

Proof. Define $B_0 = S_1 \cap \cdots \cap S_{d+1}$, $A_i = \bigcap_{j \neq i} S_j$ for $i = 1, \dots, d$. Define further $B_i = A_i - B_0$. Now the sets $B_0, \dots, B_d$ are pairwise disjoint subsets of S_{d+1} satisfying $|B_0| \leqslant l - 1$, $|B_0| + |B_i| = |A_i| \geqslant l$. This implies

$$|S_{d+1}| \geqslant \sum_{i=0}^{d} |B_i| = \sum_{i=1}^{d} (|B_0| + |B_i|) - (d-1)|B_0|$$

$$\geqslant dl - (d-1)(l-1) = d + l - 1. \quad \blacksquare \qquad (4.1)$$

To avoid trivialities we shall assume that $k \geqslant d + l - 1$ holds.

Let us define $s(n, k, d, l) = \max\{|\mathscr{F}|: \mathscr{F} \subset \binom{X}{k}, \mathscr{F}$ contains no (d, l)-simplex$\}$.

Even the case $k = d + l - 1$ is rather special. In fact, if $\{S_1, ..., S_{d+1}\}$ is a (d, l)-simplex with $|S_1| = \cdots = |S_{d+1}| = k = d + l - 1$, then (4.1) implies $|B_0| = l - 1$ and $|B_i| = 1$ and that every element of S_{d+1} is contained in at least $(d-1)$ of the remaining d sets in the simplex. Since the choice of S_{d+1} was arbitrary, we obtain $(S_1 \cup \cdots \cup S_{d+1}) - S_{d+1} = S_1 \cap \cdots \cap S_d$ and thus $|S_1 \cup \cdots \cup S_{d+1}| = |S_{d+1}| + 1 = k + 1$.

On the other hand, it is easily checked that every collection of $(d+1)$ k-element subsets of a $(k+1)$-element set forms a (d, l)-simplex with $l = k - d + 1$.

Let us define $m(n, k, t, r)$ as the maximum size of $\mathscr{F} \subset \binom{X}{k}$ such that no t vertices span r or more edges. This function was introduced by Brown, *et al.* [BES1, BES2].

The determination of $m(n, k, t, r)$ is in general a hopelessly difficult problem, even for graphs, i.e., $k = 2$.

With this definition for $k = l + d - 1$ we have

$$s(n, k, d, l) = m(n, k, k+1, d+1).$$

For $d \geqslant 2$ let us mention the bounds

$$\frac{1}{2^{k-2}} \binom{n}{k} \leqslant m(n, k, k+1, d+1) \leqslant \frac{d-1}{k} \binom{n}{k} + O(n^{k-1}). \tag{4.2}$$

In (4.2) the lower bound comes from [FF2] while the upper bound was proved by deCaen [C].

In the case $d = 1$ a (d, l)-simplex is just two sets whose intersection has size less than l. In this connection one should mention

THEOREM 4.3 (Erdös–Ko–Rado [EKR]). *Suppose that* $\mathscr{F} \subset \binom{X}{k}$, *and* $|F \cap F'| \geqslant l$ *holds for all* $F, F' \in \mathscr{F}$. *Then* $|\mathscr{F}| \leqslant \binom{n-l}{k-l}$ *holds for* $n \geqslant n_0(k, l)$.

The best possible bound $n_0(k, l) = (k - l + 1)(l + 1)$ was determined by Erdös, Ko, and Rado for $l = 1$, by Frankl [F2] for $l \geqslant 15$ and by Wilson [W] in general.

From now on we assume that $k \geqslant d + l$ and $d \geqslant 2$.

The easiest way to exclude (d, l)-simplices is to take all k-element sets containing a fixed l-element set. This gives

$$s(n, k, d, l) \geqslant \binom{n-l}{k-l}. \tag{4.3}$$

EXAMPLE 4.4. Suppose that $k \geqslant 2l$. Let $X = X_1 \cup X_2$ be a partition with $|X_1| = \lfloor \alpha n \rfloor$ where $\alpha = (d-1)/(l+d-1)$. Let $A \subset \binom{X_1}{k-l}$ be a $(d-1)$-packing of maximal size. By the theorem Rödl [R] we have $|\mathscr{A}| = (1 - o(1)) \binom{n}{d-1} / \binom{k-l}{d-1}$. Define

$$\mathscr{F} = \left\{ F \in \binom{X}{k} : (F \cap X_1) \in \mathscr{A} \right\}.$$

Clearly

$$|\mathscr{F}| = |\mathscr{A}| \binom{|X_2|}{l} = (1 - o(1)) \binom{(1-\alpha)n}{l} \binom{\alpha n}{l} \Big/ \binom{k-l}{d-1} = \Omega(n^{l+d-1}).$$

Claim 4.5. The family $\mathscr{F}$ from Example 4.4 contains no (d, l)-simplex.

Proof. Suppose that $F_1, ..., F_{d+1} \in \mathscr{F}$ form a (d, l)-simplex. Since $k \geqslant 2l$, we may suppose that, e.g., $F_1 \cap X_1 \neq F_2 \cap X_1$. Then $|F_1 \cap F_2| \leqslant l + d - 2$ holds.

On the other hand in every (d, l)-simplex the intersection of any two sets is at least $l + d - 2$. Hence $|F_1 \cap F_2| = l + d - 2$ and thus $F_1 \cap X_2 = F_2 \cap X_2$. Since for all $3 \leqslant i \leqslant d+1$ either $F_1 \cap X_1 \neq F_i \cap X_i$ or $F_2 \cap X_1 \neq F_i \cap X_1$ holds, we infer $F_1 \cap X_2 = F_2 \cap X_2 = \cdots = F_{d+1} \cap X_2$. However, this implies $|F_1 \cap \cdots \cap F_{d+1}| \geqslant |F_1 \cap X_2| = l$, a contradiction. ∎

In the case $k < 2l$ one can still take a $(l + d - 2)$-packing of maximal size to show (using again the theorem of Rödl [R])

$$s(n, k, d, l) \geqslant (1 - o(1)) \binom{n}{l+d-2} \Big/ \binom{k}{l+d-2}. \tag{4.4}$$

THEOREM 4.6. *Suppose that* $k \geqslant l + d$. *Then* (*i*) *and* (ii) *hold.*

(i) $s(n, k, d, l) = O(n^{\max\{k-l, l+d-1\}})$

(ii) *If* $k \geqslant 2l + d$ *and* $n \geqslant n_0(k)$ *then* $s(n, k, d, l) = \binom{n-l}{k-l}$, *the unique optimal family consisting of all k-sets through a fixed l-set.*

Conjecture 4.7. (ii) holds also in the case $k = 2l + d - 1$. Note that in the case $l = 1$ this is a theorem of Chvátal [Ch2]. Let us remark that for $k < 2l$ there is a gap of one in the exponent of n between the lower bounds and the upper bound in (i). The following theorem shows that it is not by chance.

THEOREM 4.8. *For every* $\varepsilon > 0$ *one has* $s(n, 5, 2, 3) \neq O(n^{4-\varepsilon})$. *On the other hand* $s(n, 5, 2, 3) = o(n^4)$.

Let us note that for $k = 5$ there is only one $(2, 3)$-simplex, consisting of the three sets $\{a, b, x_1, x_2, y_3\}$, $\{a, b, x_1, y_2, x_3\}$, and $\{a, b, y_1, x_2, x_3\}$. Thus Theorem 4.8 solves a problem of Erdös [E6].

5. Special Simplices, Intersection Condensed Families, and Traces

For a family $\mathscr{A} \subset \binom{X}{k}$ we call $\cap \mathscr{A}$ the *kernel* of $\mathscr{A}$ and denote it by $K(\mathscr{A})$. The *center* $C(\mathscr{A})$ is defined as the set of vertices of degree at least two in $\mathscr{A}$. For $|\mathscr{A}| \geqslant 2$ obviously $K(\mathscr{A}) \subset C(\mathscr{A})$ holds with equality if and only if $\mathscr{A}$ is a sunflower.

Let now $\mathscr{A}$ be a fixed k-graph, $|\mathscr{A}| \geqslant 2$ and set $p = |K(\mathscr{A})|$, $q = |C(\mathscr{A}) - K(\mathscr{A})|$.

Let us define

$$\mathrm{ex}(n, \mathscr{A}) = \max \left\{ |\mathscr{F}| : \mathscr{F} \subset \binom{X}{k}, \mathscr{F} \text{ contains no copy of } \mathscr{A} \right\}.$$

Looking at all k-subsets through a $(p+1)$-set shows that $\mathrm{ex}(n, \mathscr{A}) \geqslant \binom{n-p-1}{k-p-1}$. Now we describe a potentially larger construction.

Call a set Y *t-crosscut* of $\mathscr{A}$ if $|A \cap Y| = t$ holds for all $A \in \mathscr{A}$. Let $\mathscr{A}_Y$ be the corresponding *trace*, i.e., $\mathscr{A}_Y = \{A \cap Y : A \in \mathscr{A}\}$. We call $\mathscr{A}_Y$ a *t-trace* of $\mathscr{A}$.

Let us note that if $2p + q < k$, then every sunflower of size $|\mathscr{A}|$ and consisting of $(p+1)$-element sets occurs as a t-trace of $\mathscr{A}$.

Definition 5.1. Let $\pi(\mathscr{A})$ denote the maximum size of a $(p+1)$-graph B which contains no $(p+1)$-trace of $\mathscr{A}$.

By a theorem of Erdös and Rado ([ER], see also Sect. 2) $\pi(\mathscr{A})$ is finite, $\pi(\mathscr{A}) \leqslant (p+1)!(|\mathscr{A}| - 1)^{p+1}$.

Example 5.2. Let $\mathscr{B} = \mathscr{B}(\mathscr{A})$ be a $(p+1)$-graph achieving equality in Definition 5.1. Suppose that $Y = \cup \mathscr{B}$ is a subset of X. Define

$$\mathscr{F}(\mathscr{B}) = \left\{ F \in \binom{X}{k} : F \cap Y \in \mathscr{B} \right\}.$$

It is easy to check that $\mathscr{F}(\mathscr{B}) = (\pi(\mathscr{A}) - o(1))\binom{n-p-1}{k-p-1}$ and that $\mathscr{F}(\mathscr{B})$ contains no copy of $\mathscr{A}$.

Theorem 5.3. *Suppose that* $k \geqslant 3p + q + 2$. *Then* $\mathrm{ex}(n, \mathscr{A}) = (\pi(\mathscr{A}) - o(1))\binom{n-p-1}{k-p-1}$ *holds.*

Let us say that $\mathscr{A}$ is a special (d, l)-simplex if $|K(\mathscr{A})| = l - 1$ and $\{A - K(A) : A \in \mathscr{A}\}$ is a special simplex of dimension d, as defined in Section 3.

Let $\mathscr{A}(d, l)$ denote the special (d, l)-simplex. Simple considerations show for $k > d + l$ that $\pi(\mathscr{A}(d, 1)) = 1$, $\pi(\mathscr{A}(d, 2)) = 2$, $\pi((\mathscr{A}(2, 3)) = 2$, and $\pi(\mathscr{A}(2, 4)) = 6$. Using Theorem 5.3 one obtains the asymptotic size of

$\mathrm{ex}(n, \mathscr{A}(d, l))$ for the corresponding cases. Recall that $\phi(t, s)$ is the maximum size of a t-graph containing no sunflower of size s.

Claim 5.4. For $k > d + l$ we have

$$\phi\left(\left\lfloor \frac{l}{2} \right\rfloor, d+1\right) \leqslant \pi(\mathscr{A}(d, l)) \leqslant \phi(l, d+1).$$

Proof. Set $t = \lfloor l/2 \rfloor$ and let $\mathscr{D}$ be a collection of $\phi(t, d+1)$ sets of size t, without sunflowers of size $d+1$. Set $Y = \cup \mathscr{D}$ and for each $y \in Y$ let G_y be a 2-element set. Suppose that $G_y \cap G_{y'} = \varnothing$ for $y \neq y' \in Y$. Define

$$\mathscr{B}_0 = \{ \bigcup_{y \in D} G_y : D \in \mathscr{D} \}.$$

That is, $|\mathscr{B}_0| = |\mathscr{D}|$ and $\mathscr{B}_0$ arises by replacing in $\mathscr{D}$ every vertex by two new ones.

If l is even set $\mathscr{B} = \mathscr{B}_0$. If l is odd let z be an extra vertex and define $\mathscr{B} = \{ B \cup \{z\} : B \in \mathscr{B}_0 \}$.

We claim that $\mathscr{B}$ contains no l-trace of $\mathscr{A}(d, l)$.

Indeed, except possibly for z, the vertices in $\mathscr{B}$ came in equivalent pairs, while in $\mathscr{A}(d, l)$ two vertices are equivalent only if both are of degree 1 or $d+1$. Thus the l-trace should be a sunflower of size $d+1$ in order to be contained in $\mathscr{B}$. But $\mathscr{B}$ has no such sunflower proving our claim.

The upper bound holds for all $\mathscr{A}$ with $p + q < k$, as we pointed out above. ∎

Let us note that in [F3] there is a related conjecture which is not true as stated. We propose the following version of it.

Conjecture 5.5. Suppose that $p = 0$, $q < k$ and r is the minimal size of a 1-crosscut of $\mathscr{A}$. Then

$$\mathrm{ex}(n, \mathscr{A}) = (r - o(1)) \binom{n-1}{k-1}.$$

Note that $r < \infty$ is a consequence of $q \leqslant k$. Theorem 5.3 shows the validity of the conjecture for $k \geqslant q + 2$.

6. Tools of Proofs

a. *Shadows: The Kruskal–Katona Theorem*

Recall that for $\mathscr{F} \subset \binom{X}{k}$ and $0 \leqslant h < k$ the hth *shadow* $\Delta_h(\mathscr{F})$ of $\mathscr{F}$ is defined by

$$\Delta_h(\mathscr{F}) = \left\{ H \in \binom{X}{h} : \exists F \in \mathscr{F},\ H \subset F \right\}.$$

Given $|\mathcal{F}|$, what is the minimum of $|\Delta_h(\mathcal{F})|$? This problem was completely solved by Kruskal [Kr] and Katona [Ka1]. Since their formula for $|\Delta_h(\mathcal{F})|$ is not too convenient for computation, we shall use the following version of their result.

THEOREM 6.1 (Lovász [Lo]). *Suppose that $\mathcal{F} \subset \binom{X}{k}$ and let the real number $x \geqslant k$ be defined by $|\mathcal{F}| = \binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$ Then $|\Delta_h(\mathcal{F})| \geqslant \binom{x}{h}$ holds for all $k > h \geqslant 0$.*

For the simplest proof of these results see [F5].

b. *Families with Lots of Sunflowers.*

The main tool in proving most of the theorems is a recent result of the second author.

Recall that a family $\mathcal{B}$ is *closed under intersection* if $B, B' \in \mathcal{B}$ implies $(B \cap B') \in \mathcal{B}$.

For a family $\mathcal{B}$ and a set $B \in \mathcal{B}$ let us define the *intersection structure* of $\mathcal{B}$ on B by $\mathcal{M}(B, \mathcal{B}) = \{B \cap B' : B \neq B' \in \mathcal{B}\}$. The k-graph $\mathcal{B} \subset \binom{X}{k}$ is called k-partite if for some k-partition $X = X_1 \cup \cdots \cup X_k$ and for every $B \in \mathcal{B}$ we have $|B \cap X_i| = 1$, $1 \leqslant i \leqslant k$.

If $\mathcal{B}$ is k-partite with k-partition $X = X_1 \cup \cdots \cup X_k$ then we define for a set $A \subset B \in \mathcal{B}$ its *projection* $\pi(A) = \{i : A \cap X_i \neq \emptyset\}$ and $\pi(\mathcal{M}(B, \mathcal{B})) = \{\pi(A) : A \in \mathcal{M}(B, \mathcal{B})\}$. Note that $\mathcal{M}(B, \mathcal{B})$ and $\pi(\mathcal{M}(B, \mathcal{B}))$ are isomorphic.

THEOREM 6.2 (Füredi [Fü1]). *For any two positive integers k and s there exists a positive constant $c(k, s)$ such that every family $\mathcal{F} \subset \binom{X}{k}$ contains a subfamily $\mathcal{F}^* \subset \mathcal{F}$ satisfying* (i)–(iv)

(i) $|\mathcal{F}^*| > c(k, s)|\mathcal{F}|$.

(ii) *$\mathcal{F}^*$ is k-partite.*

(iii) *There is a family $\mathcal{J} \subset 2^{\{1, 2, \ldots, k\}}$ such that $\pi(\mathcal{M}(F, \mathcal{F}^*)) = \mathcal{J}$ holds for all $F \in \mathcal{F}^*$,*

(iv) *Every member of $\mathcal{M}(F, \mathcal{F}^*)$ is the center of a sunflower of size s formed by members of $\mathcal{F}^*$.*

Let us remark that if $s \geqslant k$ (which we will always assume), then (iv) implies that $\mathcal{M}(F, \mathcal{F}^*)$ is closed under intersection.

For the family $\mathcal{J} = \pi(\mathcal{M}(F, \mathcal{F}^*))$ we define its rank $r(\mathcal{J})$ by

$$r(\mathcal{J}) = \min\{|A| : A \subset \{1, 2, \ldots, k\},\ \nexists\, B \in \mathcal{J},\ A \subset B\}.$$

Clearly. $r(\mathcal{J}) \leqslant k$ with equality holding if and only if $\mathcal{J} = \{A : A \subsetneqq \{1, 2, \ldots, k\}\}$.

We shall often use the following simple observation.

Claim 6.3. $c(k, s)|\mathscr{F}| \leqslant |\mathscr{F}^*| \leqslant \Delta_{r(J)}(\mathscr{F}^*) \leqslant \binom{n}{r(\mathscr{J})}$.

Proof. Let $A \subset (1, 2, ..., k\}$ be an uncovered set of size $r(\mathscr{J})$, i.e., $A \not\subset B$ holds for all $B \in \mathscr{J}$. For $F \in \mathscr{F}^*$ let $A(F)$ be the unique subset of F satisfying $\pi(A(F)) = A$.

We claim that $A(F) \neq A(F')$ for $F, F' \in \mathscr{F}^*$. Indeed, the contrary would imply $A = \pi(A(F)) \subset \pi(F \cap F') \in \pi(\mathscr{M}(F, \mathscr{F}^*)) = \mathscr{J}$, a contradiction. Thus $|\mathscr{F}^*| \leqslant \Delta_{r(J)}(\mathscr{F}^*) \leqslant \binom{n}{r(J)}$ and therefore by (i) the statement follows. ∎

The next observation was essentially proved in [DEF].

Claim 6.4. Suppose that $\mathscr{A}_i \subset \mathscr{F}$ is a sunflower of size kr with center K_i for $i = 1, ..., r$. Set $R = K_1 \cup \cdots \cup K_r$. Then there exist $A_i \in \mathscr{A}_i$, $1 \leqslant i \leqslant r$ such that $A_i \cap R = K_i$ and the sets $A_i - R$ are pairwise disjoint. ∎

c. *Families with Many Intersection Conditions*

Another tool for investigating the intersection structure $\mathscr{M}(F, \mathscr{F}^*)$ is the following result of Frankl and Katona.

THEOREM 6.5 [FK]. *Suppose that $\mathscr{D} = \{D_1, ..., D_m\}$ is a collection of not necessarily distinct subsets of Y. Let s be a positive integer and suppose that for all t, $1 \leqslant t \leqslant m$ and $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant m$ we have*

$$|D_{i_1} \cap \cdots \cap D_{i_t}| \neq t - s.$$

Then $|\mathscr{D}| = m \leqslant |Y| + s - 1$ holds.

We shall need the following strengthening of this theorem.

THEOREM 6.6. [FF3]. *Suppose that $\mathscr{D}$ satisfies the assumptions of Theorem* 6.5 *and $|\mathscr{D}| = |Y| + s - 1$. If $s \geqslant 2$ then $\mathscr{D}$ consists of $|Y| + s - 1$ copies of Y.*

For $s = 1$ the situation is much more involved. Families achieving the upper bound are the disjoint unions of well-intersection designs, which in their turn are generalizations of semi biplanes.

7. MORE ON FAMILIES CLOSED UNDER INTERSECTION

Let F be a k-element set and let $\mathscr{J} \subset 2^F$ be a family closed under intersection with $F \notin \mathscr{J}$. Set $r = r(\mathscr{J})$.

LEMMA 7.1. *Suppose that $r(\mathscr{J}) \geqslant k - 1$. Then* (i) *or* (ii) *hold.*

(i) $\{A\colon x \in A \subsetneqq F\} \subset \mathscr{J}$ *for some* $x \in F$.

(ii) $2^B \subset \mathscr{J}$ *for some* $B \in \binom{F}{k-2}$.

Proof. If there is at most one $(k-1)$-element subset of F which is not in $\mathscr{J}$, then we can choose $x \in F$ such that all $k-1$ subsets of size $k-1$ of F through x are in $\mathscr{J}$. Let $A_1, ..., A_{k-1}$ be these sets. Since $\mathscr{J}$ is closed under intersection, for all choices of j and $1 \leqslant i_1 < \cdots < i_j \leqslant k-1$ the set $A_{i_1} \cap \cdots \cap A_{i_j}$ is in $\mathscr{J}$. This implies that (i) hold.

Suppose now that $F - \{x\}$ and $F - \{y\}$ are not in $\mathscr{J}$. Since $r(\mathscr{J}) \geqslant k-2$, $F - \{x, y\}$ must be in $\mathscr{J}$. Set $B = F - \{x, y\}$. For $z \in B$ define $B_z = (B - \{z\}) \cup \{x\}$. Since $r(\mathscr{J}) \geqslant k-1$, there exists $D_z \in \mathscr{J}$ satisfying $B_z \subset D_z$. Since $D_z \cup \{z\} = F - \{x\}$ is not in $\mathscr{J}$, $D_z \cap B = B_z \in \mathscr{J}$ follows.

Again, using that $\mathscr{J}$ is closed under intersection we infer $2^B \subset \mathscr{J}$, i.e., (ii) holds. ∎

LEMMA 7.2. *Suppose that $r(\mathscr{J}) \geqslant \max\{k-l, l+d\}$. Then either $\mathscr{J}$ contains a (d, l)-simplex or $k \geqslant 2l+d$ and $\{B: Y \subset B \subsetneq F\} \subset \mathscr{J}$ holds for some $Y \in \binom{F}{l}$.*

Proof. Set $r = r(\mathscr{J})$ and $Z = \{z_1, ..., z_r\}$ be an uncovered r-subset of F, i.e., $Z \not\subset A$ holds for all $A \in \mathscr{J}$. By definition for every i, $1 \leqslant i \leqslant r$, there exists $A_i \in \mathscr{J}$ with $(Z - \{z_i\}) \subset A_i$, and consequently $A_i \cap Z = (Z - \{z_i\})$.

Define $Y = F - Z$, $D_i = A_i \cap Y$.

Suppose first that for some s and $1 \leqslant i_1 < \cdots < i_s \leqslant r$ we have $|D_{i_1} \cap \cdots \cap D_{i_s}| = s - (r-l+1)$. Assume, by symmetry, that $i_j = j$, $j = 1, ..., s$. Since $r \geqslant l+d$, it follows that $s \geqslant d+1$. Now $|A_1 \cap \cdots \cap A_s| = |D_1 \cap \cdots \cap D_s| + |Z| - s = s - (r-l+1) + r - s = l-1$.

Defining $B_i = A_i \cap (\bigcap_{j=d+2}^{s} A_j)$ for $i = 1, ..., d+1$, one verifies that these $d+1$ sets form a (d, l)-simplex. Since $\mathscr{J}$ is closed under intersection, $B_i \in \mathscr{J}$ follows, $i = 1, ..., d+1$. That is, $\mathscr{J}$ contains a (d, l)-simplex.

Suppose next that

$$|D_{i_1} \cap \cdots \cap D_{i_s}| \neq s - (r-l+1) \qquad \text{holds for all } 1 \leqslant i_1 < \cdots < i_s \leqslant r.$$

Applying Theorem 6.5 gives $r \leqslant |Y| + (r-l) = k-l$. Since we assumed $r \geqslant k-l$, $r = k-l$ and $|Y| = l$ follow.

Now Theorem 6.6 yields that $D_1 = D_2 = \cdots = D_r = Y$, i.e., $A_i = F - \{z_i\}$ holds for $i = 1, ..., r$. Using that $\mathscr{J}$ is closed under intersection, we infer $\{B: Y \subset B \subsetneq F\} \subset \mathscr{J}$. ∎

Let $\mathscr{A}$ be a fixed k-graph and recall the definitions of $K(\mathscr{A})$ and $C(\mathscr{A})$ from Section 5. Also $p = |K(\mathscr{A})|$, $q = |C(\mathscr{A})| - p$.

Set $Y = C(\mathscr{A})$ and define the *full trace* of $\mathscr{A}$ as $\mathscr{C}(\mathscr{A}) = \{A \cap Y: A \in \mathscr{A}\}$.

LEMMA 7.3. *Suppose that $r(\mathscr{J}) \geqslant k-p-1$, $k \geqslant 3p+q+2$. Then either $\mathscr{J}$ contains the full trace of $\mathscr{A}$ or $\{E: B \subset E \subsetneq F\} \subset \mathscr{J}$ holds for some $B \in \binom{F}{p+1}$.*

Proof. Let us define inductively sets E_i and families $\mathcal{J}_i \subset 2^{E_i} - \{E_i\}$ satisfying the following three conditions:

(i) $\mathcal{J}_i$ is closed under intersection,

(ii) $E_i \subset E_{i-1}$, $|E_{i-1} - E_i| \geqslant 2$,

(iii) $r(\mathcal{J}_i) \geqslant r(\mathcal{J}_{i-1}) - 1$.

First set $E_0 = F$, $\mathcal{J}_0 = \mathcal{J}$.

Suppose that E_i and $\mathcal{J}_i$ were already defined and consider the maximal (for containment) sets in $\mathcal{J}_i$. If there is a maximal set E with $|E| \leqslant |E_i| - 2$ then set $E_{i+1} = E$ and $\mathcal{J}_{i+1} = \{E \cap D: D \in J_i\}$.

Otherwise stop. Let this procedure terminate with the pair E_t, $\mathcal{J}_t$. By (ii) and (iii) we have

$$k - p - 1 - t \leqslant r(\mathcal{J}_t) \leqslant |E_t| \leqslant k - 2t.$$

Thus $t \leqslant p + 1$ and consequently $|E_t| \geqslant r(\mathcal{J}_t) \geqslant k - p - 1 - t \geqslant k - 2(p+1) \geqslant p + q$. Let $B_1, ..., B_s$ be the maximal sets in $\mathcal{J}_t$: $B_i = E_t - \{x_i\}$, $i = 1, ..., s$. Then the set $\{x_1, ..., x_s\}$ is not contained in any member of $\mathcal{J}_t$. This observation yields $s \geqslant r(\mathcal{J}_t) \geqslant k - p - 1 - t \geqslant p + q$.

Define $B = B_1 \cap \cdots \cap B_s$. We have

$$|B| = |E_t| - s \leqslant (k - 2t) - (k - p - 1 - t) = p + 1 - t. \tag{7.1}$$

From (7.1) it follows that either $|B| \leqslant p$ or $|B| = p + 1$ and $t = 0$, $s = k - p - 1$. In this second case using (i) gives $\{E: B \subset E \subsetneqq F\} \subset \mathcal{J} = \mathcal{J}_t$. In the first case using $|E_t| \geqslant p + q$, $s \geqslant p + q$ and $|B| \leqslant p$ we find sets D_1, D_2 with $B \subset D_1 \subset D_2 \subset E_t$, $|D_1| = p$, $|D_2| = p + q$ and $\{E: D_1 \subset E \subset D_2\} \subset \mathcal{J}_t \subset \mathcal{J}$. Thus $\mathcal{J}$ contains the full trace of $\mathcal{A}$. ∎

Let us recall also the following statement which was proved in [FF3].

LEMMA 7.4. *Suppose that $k \geqslant 2l + 3$ and $r(\mathcal{J}) \geqslant k - l - 1$. Then either $\mathcal{J}$ contains some l-element set or for some $(l+1)$-element set B we have $\{E: B \subset E \subsetneqq F\} \subset \mathcal{J}$.*

Note that for $k \geqslant 3l + 2$ this lemma is a special case of Lemma 7.3. Its proof is very similar to the proof of Lemma 7.2.

8. The Proof of Theorems 2.7, 4.6 (i) and 5.3

We start with Theorem 4.6(i). Let us define $h = \max\{k - l, l + d - 1\}$. We are going to prove that

$$|\Delta_h(\mathcal{F})| \geqslant c(k, (d+1)k)|\mathcal{F}|. \tag{8.1}$$

Apply Theorem 6.2 (with $s=(d+1)k$) to $\mathscr{F}$ to obtain $\mathscr{F}^*$ and $\mathscr{J}$. In view of Claim 6.3 it is sufficient to show that $r(\mathscr{J})\leqslant h$. However, the contrary implies by Lemma 7.2 that $\mathscr{J}$ contains a (d, l)-simplex, and thus $\mathscr{M}(F, \mathscr{F}^*)$ as well. Since every member of $\mathscr{M}(F, \mathscr{F}^*)$ is the center of a sunflower of size $(d+1)k$, Claim 6.4 implies that $\mathscr{F}^*$ contains a (d, l)-simplex, a contradiction. ▌

Now we prove Theorems 2.7 and 5.3 together. Set $h=k-l-1$ $(h=k-p-1)$ in the case of Theorem 2.7 (5.3), respectively. Apply Theorem 6.2 to obtain $\mathscr{F}_1=\mathscr{F}^*$ and $\mathscr{J}_1=\mathscr{J}$. If $r(\mathscr{J})\leqslant h-1$ then stop. If not, apply Theorem 6.2 with $\mathscr{F}-\mathscr{F}_1$ instead of $\mathscr{F}$, and so on. This way in the mth step we obtain $\mathscr{F}_m=(\mathscr{F}-(\mathscr{F}_1\cup\cdots\cup\mathscr{F}_{m-1}))^*$ and $\mathscr{J}_m$. We stop either if there are no more sets, or if $r(\mathscr{J}_m)\leqslant h-1$. We get a partition of $\mathscr{F}$ into pairwise disjoint families $\mathscr{F}_1,\ldots,\mathscr{F}_m$, $(\mathscr{F}-(\mathscr{F}_1\cup\cdots\cup\mathscr{F}_m))\overset{\text{def}}{=}\mathscr{R}$ so that the corresponding families $\mathscr{J}_1,\ldots,\mathscr{J}_m$ satisfy $r(\mathscr{J}_i)\geqslant h$ for $1\leqslant i<m$ and $r(\mathscr{J}_m)\leqslant h-1$.

In view of Claim 6.3 $c(k,s)|(\mathscr{R}\cup\mathscr{F}_m)|\leqslant|\mathscr{F}_m|\leqslant\binom{n}{h-1}=o(\binom{n}{h})$. Thus we can forget about $\mathscr{R}\cup\mathscr{F}_m$. Set $\mathscr{F}=\mathscr{F}_1\cup\cdots\cup\mathscr{F}_{m-1}$.

Using Lemmas 7.3 and 7.4, respectively, we infer for every $F\in\mathscr{F}$ the existence of a $(k-h)$-element set $B(F)$ such that all the set E with $B(F)\subset E\subsetneqq F$ are the centers of a sunflower of size s in $\mathscr{F}$.

Set $D(F)=F-B(F)$. For a fixed h-element subset D let $B_1,\ldots,B_{r(D)}$ be all the $(k-h)$-subsets such that $D=D(D\cup B_i)$ holds, $i=1,\ldots,r(D)$. Clearly $|\mathscr{F}|=\sum_{D\in\binom{X}{h}}r(D)$ holds. Thus it is sufficient to show that $r(D)\leqslant\pi(\mathscr{A})$ in the case of Theorem 5.3 and that $r(D)\leqslant\phi(l+1,s)$ in the case of Theorem 2.7. Consider first Theorem 5.3. If $r(D)>\pi(\mathscr{A})$ for some D then we can find not necessarily distinct $B_1,\ldots,B_t$ with $B_i=B(D\cup B_i)$ and such that for an appropriate isomorphic copy of $\mathscr{A}=\{A_1,\ldots,A_t\}$ and a set Y we have $A_i\cap Y=B_i$. Since $|D|=k-p-1\geqslant 2p+q+1>p+q=|\mathscr{C}(\mathscr{A})|$, we can find not necessarily distinct subsets $D_i\subset D$, $i=1,\ldots,t$, such that $\{B_1\cup D_1,\ldots,B_t\cup D_t\}$ is a copy of $\mathscr{C}(\mathscr{A})$.

Finally, since each $B_i\cup D_i$ is the center of a large sunflower, Claim 6.4 provides us with a copy of $\mathscr{A}$ in $\mathscr{F}$, a contradiction.

The case of Theorem 2.7 is slightly simpler. Again, we have to show that $r(D)\leqslant\phi(l+1,s)$ holds. Suppose the contrary. Then we can find a sunflower $\{B_1,\ldots,B_s\}$ such that $(B_i\cup D)\in\mathscr{F}$ and $B(B_i\cup D)=B_i$, $1\leqslant i\leqslant s$.

Let c be the size of the center of the sunflower and D_0 a $(l-c)$-subset of D.

Then the sets $B_1\cup D_0,\ldots,B_s\cup D_0$ form a sunflower whose center has size l. An application of Claim 6.4 gives the desired contradiction. ▌

9. Exact Bounds for Simplices

Let k, l and r be positive integers, $k > l$. Throughout this section we consider k-graphs $\mathscr{F} \subset \binom{X}{k}$ having the following property.

> If $F \in \mathscr{F}$ and $L \in \binom{F}{l}$ are such that each set E with $L \subset E \subsetneqq F$ is the center of a sunflower in $\mathscr{F}$ of size rk, then $L \subset F'$ holds for every $F' \in \mathscr{F}$ with $|F' \cap (F - L)| \geqslant k - l - 1$. (9.1)

The proofs of this section are based on the following, rather technical lemma, which was essentially proved in [FF3]. For completeness' sake we include a (somewhat sketchy) proof.

Let c_i be a positive constant depending on k, l, and r only $(1 \leqslant i \leqslant 6)$ throughout the statement and the proof of

Lemma 9.1. *Suppose that $\mathscr{F} = \mathscr{F}_0 \cup \mathscr{F}_1$ is a partition veryfying* (i), (ii).

(i) $|\mathscr{F}_0| < c_1 n^{k-l-1}$.

(ii) *For each $F \in \mathscr{F}_1$ there exists an l-set $L = L(F) \subset F$ such that every E with $L \subset E \subsetneqq F$ is the center of some sunflower in $\mathscr{F}$ of size rk.*

(iii) $|\Delta_{k-l}(\mathscr{G})| > c_2 |\mathscr{G}|$ *for all $\mathscr{G} \subset \mathscr{F}$.*

Then for $n > n_0(k, l, r)$ either $|\mathscr{F}| < \binom{n-l}{k-l}$ or $\mathscr{F} = \{F \in \binom{X}{k} : L \subset F\}$ holds for some $L \in \binom{X}{l}$.

Proof. Without loss of generality we may assume that $\mathscr{F}_1$ is maximal, i.e., (ii) fails for all $F \in \mathscr{F}_0$. Let us assume that $|\mathscr{F}| \geqslant \binom{n-l}{k-l}$. Let $\{L_1, ..., L_t\}$ be a labeling of $\{L(F): F \in \mathscr{F}_1\}$. For $1 \leqslant i \leqslant t$ define $\mathscr{G}_i = \{F \in \mathscr{F}_1 : L(F) = L_1\}$, $\tilde{\mathscr{G}}_i = \{F - L(F): F \in \mathscr{G}_i\}$.

Then (i) implies

$$\binom{n-l}{k-l} - c_1 n^{k-l-1} < |\mathscr{F}_1| = \sum_{i=1}^{t} |\mathscr{G}_i|. \tag{9.2}$$

Using (9.1) it is not hard to see that the families $\Delta_{k-l-1}(\tilde{\mathscr{G}}_i)$ are pairwise disjoint. Define real numbers $x_i \geqslant k - l$ by $|\tilde{\mathscr{G}}_i| = |\mathscr{G}_i| = \binom{x_i}{k-l}$. Assume for convenience that $x_1 \geqslant x_j$ for $2 \leqslant j \leqslant t$. Since $\binom{x}{k-l} / \binom{x}{k-l-1} = (x - k + l + 1)/(k - l)$ is a monotone increasing function of x, Theorem 6.1 implies

$$|\mathscr{G}_i| / |\Delta_{k-l-1}(\tilde{\mathscr{G}}_i)| \leqslant \frac{x_1 - k + l + 1}{k - l}.$$

Using (9.2) we obtain

$$\binom{n-l}{k-l} - c_1 n^{k-l-1} \leqslant \sum_i |\mathscr{G}_i| \leqslant \frac{x_1 - k + l + 1}{k-l} \sum_i \Delta_{k-l-1}(\tilde{\mathscr{G}}_i)$$

$$\leqslant \frac{x_1 - k + l + 1}{k-l} \binom{n}{k-l-1}.$$

Comparing the two extreme sides gives $x_1 > n - c_3$.

This implies that

$$\sum_{i=2}^{t} |\mathscr{G}_i| < \binom{n}{k-l} - \binom{n-c_3}{k-l} < c_4 n^{k-l-1}. \tag{9.3}$$

Consequently,

$$|\mathscr{G}_1| > \binom{n-l}{k-l} - (c_1 + c_4)\, n^{k-l-1}.$$

If $\mathscr{G}_1 = \mathscr{F}$, then we have nothing to prove. Thus suppose $\mathscr{F} - \mathscr{G}_1 \neq \varnothing$ and let the partition $\mathscr{F} - \mathscr{G}_1 = \mathscr{D} \cup \mathscr{E}$ be defined by $\mathscr{E} = \{E\colon L \not\subset E \in \mathscr{F}\}$.

We distinguish two cases according whether $|\mathscr{D}|$ or $|\mathscr{E}|$ is larger.

a. $|\mathscr{D}| \geqslant |\mathscr{E}|$

Define $\tilde{\mathscr{D}} = \{D - L\colon D \in \mathscr{D}\}$. Note that $|\tilde{\mathscr{D}}| = |\mathscr{D}|$. Apply Theorem 6.2 with $s = rk$ to $\tilde{\mathscr{D}}$ and let $\mathscr{D}^*$ and $\mathscr{J}$ be the families, we obtain. Suppose that $D \in \mathscr{D}^*$. Since $D \notin \mathscr{G}_1$ and $\mathscr{J}$ is closed under intersection, $D - L$ contains a $(k-l-1)$-element set which is neither in $\Delta_{k-l-1}(\tilde{\mathscr{G}}_1)$ nor contained in $\mathscr{M}(D, \mathscr{D}^*)$.

Choosing one such set from each $D - L$ shows together with the obvious

$$\Delta_{k-l-1}(\tilde{\mathscr{G}}_1) > \frac{\binom{n-l}{k-l-1}}{\binom{n-l}{k-l}} |\tilde{\mathscr{G}}_1|$$

gives

$$\binom{n-l}{k-l-1} \geqslant \frac{\binom{n-l}{k-l-1}}{\binom{n-l}{k-l}} |\tilde{\mathscr{G}}_1| + |\mathscr{D}^*|. \tag{9.4}$$

Using $n > n_0(k, l, r)$, $|\mathscr{D}^*| > c(k-l, rk)|\mathscr{D}|$ and $|\mathscr{D}| \geqslant |\mathscr{E}|$, we infer

$$|\mathscr{D}^*| > \binom{n-l}{k-l-1}(|\mathscr{D}| + |\mathscr{E}|)\Big/\binom{n-l}{k-l}.$$

Substituting this into (9.4) and using $|\mathscr{F}| = |\tilde{\mathscr{G}}_1| + |\mathscr{D}| + |\mathscr{E}|$ gives $|\mathscr{F}| \leqslant \binom{n-l}{k-l}$, a contradiction.

b. $|\mathscr{E}| > |\mathscr{D}|$

Define $x \geqslant k-l$ by $|\mathscr{E}| = \binom{x}{k-l}$.

Then $|\mathscr{G}_1| > \binom{n-c_3}{k-l}$, (9.3) and (i) imply $x < c_6 n^{1-1/(k-l)}$. Using Theorem 6.1 we obtain for $n > n_0(k, l, r)$,

$$|\Delta_{k-l-1}(\mathscr{E})| \geqslant \frac{k-l}{x-k+l+1}|\mathscr{E}| > (|\mathscr{E}| + |\mathscr{D}|)\binom{n}{k-l-1}\Big/\binom{n-l}{k-l}. \tag{9.5}$$

Since $L \not\subset E$ holds for all $E \in \mathscr{E}$, (9.1) implies that $\Delta_{k-l-1}(\mathscr{G}_1) \cap \Delta_{k-l-1}(\mathscr{E}) = \varnothing$.

Further, by Theorem 6.1 we have

$$\Delta_{k-l-1}(\mathscr{G}_1) \geqslant \sum_j \binom{l}{j}\binom{x_1}{k-l-1-j} = \binom{x_1+l}{k-l-1}. \tag{9.6}$$

Since $x_1 < n$, (9.6) implies

$$|\Delta_{k-l-1}(\mathscr{G}_1)| > |\mathscr{G}_1|\binom{n}{k-l-1}\Big/\binom{n-l}{k-l}. \tag{9.7}$$

Adding up (9.5) and (9.7), and using $|\mathscr{F}| = |\mathscr{G}_1| + |\mathscr{E}| + |\mathscr{D}|$ gives $\binom{n}{k-l-1} \geqslant |\Delta_{k-l-1}(\mathscr{G}_1)| + |\Delta_{k-l-1}(\mathscr{E})| > |\mathscr{F}|\binom{n}{k-l-1}/\binom{n-l}{k-l}$, i.e., $|\mathscr{F}| < \binom{n-l}{k-l}$, a contradiction. ∎

Let us prove now Theorems 1.6, 3.2 and 4.6(ii) together. Set $l=1$ in the case of the first two theorems and define $h = k-l$. Consider a family $\mathscr{F} \subset \binom{X}{k}$ which contains no corresponding simplex. It is straightforward to verify that $\mathscr{F}$ satisfies (9.1). Starting with the family $\mathscr{F}$ let us apply Theorem 6.2 repeatedly as in Section 8 to obtain pairwise disjoint families $\mathscr{H}_1, ..., \mathscr{H}_m$ so that the corresponding families $\mathscr{J}_1, ..., \mathscr{J}_m$ satisfy $r(\mathscr{J}_i) \geqslant h$ for $1 \leqslant i \leqslant m-1$ and $r(\mathscr{J}_m) \leqslant h-1$.

Define $\mathscr{F}_1 = \mathscr{H}_1 \cup \cdots \cup \mathscr{H}_{m-1}$ and $\mathscr{F}_0 = \mathscr{H}_m$. Lemmas 7.1 and 7.2 imply that $\mathscr{F}_1$ has property (ii) and that $\mathscr{F}$ has property (iii). Claim 6.3 implies that $\mathscr{F}_0$ has property (i).

Now Lemma 9.1 gives the desired conclusion.

10. A Linear Independence Proof

Let us prove Theorem 1.5.

Let $\mathscr{F} = \{F_1, ..., F_v\}$ and $\Delta_{k-1}(\mathscr{F}) = \{G_1, ..., G_u\}$. Define the u by v matrix $M = (m_{ij})_{1 \leq i \leq u,\, 1 \leq j \leq v}$ by

$$m_{ij} = \begin{cases} 1 & \text{if } G_i \subset F_j \\ 0 & \text{if } G_i \not\subset F_j. \end{cases}$$

Clearly it is sufficient to show that M has rank v. Suppose the contrary and let $\alpha_j \in Q$ be the coefficients of a non-trivial linear dependence among the columns. That is for every G_i we have

$$\sum_{G_i \subset F_j} \alpha_j = 0. \tag{10.1}$$

Let us suppose by symmetry that $\alpha_1 \neq 0$. In view of Theorem in [FS] we can find F_j with $\alpha_j \neq 0$ such that $|F_1 \cap F_j| = d$. Again by symmetry suppose that $j = 2$ and, say, $\alpha_2 > 0$. Set $F_1 \cap F_2 = \{x_1, ..., x_d\}$.

In view of (10.1) we can find sets $F_{j(i)}$ such that $F_{j(i)} \cap F_2 = F_2 - \{x_i\}$, $i = 1, ..., d$ and $\alpha_{j(i)} < 0$. Note that $F_{j(i)} \cap \cdots \cap F_{j(d)} \supset F_2 - F_1$ and $x_i \in (F_1 \cap (\bigcap_{v \neq i} F_{j(v)}))$. Thus $F_1, F_{j(1)}, ..., F_{j(d)}$ form a d-dimensional simplex if $F_1 \cap F_{j(1)} \cap \cdots \cap F_{j(d)} = \emptyset$. Thus we may assume that some element $x \in F_1$ is common to all the $F_{j(i)}$. That is, $F_{j(i)} = (F_2 - \{x_i\}) \cup \{x\}$. Apply now (10.1) to the $(k-1)$-element set $F_{j(1)} \cap F_{j(2)}$. Since $\alpha_{j(1)} < 0$ and $\alpha_{j(2)} < 0$, there exists some F_j, say F_3 with $\alpha_3 > 0$ and $F_{j(1)} \cap F_{j(2)} \subset F_3$. Now $F_1, F_2, F_3, F_{j(3)}, F_{j(4)}, ..., F_{j(d)}$ form a d-dimensional simplex, a contradiction. ∎

11. Weight Functions and Theorem 3.4

First we prove Theorem 3.4 in the case $k = d + 1$. For a d-element set D let $\deg_F(D)$ denote the number of $F \in \mathscr{F}$ with $D \subset F$. Define a weight function $w\colon \binom{X}{d} \times \mathscr{F} \to R^+$ by

$$w(D, F) = \begin{cases} 1/\deg_F(D) & \text{for } D \subset F \\ 0 & \text{for } D \not\subset F. \end{cases}$$

By definition

$$\sum_{D \in \binom{X}{d}} \sum_{F \in \mathscr{F}} w(D, F) = |\Delta_d(\mathscr{F})|. \tag{11.1}$$

PROPOSITION 11.1. *For every $F \in \mathscr{F}$ we have*

$$\sum_{D \subset F} w(D, F) \geqslant 1. \tag{11.2}$$

Proof. Let $D_1, ..., D_{d+1}$ be the d-subsets of some $F \in \mathscr{F}$. For $1 \leqslant i \leqslant d+1$ let A_i be the set of those $x \in (X-F)$ that $(D_i \cup \{x\}) \in \mathscr{F}$.

The sets $A_1, ..., A_{d+1}$ could not have a system of distinct representatives say $x_1, ..., x_{d+1}$ because then $D_1 \cup \{x_1\}, ..., D_{d+1} \cup \{x_{d+1}\}$ is a d-dimensional special simplex. Thus Hall's theorem implies that for some non-empty $I \subset \{1, 2, ..., d+1\}$ we have $|\bigcup_{i \in I} A_i| \leqslant |I| - 1$. Then $\deg_F(D_i) = |A_i| + 1 \leqslant |I|$ holds for $i \in I$. Consequently,

$$\sum_{D \subset F} w(D, F) \geqslant \sum_{i \in I} 1/\deg_{\mathscr{F}}(D_i) \geqslant 1. \quad \blacksquare$$

Now, reversing the order of summation in (11.1) gives

$$|\Delta_d(\mathscr{F})| = \sum_{F \in \mathscr{F}} \sum_{D \subset F} w(D, F) \geqslant \sum_{F \in \mathscr{F}} 1 = |\mathscr{F}|. \quad \blacksquare$$

Now we prove Theorem 3.4 for fixed d by induction on k. Suppose that $k > d+1$ and the statement is true for $k-1$.

Let us set $\mathscr{F}_0 = \mathscr{F}$. Suppose that $\mathscr{F}_i$ is defined already. Consider $\Delta_{k-1}(\mathscr{F}_i)$. If there is some $G \in \Delta_{k-1}(\mathscr{F}_i)$ which is contained in at most $c(k, d)$ members of $\mathscr{F}$, then define $\mathscr{F}_{i+1} = \mathscr{F}_i - \{F \in \mathscr{F}_i : G \subset F\}$ and continue.

Finally we obtain a family $\mathscr{F}_s$ so that each $G \in \Delta_{k-1}(\mathscr{F}_s)$ is contained in more than $c(k, d)$ members of $\mathscr{F}_s$. The definition implies

$$s \leqslant |\Delta_{k-1}(\mathscr{F}) - \Delta_{k-1}(\mathscr{F}_s)|, \ |\mathscr{F} - \mathscr{F}_s| \leqslant c(k, d)\, s \tag{11.3}$$

Now let us consider $\mathscr{F}_s$.

Claim 11.2. $\Delta_{k-1}(\mathscr{F}_s)$ contains no special simplex of dimension d.

Proof of the Claim. Suppose for contradiction that $G_1, ..., G_{d+1}$ is a d-dimensional special simplex in $\Delta_{k-1}(\mathscr{F}_s)$. Set $A = G_1 \cup \cdots \cup G_{d+1}$. Then $|A| = (d+1)(k-d)$. We want to show that there exist $F_1, ..., F_{d+1}$ with $G_i \subset F_i$, $F_i \cap F_{i'} = G_i \cap G_{i'}$, $1 \leqslant i \neq i' \leqslant d+1$.

Suppose F_i is defined for $i \leqslant j$. Set $A_j = F_1 \cup \cdots \cup F_j \cup G_{j+1} \cup \cdots \cup G_s$, $|A_j| = |A| + j$, $|A_j - G_{j+1}| = |A| + j - (k-1) \leqslant d(k-d) + 1 \leqslant c(k, d)$. Therefore there exists $F_{j+1} \in F_s$ with $G_{j+1} \subset F_{j+1}$ and $F_{j+1} \cap A_j = G_{j+1}$, as desired. Now $F_1, ..., F_{d+1}$ form a special simplex of dimension d, a contradiction. $\blacksquare$

Applying the induction hypothesis to $\Delta_{k-1}(\mathscr{F}_s)$ we infer

$$|\Delta_{k-2}(\mathscr{F}_s)| \geqslant c(k-1, d)^{-1} |\Delta_{k-1}(\mathscr{F}_s)|. \tag{11.4}$$

On the other hand each $G \in \Delta_{k-1}(\mathscr{F}_s)$ has only $k-1$ $(k-2)$-element subsets while each $H \in \Delta_{k-2}(\mathscr{F}_s)$ is covered by at least $c(k, d)+2$ members of $\Delta_{k-1}(\mathscr{F}_s)$. This yields $|\Delta_{k-2}(\mathscr{F}_s)| \leqslant (k-1)/(c(k, d)+2) |\Delta_{k-1}(\mathscr{F}_s)|$. Comparing this with (11.4) gives $c(k, d)+2 \leqslant (k-1)\, c(k-1, d)$, a contradiction. Consequently we must have $\mathscr{F}_s = \emptyset$. Therefore (11.3) implies $|\Delta_{k-1}(\mathscr{F}) \geqslant (1/c(k, d)) |\mathscr{F}|$ ∎

12. Star-Shaped Trees

Here we prove Theorem 3.8. Define again a weight function $w\colon \binom{X}{k-1} \times \mathscr{F} \to R^+$ by

$$w(G, F) = \begin{cases} 1/\deg_{\mathscr{F}}(G) & \text{if } G \subset F \\ 0 & \text{if } G \not\subset F. \end{cases} \tag{12.0}$$

As we saw in Section 11 we have

$$\sum_{G \in \Delta_{k-1}(\mathscr{F})} \sum_{G \subset F \in \mathscr{F}} w(G, F) = |\Delta_{k-1}(\mathscr{F})| \leqslant \binom{n}{k-1}.$$

Reversing the order of summation gives

$$\sum_{F \in \mathscr{F}} \left(\sum_{G \subset F} w(G, F) \right) \leqslant \binom{n}{k-1}.$$

Using $|\mathscr{F}| > u\binom{n}{k-1}/k$, the above inequality implies that for some $F \in \mathscr{F}$,

$$\sum_{G \subset F} w(G, F) < k/u. \tag{12.1}$$

Let $G_1, \ldots, G_k$ be the $(k-1)$-subsets of F, define $a_i = \deg_{\mathscr{F}}(G_i)$. Suppose by symmetry that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$ holds.

Substituting into (12.1) yields

$$\sum_{i=1}^{k} 1/a_i < k/u. \tag{12.2}$$

Examining (12.2) shows that

$$a_i > \frac{i}{k} u \qquad \text{holds for } 1 \leqslant i \leqslant k. \tag{12.3}$$

Let $1 \leqslant d_1 \leqslant \cdots \leqslant d_k$ be the degrees of the star-shaped tree with $u+1$ edges. Note that $\sum_{i=1}^{k} (d_i - 1) = u$.

Let A_i be the set of vertices outside F such that $(G_i \cup \{x\}) \in \mathscr{F}$ holds for $x \in A_i$. By definition $|A_i| = a_i - 1$. Using (12.3) we infer

$$|A_i| \geqslant \left\lceil \frac{iu+1}{k} \right\rceil - 1 \geqslant \left\lfloor \frac{iu}{k} \right\rfloor \geqslant (d_1 - 1) + \cdots + (d_i - 1).$$

Consequently we can choose successively sets $D_1 \subset A_1, \ldots, D_k \subset A_k$ such that $|D_i| = d_i - 1$ and $D_1, \ldots, D_k$ are pairwise disjoint.

Now F together with the edges $\{G_i \cup \{y_i\} \colon y_i \in D_i,\ 1 \leqslant i \leqslant k\}$ forms the sought after star-shaped tree. ∎

Remark. More careful analysis shows that the conclusion of the theorem holds for $|\mathscr{F}| = u\binom{n}{k-1}/k$, unless $\mathscr{F} = \Delta_k(\mathscr{P})$ for a $(k-1)$-packing $\mathscr{P} \subset \binom{X}{u+k-1}$ with

$$|\mathscr{P}| = \binom{n}{k-1} \Big/ \binom{u+k-1}{k-1}.$$

Namely, the above argument gives desired star-shaped tree unless $|\Delta_{k-1}(\mathscr{F})| = \binom{n}{k-1}$ and for every $F \in \mathscr{F}$ we have equality in (12.2)

$$\sum_{i=1}^{k} 1/a_i = k/u.$$

This still implies the inequality (12.3) for $1 \leqslant i \leqslant k-1$. If (12.3) holds for $i=k$ as well, then the argument works. The only case it fails is if $a_1 = a_2 = \cdots = a_k = u$. More exactly $A_1 = A_2 = \cdots = A_k$.

Repeating this argument with F replaced by $G_i \cup \{y_i\}$, $y_i \in A_i$, $1 \leqslant i \leqslant k$, etc., gives that $\binom{F \cup A_i}{k} \subset \mathscr{F}$.

Since the same holds for all $F \in \mathscr{F}$ and since for $F \neq F'$, $F \cup A_1$ and $F' \cup A_1'$ either coincide or overlap in at most $k-2$ vertices, $\mathscr{F} = \Delta_k(\mathscr{P})$ holds for some $(k-1)$-packing $\mathscr{P} \subset \binom{X}{u+k-1}$. Now $|\mathscr{F}| = \binom{u+k-1}{k}|\mathscr{P}|$ implies $|\mathscr{P}| = \binom{n}{k-1}/\binom{u+k-1}{k-1}$, i.e., $\mathscr{P}$ is a *perfect packing.*

13. Triple-Systems without Special Triangles

In this section we prove Theorem 3.3 for the case $k=3$. The proof is based on a refinement of the weight function argument from Section 12. Recall the definition of the weight function

$$w(G, F)\colon \binom{X}{k-1} \times \mathscr{F} \to R^+ \colon w(G, F) = \begin{cases} 1/\deg_{\mathscr{F}}(G) & \text{if } G \subset F \\ 0 & \text{otherwise.} \end{cases}$$

To give the flavor of the argument first we give a short proof of Chvátal's theorem, i.e., the case $k=d+1$ of Conjecture 1.3.

Suppose that $\mathcal{F} \subset \binom{X}{k}$, $k=d+1 \geqslant 3$ and $\mathcal{F}$ contains no d-dimensional simplex.

We are going to show that for $n>k+2$ necessarily $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$ holds.

Claim 13.1.

$$\sum_{G \subset F} w(G, F) \geqslant 1+\frac{d}{n-d} \qquad \text{holds for all } F \in \mathcal{F}. \tag{13.0}$$

Proof. Note that $\deg_{\mathcal{F}}(G) \leqslant n-d$ holds for every $G \in \binom{X}{d}$. Thus (13.0) follows immediately for all $F \in \mathcal{F}$ which contain some $G \in \binom{X}{d}$ with $\deg_{\mathcal{F}}(G)=1$. Suppose next that $\deg_{\mathcal{F}}(G) \geqslant 2$ holds for some $F \in \mathcal{F}$ and all $G \in \binom{F}{d}$. We claim that if for some $G \in \binom{F}{d}$ the inequality is strict, i.e., if $\deg_{\mathcal{F}}(G) \geqslant 3$ then $\mathcal{F}$ contains a d-dimensional simplex. Indeed, let $\{G_1, G_2, ..., G_k\} = \binom{F}{d}$ with $\deg_{\mathcal{F}}(G_1) \geqslant 3$. Since $\deg_{\mathcal{F}}(G_i) \geqslant 2$, we may choose $x_i \notin F$ such that $F_i = G_i \cup \{x_i\}$ is in $\mathcal{F}$. If $\{F_1, ..., F_k\}$ is a simplex then we are done, if not then $F_1 \cap \cdots \cap F_k \neq \emptyset$. Consequently, $x_1 = \cdots = x_k$. Using $\deg_{\mathcal{F}}(G_1) \geqslant 3$ choose $y_1 \neq x_1$, $y_1 \notin F$ such that $F_1' = G_1 \cup \{y_1\} \in \mathcal{F}$. Now $\{F_1', F_2, ..., F_k\}$ is a simplex in $\mathcal{F}$, the desired contradiction.

Therefore $\deg_{\mathcal{F}}(G_i)=2$ for $i=1, ..., k$, yielding

$$\sum_{G \subset F} w(G, F) = \frac{k}{2} = 1 + \frac{k-2}{2}.$$

Now (13.0) follows for $n \geqslant k+2$ if $k \geqslant 4$ and $n \geqslant 6$ if $k=3$. Moreover, for $n \geqslant k+3$ the inequality is strict unless $\deg_{\mathcal{F}}(G)=1$ for some $G \subset F$ and this degree is $n-d$ for the d remaining sets $\binom{F}{d}$.

Summing (13.0) for all $F \in \mathcal{F}$ gives

$$|\mathcal{F}| \frac{n}{n-d} \leqslant \sum_{F \in \mathcal{F}} \sum_{G \subset F} w(G, F) = \sum_{G \in \Delta_{k-1}(\mathcal{F})} \sum_{G \subset F \in \mathcal{F}} 1/\deg_{\mathcal{F}}(G)$$
$$= |\Delta_{k-1}(\mathcal{F})| \leqslant \binom{n}{d}.$$

Comparing the extreme sides gives $|\mathcal{F}| \leqslant ((n-d)/n)\binom{n}{d} = \binom{n-1}{d}$, as desired.

Let us recall the following special case of a theorem of Bollobás [Bo1].

Proposition 13.2. *Suppose that $\mathcal{F} \subset \binom{X}{k}$ and for every $F \in \mathcal{F}$ there is some $G \in \binom{F}{k-1}$ with $\deg_{\mathcal{F}}(G)=1$. Then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$ with equality holding if and only if $\mathcal{F}$ consists of all k-subsets of X through some fixed element of X.*

Combining this proposition with our preceding observations shows the uniqueness of the optimal families for $n \geqslant k+3$.

Now we turn to Theorem 3.3, $k=d+1=3$. Since $\mathscr{F}$ does not contain the special simplex $\{\{1,2,4\},\{1,3,5\},\{2,3,6\}\}$, it contains even less the star-shaped tree $\{\{1,2,3\},\{1,2,4\},\{1,3,5\},\{2,3,6\}\}$.

Let, with the notation of Section 12, a_1, a_2, a_3 be, in non-increasing order, the degrees of the 2-subsets of some fixed $F \in \mathscr{F}$. We call (a_1, a_2, a_3) the type of F.

Then, as we proved in Section 12, either $a_1 = 1$ or $a_1 = a_2 = 2$ or $a_1 = a_2 = a_3 = 3$ hold.

Unfortunately, in the last two cases (13.0) need not be satisfied. This makes our argument more complicated. Call F, $F' \in \mathscr{F}$ neighbors if $|F \cap F'| = 2$.

We will get by this difficulty by transferring some of the weights from sets for which (13.0) is "generously" satisfied to its neighbors and showing that (13.0) holds for these modified weights.

Define the weight $w(F)$ of $F \in \mathscr{F}$ by $w(F) = (1/a_1) + (1/a_2) + (1/a_3)$. We first define auxiliary functions $W_F(H)\colon \mathscr{F} \to R^+$ for all $F \in \mathscr{F}$.

The definition of $W_F(H)$ will depend on the type of F.

(a) F has type $(1, 1, a_3)$ with $a_3 \geqslant 4$ then

$$W_F(H) = \begin{cases} 1/(n-2) & \text{if } |H \cap F| = 2 \\ w(F) - (a_3 - 1)/(n-2) & \text{if } H = F \\ 0 & \text{otherwise.} \end{cases}$$

(b) F has type $(1, 2, a_3)$ or $(1, 3, a_3)$ with $a_3 \geqslant 4$

$$W_F(H) = \begin{cases} 2/(n-2) & \text{if } |H \cap F| = 2 \text{ and } \deg_{\mathscr{F}}(H \cap F) \leqslant 3 \\ w(F) - 2(a_3 - 1)/(n-2) & \text{if } H = F \\ 0 & \text{otherwise.} \end{cases}$$

(c) F has type (a_1, a_2, a_3) with $a_1 < 3$, $2 \leqslant a_3 \leqslant 3$ (i.e., one of (1, 1, 2), (1, 1, 3), (1, 2, 2), (1, 2, 3), (1, 3, 3), (2, 2, 2), (2, 2, 3), (2, 3, 3)), then

$$W_F(H) = \begin{cases} 2/(n-2) & \text{if } |H \cap F| = 2 \\ w(F) - 2(\sum a_i - 3)/(n-2) & \text{if } H = F \\ 0 & \text{otherwise.} \end{cases}$$

(d) For the remaining $F \in \mathscr{F}$ we define

$$W_F(H) = \begin{cases} w(F) & \text{for } H = F \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define the new weight function by

$$W(H) = \sum_{F \in \mathscr{F}} W_F(H).$$

It should be noted that

$$\sum_{H \in \mathscr{F}} W(H) = \sum_{H \in \mathscr{F}} w(F) \text{ holds.} \tag{13.1}$$

Hence if $W(H) \geqslant 1 + 2/(n-2)$ held for all $H \in \mathscr{F}$, we would obtain as above

$$|\mathscr{F}| \left(1 + \frac{2}{n-2}\right) \leqslant \sum_{H \in \mathscr{F}} W(H) = \sum_{H \in \mathscr{F}} w(H) = |\Delta_2(\mathscr{F})| \leqslant \binom{n}{2}, \tag{13.2}$$

which yields $|\mathscr{F}| \leqslant \binom{n-1}{2}$, as desired. Since we are looking for the maximal families, we may assume that

$$|\mathscr{F}| \geqslant \binom{n-1}{2}. \tag{13.3}$$

Divide $\mathscr{F}$ into four parts, $\mathscr{F} = \mathscr{F}_0 \cup \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$:

$\mathscr{F}_0 = \{F \in \mathscr{F}$: the type of F falls in a), b) or c), i.e., either $(1, 1, a)$, $(1, 2, a)$, $(1, 3, a)$, $(1, 1, 2)$, $(1, 1, 3)$, $(1, 2, 2)$, $(1, 2, 3)$, $(1, 3, 3)$, $(2, 2, 2)$, $(2, 2, 3)$ or $(2, 3, 3)$, where $a \geqslant 4\}$,

$$\mathscr{F}_1 = \{F \in \mathscr{F}\text{: the type of } F \text{ is } (1, a_2, a_3)\text{, where } a_2, a_3 \geqslant 4\},$$

$$\mathscr{F}_2 = \{F \in \mathscr{F}\text{: the type of } F \text{ is } (2, 2, a)\text{, where } a \geqslant 4\}$$

and

$$\mathscr{F}_3 = \{F \in \mathscr{F}\text{: the type of } F \text{ is } (3, 3, 3)\}.$$

A simple case by case analysis gives

Proposition 13.3. *If $F \in \mathscr{F}_0$ then for $n \geqslant 75$*

$$W(F) \geqslant 1 + 2/(n-2).$$

Here equality holds only if F has type $(1, 1, n-2)$.

Proof. If the type of F is not $(1, 1, a)$ then $W(F) \geqslant w(F) - 5(2/(n-2))$. However, $w(F) \geqslant \frac{1}{2} + \frac{1}{3} + \frac{1}{3}$. ∎

PROPOSITION 13.4. *If $F \in \mathscr{F}_1$, then*

$$W(F) \geqslant 1 + 2/(n-2).$$

Here equality holds only if F has type $(1, n-2, n-2)$.

Proof. For $F \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$ we have $W(F) \geqslant w(F) (\geqslant 1)$. In the case of $F \in \mathscr{F}_1$, if its type is $(1, a_2, a_3)$, we get $W(F) \geqslant 1 + 1/a_2 + 1/a_3 \geqslant 1 + 2/(n-2)$, (because $a_i \leqslant n-2$). ∎

Now consider a set $F \in \mathscr{F}_2$ with $W(F) \leqslant 1 + 2/(n-2)$, $F = \{1, 2, 3\}$, $\deg_{\mathscr{F}}(\{1, 2\}) \geqslant 4$.

Claim 13.5. There exists a (unique) element of X, say 4, such that $\binom{\{1,2,3,4\}}{3} \subset \mathscr{F}_2$, and $\deg_{\mathscr{F}}(\{3, 4\}) \geqslant 4$. Then, by definition of $\mathscr{F}_2$, we have $\deg_{\mathscr{F}}(\{i, j\}) = 2$ for $1 \leqslant i \leqslant 2$, $3 \leqslant j \leqslant 4$.

Proof. Consider the triples $\{1, 3, x\}$ and $\{2, 3, y\}$, $x, y \in X \to F$. We claim $x = y$. Indeed, $\deg_{\mathscr{F}}(\{1, 2\}) \geqslant 4$ implies that there exists a triple $\{1, 2, u\} \in \mathscr{F}$, $u \neq x$, $u \neq y$, $u \neq 3$, hence $\{1, 2, u\}$, $\{1, 3, x\}$, $\{2, 3, y\}$ form a special triangle. So we have that e.g., $F_2 = \{1, 3, 4\}$, $F_3 = \{2, 3, 4\} \in \mathscr{F}$. As $\deg_{\mathscr{F}}(\{1, 3\}) = 2$ we have $F_2 \notin \mathscr{F}_1 \cup \mathscr{F}_3$. Moreover $F_2 \in \mathscr{F}_0$ would imply $W(F) > w(F) + 2/(n-2)$, so the only possibility is that $F_2 \in \mathscr{F}_2$. Similarly $F_3 \in \mathscr{F}_2$. The type of F_2 is $(2, 2, a)$ where $a \geqslant 4$. We claim that $\deg_{\mathscr{F}}(\{1, 4\}) = 2$, $\deg_{\mathscr{F}}(\{3, 4\}) \geqslant 4$. Suppose the opposite, and consider $F_3 = \{2, 3, 4\}$. As $\deg_{\mathscr{F}}(\{2, 3\}) = \deg_{\mathscr{F}}(\{3, 4\}) = 2$ and its type $(2, 2, b)$ $(b \geqslant 4)$ we have $\deg_{\mathscr{F}}(\{2, 4\}) = b$. Then each of the pairs $\{1, 2\}$, $\{1, 4\}$ and $\{2, 4\}$ has degree $\geqslant 4$, a contradiction.

So we have $\deg_{\mathscr{F}}(\{3, 4\}) = a \geqslant 4$, $\deg_{\mathscr{F}}(\{1, 2\}) = b \geqslant 4$, and $\deg_{\mathscr{F}}(\{i, j\}) = 2$ hold for $i \in \{1, 2\}$, $j \in \{3, 4\}$. Then there exists an $F_4 \neq F_2$ containing $\{1, 4\}$. We claim $F_4 \subset \{1, 2, 3, 4\}$, i.e., $F_4 = \{1, 2, 4\}$. Otherwise, if $F_4 = \{1, 4, 5\}$ then F_4, F_3 and a triple F_5 (distinct from F_1, F_4 and $\{1, 2, 5\}$) through $\{1, 2\}$ would form a triangle. ∎

PROPOSITION 13.6. *Let $\{F_1, F_2, F_3, F_4\} = \binom{\{1,2,3,4\}}{3} \subset \mathscr{F}_2$ as in the previous claim. Suppose that $W(F_1) \leqslant 1 + 2/(n-2)$. Then*

$$\sum_{1 \leqslant i \leqslant 4} W(F_i) > 4(1 + 2/(n-2)).$$

Proof. Clearly, $\sum W(F) = 2(\frac{1}{2} + \frac{1}{2} + (1/a)) + 2(\frac{1}{2} + \frac{1}{2} + (1/b))$, so $\sum W(F) \leqslant 4(1 + 2/(n-2))$ implies

$$a + b \geqslant n - 2. \tag{13.4}$$

Let $A =: \{x \in F - \{1, 2, 3, 4\}: \{3, 4, x\} \in \mathscr{F}\}$, $B =: \{x \in F - \{1, 2, 3, 4\}: \{1, 2, x\} \in \mathscr{F}\}$. First we show that $|A \cap B| \leqslant 1$. Indeed, $x \in A \cap B$ implies

$\deg_{\mathscr{F}}(\{i, x\}) = 1$, i.e., the type of $\{1, 2, x\}$ is $(1, 1, b)$. So $\{1, 2, x\}$ (by the definition of W) transfers a weight $1/(n-2)$ to F_1 (and to F_4 also). As

$$1 + 1/(n-2) \leqslant w(F) \leqslant W(F) \leqslant 1 + 2/(n-2),$$

there exists at most one such an $x \in A \cup B$.

Consider a vertex $y \in A \Delta B$, say $y \in A - B$, $\{3, 4, y\} \in \mathscr{F}$. Then $\deg_{\mathscr{F}}(\{i, y\}) = 0$ for $i = 1, 2$. (Otherwise, if, e.g., $\{1, y, u\} \in \mathscr{F}$ then $\{1, y, u\}$, $\{3, 4, y\}$ and $\{1, 2, 3\}$ form a traingle.) So (13.4) implies that there are at least $2|A \Delta B| \geqslant 2(a + b - 5) \geqslant 2(n-7)$ uncovered pairs. Hence by (13.2)

$$|\mathscr{F}| \leqslant |\Delta_2(\mathscr{F})| \leqslant \binom{n}{2} - 2(n-7).$$

This contradicts (13.3). ▮

As a triple $F \in \mathscr{F}_2$ can belong to only one configuration given in Claim 13.5 we have

PROPOSITION 13.7. *If* $\mathscr{F}_2 \neq \emptyset$ *then* $(1/|\mathscr{F}_2|) \sum_{F \in \mathscr{F}_2} W(F) > 1 + 2/(n-2)$. ▮

PROPOSITION 13.8. *For* $F \in \mathscr{F}_3$ *we have* $W(F) \geqslant 1 + 2/(n-2)$.

Proof. Suppose on the contrary that for $F = \{1, 2, 3\} \in \mathscr{F}_3$ $W(F) < 1 + 2/(n-2)$ holds. Then (by the definition of W) we have

$$W(F) = 1. \tag{13.5}$$

For every pair $G \subset F$, there are two elements x_G^1 and x_G^2 such that $(G \cup \{x_G^i\}) \in \mathscr{F}$ $(i = 1, 2)$. An easy case by case checking (a similar one to Proposition 13.5) shows that $\{x_G^1, x_G^2\}$ must be the same pair for all three $G \subset F$. For example, $\{i, j, k\} \in \mathscr{F}$ for $1 \leqslant i < j \leqslant 3$, $k = 4, 5$. Consider the triple $F_2 = \{1, 2, 4\}$. As $\deg_{\mathscr{F}}(\{1, 2\}) = 3$ and $\deg_{\mathscr{F}}(\{1, 4\}) \geqslant 2$, $\deg_{\mathscr{F}}(\{2, 4\}) \geqslant 2$ its type is either $(2, 2, 3)$, $(2, 3, 3)$, or $(3, 3, 3)$. But in the first two cases F_2 transfers some weight to F which contradicts to (13.5). Hence all the 6 triples intersecting F in 2 elements have type $(3, 3, 3)$. This implies that $\binom{\{1, 2, 3, 4, 5\}}{3} \subset \mathscr{F}_3$.

Denote $\{1, 2, 3, 4, 5\}$ by K. For every point $x \in X - K$ one of the following holds:

(i) there exist two elements $i, j \in K$ such that $\deg_{\mathscr{F}}(\{x, i\}) = \deg_{\mathscr{F}}(\{x, j\}) = 0$, or

(ii) there exists an element $y \in X - K - \{x\}$ such that $\{x, y, i\} \in \mathscr{F}$ for at least 4 elements $i \in K$, and these triples have type $(1, 1, a)$ (where $a \geqslant 4$).

Suppose (i) does not hold. Then there are $i_1, i_2, i_3, i_4 \in K$ such that $\deg_{\mathscr{F}}(\{x, i_j\}) \geqslant 1$, i.e., $\{x, i_j, y_j\} \in \mathscr{F}$ $(1 \leqslant j \leqslant 4)$. Then $y_1 = \cdots = y_4$ follows. This also implies that $\deg_{\mathscr{F}}(\{x, i_j\}) = 1$, hence $\deg_{\mathscr{F}}(\{y, i_j\}) = 1$, i.e., (ii) holds. Denote the number of elements of $X - K$ satisfying (i) by p. Then $(n-5-p)/2$ pairs satisfy (ii). In the second case delete four triangles of the form $\{x, y, i\}$ $(i \in K)$. We obtain the family $\mathscr{F}'$

$$|\mathscr{F}'| = |\mathscr{F}| - 2(n-p-5),$$

and

$$|\Delta_2(\mathscr{F}')| \leqslant \binom{n}{2} - 2p - 4(n-p-5).$$

Using $|F'| \leqslant |\Delta_2(\mathscr{F}')|$ we obtain

$$|\mathscr{F}| - 2(n-p-5) \leqslant \binom{n}{2} - 2p - (n-p-5),$$

i.e., $|\mathscr{F}| \leqslant \binom{n}{2} - 2(n-5) < \binom{n-1}{2}$ which contradicts (13.3). ∎

The Proof of Theorem 3.3. *for* $k = 3$. Proposition 13.3, 13.4, 13.7, and 13.8 imply that

$$\sum_{F \in \mathscr{F}_i} W(F) \geqslant \left(1 + \frac{2}{n-2}\right) |\mathscr{F}_i|$$

holds for $i = 0, 1, 2, 3$ (resp.). Then the argument given in (13.2) gives $|\mathscr{F}| \leqslant \binom{n-1}{2}$, as desired. Moreover, if equality holds here then

$$\text{for } F \in \mathscr{F}_0 \text{ its type is } (1, 1, n-2),$$

$$\text{for } F \in \mathscr{F}_1 \text{ its type is } (1, n-2, n-2),$$

$$\mathscr{F}_2 = \varnothing \quad \text{and} \quad \mathscr{F}_3 = \varnothing.$$

The first 3 statements are implied by Propositions 13.3, 13.4, and 13.7, respectively. To prove $\mathscr{F}_3 = \varnothing$ consider a triple $F \in \mathscr{F}_3$ with $W(F) = 1 + 2/(n-2)$. Then F has received a weight $2/(n-2)$ from an edge $H \in \mathscr{F}_0$. But this is impossible because all edges in $\mathscr{F}_0$ have type $(1, 1, n-2)$.

Summarizing, we have that in the case of $|\mathscr{F}| = \binom{n-1}{2}$,

$$\text{every } F \in \mathscr{F} \text{ has a } G \in \Delta_2(F) \text{ with } \deg_{\mathscr{F}}(G) = 1. \tag{13.6}$$

Then Proposition 13.2 gives that $\cap \mathscr{F} \neq \varnothing$. ∎

14. Quadruple-Systems without Special Triangles

Here we prove Theorem 3.3 for the case $k=4$. We will proceed as in Section 9. Consider a family $\mathscr{F} \subset \binom{X}{4}$ which does not contain a configuration isomorphic to $\{\{1, 2, a, b\}, \{1, 3, c, d\}, \{2, 3, e, f\}\}$. We can suppose that

$$|\mathscr{F}| \geqslant \binom{n-1}{3}. \tag{14.1}$$

Starting with the family $\mathscr{F}$ let us apply Theorem 6.2 (with $s=20$) repeatedly as in Section 8 to obtain pairwise disjoint families $\mathscr{H}_1 = \mathscr{F}^*$, $\mathscr{H}_2 = (\mathscr{F} - \mathscr{H}_1)^*, \ldots, \mathscr{H}_m = (\mathscr{F} - \mathscr{H}_1 - \cdots - \mathscr{H}_{m-1})^*$, $\mathscr{R} = \mathscr{F} - (\mathscr{H}_1 \cup \cdots \cup \mathscr{H}_m)$ so that the corresponding families $\mathscr{J}_1, \ldots, \mathscr{J}_m$ satisfy $r(\mathscr{J}_i) = 3$ for $1 \leqslant i \leqslant m-1$ and $r(\mathscr{J}_m) \leqslant 2$. In view of Claim 6.3,

$$|\mathscr{R} \cup \mathscr{H}_m| \leqslant \frac{1}{c(4, 20)} \binom{n}{2} = O(n^2). \tag{14.2}$$

Instead of Lemma 7.1 we need the following more exact statement: Let F be a 4-element set $F = \{1, 2, 3, 4\}$, and let $\mathscr{J} \subset 2^F - \{F\}$ be a family closed under intersection. Suppose that $r(\mathscr{J}) = 3$, and $\mathscr{J}$ does not contain a subfamily isomorphic to either $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ or $\mathscr{A} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$.

Lemma 14.1. *Either*

(i) $\mathscr{J} = \{A : x \in A \subsetneqq F\}$ for some $x \in F$, or

(ii) The family of maximal members in $\mathscr{J}$ is isomorphic to

$$\{\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4\}\}.$$

Proof. If $\mathscr{J}$ contains 2 or 3 3-element subsets then $\mathscr{J}$ is isomorphic to the family given by (ii) or (i), respectively. If $\mathscr{J}$ contains 0, or 1 3-sets then (because every 2-set is covered by $\mathscr{J}$) it contains a triangle, or a family isomorphic to $\mathscr{A}$, respectively. Finally, if $\mathscr{J}$ contains 4 3-sets, then it contains a triangle also because it is closed under intersection. ∎

Continuing the proof of Theorem 3.3 define

$$\mathscr{F}_1 = \cup \{\mathscr{H}_i : \mathscr{J}_i \text{ fulfills } (i)\},$$
$$\mathscr{F}_2 = \cup \{\mathscr{H}_i : \mathscr{J}_i \text{ fulfills (ii)}\} \qquad (1 \leqslant i < m).$$

The only thing that we have to prove, that $\mathscr{F}$ fulfills the constraints of Lemma 9.1 with $k=4$, $l=1$, $(r=20)$ and $\mathscr{F}_0 =: \mathscr{F}_2 \cup \mathscr{H}_m \cup \mathscr{R}$, $\mathscr{F}_1 =: \mathscr{F}_1$.

It is clear that (9.1) holds. Condition (ii) holds by definition, and (iii) is proved by Theorem 3.4 ($k=4, d=2$). So we have to prove (i), i.e., $|\mathscr{F}_2 \cup \mathscr{H}_m \cup \mathscr{R}| = O(n^2)$. By (14.2), it is sufficient to prove

$$|\mathscr{F}_2| = O(n^2). \tag{14.3}$$

To prove (14.3) we need

PROPOSITION 14.2. $|\mathscr{F}_1| + 2|\mathscr{F}_2| \leqslant \binom{n}{3}$.

Proof. We associate i 3-sets to every $F \in \mathscr{F}_i$ ($i=1,2$). These will all be distinct implying $\sum i|\mathscr{F}_i| \leqslant \binom{n}{3}$.

For $F \in \mathscr{F}_1$ let $B(F) = \binom{F}{3} \backslash \mathscr{J}_F$, and for $F \in \mathscr{F}_2$ let $\{C(F), C'(F)\} = \binom{F}{3} \backslash \mathscr{J}_F$. We claim that these 3-sets are all distinct. Suppose on the contrary, we will get a contradiction with the fact that $\mathscr{F}_1 \cup \mathscr{F}_2$ is triangle-free. Without loss of generality we can suppose $F = \{1,2,3,4\}$, $F' = \{2,3,4,5\}$, $\{2,3,4\} \notin \mathscr{J}_F \cup \mathscr{J}_{F'}$. We distinguish three cases. Denote by $\mathscr{J}^*$ the maximal members of $\mathscr{J}$.

If $F, F' \in \mathscr{F}_1$ then $\mathscr{J}_F^* = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}\}$. Hence $\{1,2\}$, $\{1,3\} \in \mathscr{J}_F$, thus there exist an F_2 and $F_3 \in \mathscr{F}$ such that $F_2 \cap F = \{1,2\}$, $F_2 \cap F' = \{2\}$ and $F_3 \cap F = \{1,3\}$, $F_3 \cap F' = \{3\}$, $F_3 \cap F_2 = \{1\}$. Then F', F_2, and F_3 form a triangle a contradiction.

If $F \in \mathscr{F}_1$ and $F' \in \mathscr{F}_2$ then we can suppose that $\mathscr{J}_{F'}^* = \{\{3,4\}, \{2,3,5\}, \{2,4,5\}\}$. Then $\{1,3\}$, $\{1,4\} \in \mathscr{J}_F$, $\{3,4\} \in \mathscr{J}_{F'}$, these form a triangle and all of them are centers of sunflowers. Thus there are F^1, F^2, and $F^3 \in \mathscr{F}$, through them forming a triangle.

If $F, F' \in \mathscr{F}_2$ then we have two possibilities for $\mathscr{J}_F^*$. Either $\mathscr{J}_F^* = \{\{3,4\}, \{1,2,3\}, \{1,2,4\}\}$ or $\mathscr{J}_F^* = \{\{2,3\}, \{1,2,4\}, \{1,3,4\}\}$. In the first case $\{1,2,3\}$, $\{3,4\}$, and $\{2,4,5\}$ form a special triangle (consequently there exists a special triangle through them), and in the second case $\{2,3\}$, $\{3,4\}$, and $\{2,4,5\}$ form a triangle. ∎

Proof of (14.3). Proposition 14.2 implies that

$$|\mathscr{F}_1| + |\mathscr{F}_2| \leqslant \binom{n}{3} - |\mathscr{F}_2|.$$

Hence (14.1) and (14.2) give

$$\binom{n-1}{3} \leqslant |\mathscr{F}| = |\mathscr{F}_1| + |\mathscr{F}_2| + |\mathscr{R} \cup \mathscr{H}_m| \leqslant \binom{n}{3} - |\mathscr{F}_2| + \frac{1}{c}\binom{n}{2},$$

i.e., $|\mathscr{F}_2| \leqslant (1 + \frac{1}{c})\binom{n}{2}$.

15. An Extremal Problem with No Exponent

More than 10 years ago Ruzsa and Szemerédi proved the following result:

Let us call a k-graph $\mathscr{F}$ *linear* if $|F \cap F'| \leqslant 1$ holds for any two distinct F, $F' \in \mathscr{F}$.

Let $\mathscr{S} = \{S_1, S_2, S_3\}$ be a 2-dimensional special simplex with $|S_1| = |S_2| = |S_3| = 3$.

Let us denote by $m(n)$ the maximum size of $\mathscr{H} \subset \binom{X}{3}$, such that $\mathscr{H}$ is linear and $\mathscr{H}$ contains no 2-dimensional special simplex.

Theorem 15.1 [RS]. *$m(n) = o(n^2)$ but $m(n) \neq O(n^{2-\varepsilon})$ for every $\varepsilon > 0$.*

Now we shall use this result to prove Theorem 4.8. First we show that $s(n, 5, 2, 3) \neq O(n^{4-\varepsilon})$ holds for every $\varepsilon > 0$. Let $X = X_1 \cup X_2 \cup X_3$ be a partition with $|X_1| = |X_2| = \lfloor n/3 \rfloor$. Let further $\mathscr{H} \subset \binom{X_1}{3}$ be a linear 3-graph without a special triangle and with $|\mathscr{H}| = m(\lfloor n/3 \rfloor)$. Define

$$\mathscr{F} = \left\{F \in \binom{X}{5}: (F \cap X_1 \in \mathscr{H},\ |F \cap X_2| = |F \cap X_3| = 1\right\}.$$

Clearly, $|\mathscr{F}| = |\mathscr{H}|\ |X_2|\ |X_3| \geqslant m(\lfloor n/3 \rfloor)\lfloor n/3 \rfloor^2 \neq O(n^{4-\varepsilon})$. We have to show that $\mathscr{F}$ contains no $(2, 3)$-simplex. Suppose the contrary and let F_1, F_2, F_3 form such a simplex.

Let us consider the sets $H_i = F_i \cap X_1$, $A_i = F_i \cap (X_2 \cup X_3)$, $i = 1, 2, 3$. If $A_i \neq A_j$ then $|F_i \cap F_j| \geqslant 3$ and the linearity of $\mathscr{H}$ imply $H_i = H_j$. But then the third F cannot meet both F_i and F_j in at least three vertices. Thus $A_1 = A_2 = A_3$ and $|H_i \cap H_j| \geqslant 1$ for $1 \leqslant i < j \leqslant 3$. Since $\mathscr{H}$ contains no triangle, we infer $H_1 \cap H_2 \cap H_3 \neq \emptyset$. However, this yields $|F_1 \cap F_2 \cap F_3| \geqslant 3$, a contradiction.

To prove $s(n, 5, 2, 3) = o(n^4)$ suppose that $\mathscr{F} \subset \binom{X}{5}$ contains no $(2, 3)$-simplex.

Let us apply Theorem 6.2 to obtain $\mathscr{F}^*$ and $\mathscr{J}$. If $r(\mathscr{J}) \leqslant 3$, then $|\mathscr{F}^*| \leqslant \binom{n}{3}$. Thus we may assume that $r(\mathscr{J}) \geqslant 4$ holds.

If $\mathscr{J}$ contains 3 or more 4-element sets, then in $F \in \mathscr{F}^*$ we can choose A_1, A_2, A_3 distinct 4-subsets which are centers of large sunflowers. Since A_1, A_2, A_3 form a $(2, 3)$-simplex, Claim 6.4 implies that $\mathscr{F}^*$ contains a $(2, 3)$-simplex, a contradiction.

Thus we may suppose that, e.g., $\{1, 2\}$ is contained in no 4-element member of $\mathscr{J}$. This implies by Theorem 6.2(iii) that for all $x_1 \in X_1$ and $x_2 \in X_2$ the 3-graph $\mathscr{F}(x_1, x_2) = \{H \in \binom{X}{3}: (\{x_1, x_2\} \cup H) \in \mathscr{F}\}$ is linear. Now Theorem 15.1 implies $|\mathscr{F}(x_1, x_2)| = o(n^2)$. Consequently, $|\mathscr{F}| \leqslant n^2 o(n^2) = o(n^4)$ holds. ▮

16. Open Problems

It is not hard to find open problems in this area. Actually, the determination of $\mathrm{ex}(n, \mathscr{H})$ is unsolved for almost all k-graphs $\mathscr{H}$.

We believe that the main progress in the near future will come by singling out those (possibly very few) $\mathscr{H}$'s for which the determination of $\mathrm{ex}(n, \mathscr{H})$ is a solvably difficult problem.

As mentioned in the Introduction, this does not seem to be the case for even such simple graphs as the cycle of length $2l$, $l \geqslant 6$.

An important distinction between k-graphs is whether they are k-partite or not.

For every k-partite k-graph $\mathscr{H}$ we know by a theorem of Erdös [E2] that $\mathrm{ex}(n, \mathscr{H}) = O(n^{k-\varepsilon(\mathscr{H})})$, where $\varepsilon(\mathscr{H})$ is a positive real, depending only on $\mathscr{H}$.

The determination of the best possible value of $\varepsilon(\mathscr{H})$ is an open problem even for such simple 3-graphs as $K(t, t, t) = \{\{x, y, z\}: x \in X, y \in Y, z \in Z\}$, X, Y, and Z being pairwise disjoint t-element sets.

Or for $\mathscr{H}_1 = \{\{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$. For $\mathscr{H}_1$ it is known that $\frac{1}{4} \leqslant \varepsilon(\mathscr{H}_1) \leqslant \frac{1}{2}$; cf. [F3]. If $\mathscr{H}$ is not k-partite, then obviously $\mathrm{ex}(n, \mathscr{H}) \geqslant \lfloor n/k \rfloor^k = \Omega(n^k)$ and by an averaging argument of Katona, *et al.* [KNS] there exists

$$\beta(\mathscr{H}) = \lim_{n \to \infty} \mathrm{ex}(n, \mathscr{H}) \Big/ \binom{n}{k}.$$

Turán's problem [T2], one of the outstanding open problems in combinatorics is the case when $\mathscr{H} = \binom{Y}{k}$ for some fixed Y, $|Y| > k$.

Turán conjectured that $\beta(\binom{\{1,2,3,4\}}{3}) = \frac{5}{9}$ and $\beta(\binom{\{1,2,3,4,5\}}{3}) = \frac{3}{4}$. Kalai [Ka1] proposed an interesting approach to the first one, while some positive evidence in support of the second is provided by [F6].

Let us mention that the only case of a k-graph $\mathscr{H}$ with $k \geqslant 3$ for which the exact value of $\mathrm{ex}(n, \mathscr{H})$ is known is $\mathscr{H}_0 = \{\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4, 5\}\}$.

It is proved in [FF1] that $\mathrm{ex}(n, \mathscr{H}_0) = \lfloor n/3 \rfloor \lfloor (n+1)/3 \rfloor \lfloor (n+2)/3 \rfloor$ for $n \geqslant 3{,}000$.

Sometimes it is easier and more natural to exclude a finite set of k-graphs and not only one. For example, the determination of $s(n, k, d, l)$ falls into this category.

An other example is the following:

Determine $\max\{|\mathscr{F}|: \mathscr{F} \subset \binom{X}{k}, \mathscr{F}$ contains no three members F_1, F_2, F_3 with $F_1 \Delta F_2 \subset F_3\} =: \delta(n, k)$.

Solving a problem of Katona [Ka2], Bollobás [Bo2] showed that $\delta(n, 3) = \lfloor n/3 \rfloor \lfloor (n+1)/3 \rfloor \lfloor (n+2)/3 \rfloor$ and he conjectured that $\delta(n, k) =$

$\lfloor n/k \rfloor \lfloor (n+1/k \rfloor \cdots \lfloor (n+k-1)/k \rfloor$ in general. The lower bound follows by considering the complete k-partite k-graph on n vertices. Recently Sidorenko [Si] established the validity of this conjecture for $k=4$.

Now let us list some of the most challenging open problems related to simplices.

First, one should solve Conjecture 1.3 for all $n > ((d+1)/d)\, k$, $k > d$, i.e., to show that all $\mathscr{F} \subset \binom{X}{k}$ with $|\mathscr{F}| > \binom{n-1}{k-1}$ contain a d-simplex.

Similarly, we believe that the following is true.

Conjecture 16.1 Suppose that $\mathscr{F} \subset \binom{X}{k}$, $|X| = n \geqslant 2k$, $|\mathscr{F}| > \binom{n-1}{k-1}$. Then $\mathscr{F}$ contains a special $(k-1)$-simplex.

Let us note that Theorem 3.3 shows the validity of Conjecture 16.1 for $k=3$ and $n \geqslant 75$. From results in [BF] it follows for $k=3$ and $n=6$ or 7.

We should not leave unmentioned the following result of Erdös and Milner.

THEOREM 16.2. [EM]. *Suppose that* $\mathscr{F} \subset 2^X$, $|\mathscr{F}| > 2^{n-1}+n$. *Then* $\mathscr{F}$ *contains a triangle, i.e., three sets* F_1, F_2, F_3 *with* $F_1 \cap F_2 \cap F_3 = \varnothing$ *but* $F_i \cap F_j \neq \varnothing$ *for* $1 \leqslant i < j \leqslant 3$.

Conjecture 16.3. Suppose that $\mathscr{F} \subset 2^X$, $|\mathscr{F}| > \sum_{i \geqslant d} \binom{n-1}{d-1} + \sum_{i<d} \binom{n}{d}$. Then for $n \geqslant n_0(d)$, $\mathscr{F}$ must contain a d-simplex.

To see that this conjecture, if true, is best possible, consider $\mathscr{F} = \{F \subset X\colon x \in F \text{ or } |F| < d\}$, for some fixed element $x \in X$.

In Theorem 4.8 we showed that for $\mathscr{S}$ the 5-uniform special (2, 3)-simplex $\mathrm{ex}(n, \mathscr{S})$ has no exponent, i.e., $\mathrm{ex}(n, \mathscr{S}) = o(n^4)$ but $\lim_{n\to\infty} \mathrm{ex}(n, \mathscr{S})/n^{4-\varepsilon} = \infty$ for all $\varepsilon > 0$. This solves a problem of Erdös [E6].

Actually, we can show that this is the smallest example in the sense that for all 2–3-, and 4-graphs $\mathscr{H}$ with $|\mathscr{H}| \leqslant 3$ there exists a real number (actually an integer) r such that $c_1 n^r \leqslant \mathrm{ex}(n, \mathscr{H}) \leqslant c_2 n^r$ holds with c_1 and c_2 being constants depending only on $\mathscr{H}$.

Using Theorem 4.8 one deduces easily $s(n, k, 2, k-2) = o(n^{k-1})$ for all $k \geqslant 6$ as well.

PROPOSITION 16.4. *Does* $\lim_{n\to\infty} s(n, k, 2, k-2)/n^{k-1-\varepsilon} \to \infty$ *hold for all* $k \geqslant 6$ *and* $\varepsilon > 0$?

Finally, let us call the interested and/or courageous reader's attention to Conjecture 2.6 ($\mathrm{ex}(n, \mathscr{H})$ for $\mathscr{H}$ a fixed sunflower) and Conjecture 5.5 ($\mathrm{ex}(n, \mathscr{H})$, where $\mathscr{H}$ is an intersection-condensed k-graph with $p=0$, $q=k-1$).

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