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COVERING ALL SECANTS OF A SQUARE

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Suppose that n points are given in the unit square. Then there exists an intersecting line whose L_∞-distance is at least $2/3(n+1)$ from each point. This is a slight improvement on the trivial lower bound $1/2n$ but it is still far from the best possible value $1/(n+1)$ conjectured by L. Fejes Tóth.

1. INTRODUCTION

Let S be a square on the plane with side length n (≥ 1), and let $\mathscr{S} = \{S_1, S_2, \ldots, S_t\}$ be a collection of unit squares whose sides are parallel to those of S. We say that $\mathscr{S}$ *covers* the lines intersecting S if for every line L (on the plane) which intersects S intersects some of the S_i's (i.e., $L \cap S \neq \emptyset$ implies $L \cap S_i \neq \emptyset$ for some i). Let $\tau(n) = \tau(n, S)$ denote the minimum cardinality of a cover, and let $\tau_{in}(n)$ denote the minimum cardinality of a covering system whose members are located inside S.

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L. Fejes Tóth [3,7] conjectured that for an odd integer n

$$\tau_{in}(n) = 2n-1$$

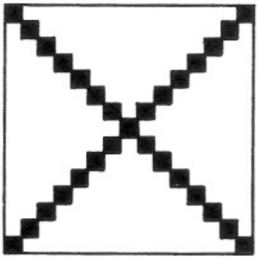

Figure 1

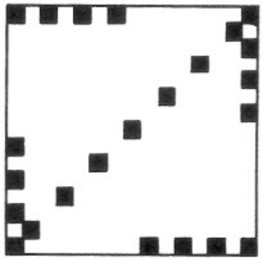

Figure 2

(see Figure 1.). Clearly, $\tau(n) \le \tau_{in}(n) \le 2\lceil n \rceil$ where $\lceil x \rceil$ denotes the upper integer part of the real x. The aim of this note is to improve on the trivial lower bound $\tau(n) \ge \lceil n \rceil$. Namely, we will prove $\tau(n) > (13n-1)/12$ (Theorem 2.1) and $\tau_{in}(n) > (4n-1)/3$ (Theorem 2.3).

The exact results are stated in Section 2. That section also contains examples showing the limit of our methods. Section 3 is devoted to the proof of the lower bounds. These proofs use weight functions, actually we calculate the fractional covering number of a hypergraph. In Section 4 we mention related problems and results.

2. INTERSECTING LINES PARALLEL TO THE SIDES OR THE DIAGONALS

THEOREM 2.1. *Let* S *be a square with side length* n (n ≥ 1, *real*) *and let* $\mathcal{S} = \{S_1, \ldots, S_t\}$ *be a collection of unit squares in* S *whose sides are parallel to those of* S. *If* $t \le (4n-1)/3$ *then there exists a line parallel to either a side or a diagonal of* S, *which intersects* S *and avoids every* S_i.

The Example 2.2 shows that for $t \ge (3n+1)/2$, Theorem 2.1 does not remain true.

EXAMPLE 2.2. Let k be a positive integer, $n = 4k-1$. Suppose that the four vertices of S are given by their coordinates: (0,0), (0,n), (n,0) and (n,n). We will denote by S(i,j) the unit square $\{(x,): i \le x \le i+1, j \le y \le j+1\}$. Then the following set of squares, $\mathcal{S}$ covers every intersecting line of S with slope 0, 45°, 90° or 135°.

$\mathcal{S} = \{S(i,j)$: where $i,j \ge 0$ integers such that $i = 0$, $j = 2t$, $0 \le t \le k-1$ or $j = 0$, $i = 2(k+t)$, $0 \le t \le k-1$ or $i = 2k-2$, $j = 2(k+t)$, $0 \le t \le k-1$ or $j = 2k-2$, $i = 2t$, $0 \le t \le k-1$ or finally $i = j = 2t+1$, $0 \le t \le 2k-2\}$. See Figure 2.

If n is not an integer of the form 4k-1, then a minor modification of the above example (e.g., let $k = \lfloor (n-1)/4 \rfloor$) demands less than $(3n+9)/2$ unit squares. Denote by $t_{in}(n)$ the minimum value of th for which 2.1 does not hold. Similarly, let t(n) denote the minimum t such that there exists a cover consisting of t unit squares (located arbitrarily, not only inside S) which meets every intersecting line with slope 0, 45°, 90° or 135°.

THEOREM 2.3 $\frac{13}{12}n - \frac{1}{12} < t(n) < \frac{4}{3}n + O(1)$.

The upper bound follows from the following example.

EXAMPLE 2.4. Suppose $n = 6k+3$, where k is an integer. Let $\mathcal{S} = \{S(i,j)$: where i,j are integers and either $i = 3j$, $0 \le j \le 3k+1$ or $j = 3i-2$, $1 \le i \le 3k+1$ or $(i,j) = (3k+2, 6k+2)$ or $j = i-2$, $i = 3k+3+t$, $0 \le t \le 3k-1$, $t \not\equiv 2 \pmod 3)\}$. Then $|\mathcal{S}| = 8k+4$. See Figure 3.

These examples show that our method, i.e., to consider only 4 directions, can not lead to the proof of Fejes Tóth's conjecture.

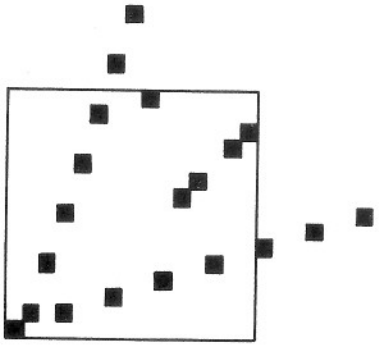

Figure 3

3. PROOFS

Suppose that $S_1, S_2, \ldots, S_t$ meet every line intersecting S with angle 0, 45^o, 90^o or 135^o. We will show that $t > (4n-1)/3$. Consider a coordinate-system whose axes are parallel to the sides of S. Choose the unit and the origin of this system in such a way that the vertices of S have the coordinates $(\pm 1, \pm 1)$. Then the side length of a square S_i is $2/n$ denoted by 2ε. We define a *weight* function $w(L)$ on the set of intersecting lines L with slopes 0, 45^o, 90^o or 135^o as follows. Actually, this weight-function is a measure on the set of these lines. If the equation of the line L is $y = c$ or $x = c$ then

$$w(L) = \frac{1}{2} - \frac{1}{2}c^2$$

and, if the form of the line L is $y = x+h$ or $y = -x+h$ then

$$w(L) = \frac{1}{8}h^2.$$

As for an intersecting line $|c| \leq 1$, $|h| \leq 2$ hold we have $\frac{1}{2} \geq w(L) \geq 0$. The total weight of the lines in these four directions is:

$$(1) \qquad 2\int_{-1}^{+1}\left(\frac{1}{2}-\frac{1}{2}c^2\right)dc + 2\int_{-2}^{2}\frac{1}{8}h^2\,dh = \frac{8}{3}.$$

Now consider a square $Q = Q(a,b)$ with center (a,b) ($|a|$, $|b| \leq 1-\varepsilon$) and side length 2ε.

We will show that the weight of the lines intersecting Q is

$$(2) \qquad 2\varepsilon + \frac{2}{3}\varepsilon^3.$$

Hence (1) and (2) yield that for $n > 1$

$$t \geq \frac{8}{3} / \left(2\varepsilon + \frac{2}{3}\varepsilon^3\right) = \frac{4}{3}n - \frac{4}{9n + (3/n^2)} > \frac{4n-1}{3}$$

proving Theorem 2.1. The proof of (2) is simple because the weight of the lines intersecting Q and parallel to the axis $x = 0$ is

$$(3) \qquad \int_{a-\varepsilon}^{a+\varepsilon}\left(\frac{1}{2} - \frac{1}{2}c^2\right) dc = \varepsilon - a^2\varepsilon - \frac{1}{3}\varepsilon^3.$$

See Figure 4. Similarly the weights of the lines intersecting Q and parallel to the lines $y = 0$, $y = x$, $y = -x$ are

$$(4) \qquad \int_{b-\varepsilon}^{b+\varepsilon}\left(\frac{1}{2} - \frac{1}{2}c^2\right) dc = \varepsilon - b^2\varepsilon - \frac{1}{3}\varepsilon^3 ,$$

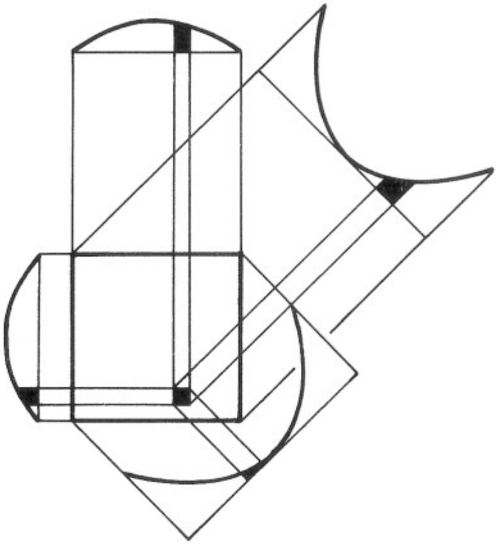

Figure 4

$$(5) \qquad \int_{b-a-2\varepsilon}^{b-a+2\varepsilon} \tfrac{1}{8}h^2 dh = \tfrac{1}{2}\varepsilon(b-a)^2 + \tfrac{2}{3}\varepsilon^3,$$

$$(6) \qquad \int_{a+b-2\varepsilon}^{a+b+2\varepsilon} \tfrac{1}{8}h^2 dh = \tfrac{1}{2}\varepsilon(a+b)^2 + \tfrac{2}{3}\varepsilon^3,$$

Summing up (3) - (6) we get (2).

The proof of 2.3 is analogous to the above. We modify the weight functions of the lines, because in the previous case a small square outside S, e.g., Q(0,2) could get too much weight.

If $y = c$ or $x = c$ then $w(L) = \begin{cases} \frac{1}{2} - \frac{1}{8}c^2 & \text{for } |c| \le 1 \\ 0 & \text{otherwise} \end{cases}$

and if $y = \pm x + h$ then $w(L) = \begin{cases} \frac{1}{32}h^2 & \text{for } |h| \le 2, \\ 0 & \text{otherwise.} \end{cases}$

Then the total weight of the lines is 13/6 and every small square covers lines with weight at most $2\varepsilon + \frac{1}{6}\varepsilon^3$. Hence $t < 13/(12\varepsilon + \varepsilon^3) = \frac{13}{12}n - 1/12(12n^2 + 1)$.

4. RELATED PROBLEMS AND RESULTS

We have the following conjectures:

$$t(n) = \frac{4}{3}n + O(1),$$

$$t_{in}(n) = \frac{3}{2}n + O(1).$$

We could not even prove that $\lim_{n\to\infty} t(n)/n$ exists (or $\lim t_{in}(n)/n$, or $\lim \tau(n)/n$ or $\lim \tau_{in}(n)/n$.) The only result we have is if we consider 8 directions of the lines, and define a more sophisticated weight-function, then we obtain

THEOREM 4.1. $\tau_{in}(n) > 1.43n - O(1)$.

Paul Endös asked what is the minimum number of covering unit squares *outside* S? It is very likely $3n + O(1)$.

Our problem is a particular case of a problem of Fejes Tóth [2]. Assume K is a convex body on the plane and $\lambda > 0$. Consider a set $\mathcal{S}$ of λ-homothetic copies of K having the property that each line intersecting K intersects at least one member of $\mathcal{S}$. What is the minimum cardinality of such a set? Fejes Tóth [3] points out further that this question is closely related to the dual of Tarski's plank problem (see Bang [1] or Fenchel [4]).

Another related problem is the following, considered by Makai and Pach [6]. Let $\mathcal{F}$ be a class of functions $f: R \to \mathbb{R}^d$. A set of points $\{(x_i, y_i) \in R \times \mathbb{R}^d,\ i = 1,2,\ldots\}$

is said to be $\mathscr{F}$-controlling system if for each $f \in \mathscr{F}$ there is an i with $\|f(x_i) - y_i\| \leq 1$. So an $\mathscr{F}$-controlling system is a set of points P in $\mathbb{R}^1 \times \mathbb{R}^d$ with the property that for each $f \in \mathscr{F}$ one can find a point in P sufficiently close to the graph of f. The problem is to find an $\mathscr{F}$-controlling system with "few" points (or with small density if P must be inifinite). Makai and Pach [6], and Groemer [5] prove several results concerning this problem. In their case the norm is always the Euclidean norm.

When we take in the above formulation $d = 1$, $\mathscr{F}$ to be the class of all linear functions whose graphs intersect the square S, and $\| \ \|$ to be the L_∞ norm, then what we arrive to is exactly our problem about $\tau(n,S)$.

We end this paper by mentioning a question of Fejes Tóth [2] which we find very appealing and which belongs to the sort of questions considered here. A *zone* of widht w is defined as the parallel domain of a great circle (of the sphere) with angular distance w/2. Prove (or disprove) that the total width of any set of zones covering the sphere is at least π.

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