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RAMSEY–SPERNER THEORY

Zoltán FÜREDI
Mathematical Institute of the Hungarian Academy of Sciences, 1364 Budapest, Hungary

Jerrold R. GRIGGS*
Department of Mathematics, University of South Carolina, Columbia, SC 29208, U.S.A.

Andrew M. ODLYZKO
AT & T Bell Laboratories, Inc., Murray Hill, New Jersey 07974 U.S.A.

James B. SHEARER
I.B.M. Thomas J. Watson Research Center, Yorktown Heights, NY 10598, U.S.A.

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Let $[n]$ denote the n-set $\{1, 2, \ldots, n\}$, let k, $l \geq 1$ be integers. Define $f_l(n, k)$ as the minimum number f such that for every family $F \subseteq 2^{[n]}$ with $|F| > f$, for every k-coloring of $[n]$, there exists a chain $A_1 \subsetneq \cdots \subsetneq A_{l+1}$ in F in which the set of added elements, $A_{l+1}-A_1$, is monochromatic.

We survey the known results for $l = 1$. Applying them we prove for any fixed l that there exists a constant $\varphi_l(k)$ such that as $n \to \infty$

$$f_l(n, k) \sim \varphi_l(k)\binom{n}{\lfloor \frac{1}{2}n \rfloor} \quad \text{and} \quad \varphi_l(k) \sim l\sqrt{\frac{\pi k}{4 \log k}} \quad \text{as } k \to \infty.$$

Several problems remain open.

Dedicated to the memory of Professor H. J. Ryser.

1. Introduction

The purpose of this paper is to survey and extend known results and open problems in the fields of 'Ramsey–Sperner theory' with particular emphasis on two recent papers by Füredi [5] and by Griggs, Odlyzko, and Shearer [10] that concern the asymptotic size of k-color Sperner families.

Let $[n]$ denote the n-set $\{1, \ldots, n\}$. A *k-coloring* of $[n]$ is a partition of $[n]$ into at most k parts. A subset $A \subseteq [n]$ is *monochromatic* with respect to a coloring if all of its elements belong to the same color class in the partition.

Fix a k-coloring of $[n]$ with color class sizes $n_1, \ldots, n_k \geq 0$, $\sum n_i = n$. A family of subsets $F \subseteq 2^{[n]}$ has *property* X_l with respect to this coloring if it contains no $l+1$ sets $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{l+1}$ such that $A_{l+1}-A_1$ is monochromatic. A family F

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which has property X_1 is said to be *k-color Sperner*. Let $f_l(n_1 \mid n_2 \mid \cdots \mid n_k)$ be the largest size of any family with property X_l with respect to this given coloring. Let $f_l(n, k)$ denote the maximum value of $f_l(n_1 \mid \cdots \mid n_k)$ over all k-colorings of $[n]$. A Ramsey-theoretic way of defining $f_l(n, k)$ is to say that it is the minimum number f such that for every $F \subseteq 2^{[n]}$ with $|F| > f$, for every k-coloring of $[n]$, there exists a chain $A_1 \subsetneq \cdots \subsetneq A_{l+1}$ in F in which the set of added elements, $A_{l+1} - A_1$, is monochromatic.

A stronger condition related to a given k-coloring of $[n]$ is the following: A family of subsets $F \subseteq 2^{[n]}$ has *property* Y_l with respect to a coloring if it contains no $l+1$ sets $A_1 \subsetneq \cdots \subsetneq A_{l+1}$ such that for all i $A_{i+1} - A_i$ is monochromatic. Of course, the full set of added elements $A_{l+1} - A_1$, need not be monochromatic if k, $l \geq 2$. Let $c_l(n_1 \mid n_2 \mid \cdots \mid n_k)$ denote the maximum of $|F|$ over all F with property Y_l with respect to this coloring, and let $c_l(n, k)$ be the maximum of $c_l(n_1 \mid \cdots \mid n_k)$ over all k-colorings of $[n]$.

For comparison with the functions f_l and c_l above, we also define $d_l(n_1 \mid \cdots \mid n_k)$ to be the maximum size of the union $\bigcup_{i=1}^{l} F_i$ of l k-color Sperner families F_i with respect to a given coloring. Then $d_l(n, k)$ denotes the maximum of $d_l(n_1 \mid \cdots \mid n_k)$ over all k-colorings of $[n]$.

Clearly properties X_1 and Y_1 are identical, and thus $c_1(n, k) = d_1(n, k) = f_1(n, k)$.

In the next section we discuss some useful results about c_l, d_l, f_l for general l, n, k. The following section reviews the $k = 1$ color case, which is Sperner's theorem and Erdös' generalization of it. We next treat the case of $k = 2$ colors, first reviewing the Two-Part Sperner Theorem and Katona's generalization which gives $c_l(n, 2)$, and then giving a new generalization which gives $d_l(n, 2)$ and $f_l(n, 2)$. The main thrust of [5] and [10] was the asymptotic behavior of $f_1(n, k)$; we review those results and extend them to $d_l(n, k)$ and $f_l(n, k)$ for arbitrary l. Another section discusses k-color Sperner theorems for products of symmetric chain orders, especially a recent theorem of Sali [18]. The paper is concluded with a list of problems still outstanding.

2. Results for arbitrary l, n, k

We begin with a relationship among c_l, d_l, and f_l.

Theorem 1. *For all* l, $k \geq 1$, $n_1, \ldots, n_k \geq 0$,

$$c_l(n_1 \mid \cdots \mid n_k) \leq d_l(n_1 \mid \cdots \mid n_k) \leq f_l(n_1 \mid \cdots \mid n_k).$$

For all n, l *and* $k \geq 1$,

$$c_l(n, k) \leq d_l(n, k) \leq f_l(n, k).$$

Proof. The second statement follows from the first which we now prove. Fix a k-coloring of $[n]$ with color class sizes n_i.

First suppose $F \subseteq 2^{[n]}$ attains $c_l(n_1 \mid \cdots \mid n_k)$. Then F can be partitioned into at most l k-color Sperner families F_i as follows: For each $A \in F$ let $h(A)$ denote the largest number r of sets in any chain of sets in F with A at the top, $A_1 \subsetneq \cdots \subsetneq A_r = A$, such that for all i, $A_{i+1} - A_i$ is monochromatic. Let $F_i = \{A \in F \mid h(A) = i\}$. Clearly each F_i has property X_1 and $F = \bigcup_{i=1}^{l} F_i$. It follows that $c_l(n_1 \mid \cdots \mid n_k) \leq d_l(n_1 \mid \cdots \mid n_k)$.

Since any union of at most l k-color Sperner families has property X_l, it follows that $d_l(n_1 \mid \cdots \mid n_k) \leq f_l(n_1 \mid \cdots \mid n_k)$. $\square$

Whether $d_l(n_1 \mid \cdots \mid n_k) = f_l(n_1 \mid \cdots \mid n_k)$ in general is not clear. There do exist families with property X_l which are not the union of at most l k-color Sperner families.

Example 1. Take $n = 4$, $k = 2$, and the 2-coloring $\{1, 2\} \mid \{3, 4\}$. The family $F \subseteq 2^{[4]}$ below has property X_2 but is not the union of any 2 families with property X_1.

$$F = \{\emptyset, \{1\}, \{3\}, \{2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}.$$

The next result simplifies the study of $c_l(n, k)$ and $f_l(n, k)$. It is a natural extension of a result in [10] which was itself a nice generalization of Sperner's Theorem. A subset of $[n]$ is said to be of *type* $(r_1 \mid \cdots \mid r_k)$ for a coloring of $[n]$ if it contains precisely r_i elements of color i, $1 \leq i \leq k$. The form of Sperner's Theorem we generalize here states that the subsets of $[n]$ of size $\lfloor \frac{1}{2}n \rfloor$ (or type $(\lfloor \frac{1}{2}n \rfloor)$, with $k = 1$) form an antichain (1-color Sperner family) of maximum size.

Theorem 2. *There exists a family F achieving $f_l(n_1 \mid \cdots \mid n_k)$ (respectively, $c_l(n_1 \mid \cdots \mid n_k)$, $d_l(n_1 \mid \cdots \mid n_k)$) with the property that if $A \in F$, then F contains all subsets of the same type as A. There exists a family F achieving $f_l(n, k)$ (respectively, $c_l(n, k)$, $d_l(n, k)$) with this property with respect to some coloring.*

Thus each of these parameters is achieved by a family which is the union of orbits of $2^{[n]}$ under the group of automorphisms generated by the permutations of $[n]$ that preserve the color classes. The proof of the result is an immediate extension of the proof in [10] for f_1. The averaging argument given there, which uses all maximal chains of subsets in each color, can be viewed as an extension of Lubell's [15] proof of Sperner's Theorem. It is also in [10] that there exist families achieving $f_1(n, k)$ which do not have this homogeneity property of Theorem 2.

3. One color

The fundamental result for one color, which is for antichains ($l = 1$), is Sperner's Theorem [20], discovered in 1928. In our notation it states that $f_1(n, 1) = \binom{n}{\lfloor \frac{1}{2}n \rfloor}$.

In 1945 Erdös considered families of subsets of $[n]$ in which no $l+1$ sets form a chain. Sperner's Theorem is the case $l=1$. Erdös proved

Theorem 3 ([3]). *$c_l(n, 1) = d_l(n, 1) = f_l(n, 1) =$ the sum of the l largest binomial coefficients in n.*

Erdös proved in fact that the only extremal families are obtained by taking all subsets of $[n]$ of the l middle sizes.

The asymptotic behavior of our parameters as $n \to \infty$ follows from Theorem 3.

Corollary. *For fixed $l \geqslant 1$ as $n \to \infty$,*

$$c_l(n, 1) = d_l(n, 1) = f_l(n, 1) \sim l\binom{n}{\lfloor \frac{1}{2}n \rfloor} = lf_1(n, 1).$$

4. Two colors

Around 1965 Katona and Kleitman independently discovered the Two-Part Sperner Theorem, each in connection with a problem of Littlewood and Offord concerning the distribution of sums of random vectors.

Theorem 4 ([11, 14]). $c_1(n, 2) = d_1(n, 2) = f_1(n, 2) = \binom{n}{\lfloor \frac{1}{2}n \rfloor}$.

Several years later Katona introduced and determined the value we call here $c_l(n, 2)$ for arbitrary l.

Theorem 5 ([12]). *$c_l(n, 2) =$ the sum of the l largest binomial cofficients in n.*

This result is a stronger form of Erdös' Theorem 3: Families F with property Y_l may contain chains of $l+1$ sets, unlike before, as long as not every jump in the chain $A_{i+1}-A_i$, is monochromatic, yet the maximum size $|F|$ is not increased compared to Theorem 3.

This is the idea of the proof of Theorem 5. Fix a 2-coloring of $[n]$. For each color the collection of subsets of that color can be partitioned into symmetric chains (see [2]). This induces a partition of $2^{[n]}$ into 'symmetric rectangles', i.e., each is a product of a symmetric chain in each color. The l middle ranks in $2^{[n]}$ intersect such a rectangle R at its middle l ranks, which correspond to the l largest different diagonals in R. Thus it suffices to prove that if a collection $F \subseteq R$ satisfies Y_l, then $|F|$ is at most the sum of the sizes of the l largest different diagonals. Katona actually proves this under a weaker condition than Y_l, which yields a stronger result than Theorem 5: The bound in Theorem 5 holds for $F \subseteq 2^{[n]}$ which contain no $l+1$ sets $A_1 \subsetneq \cdots \subsetneq A_{l+1}$ such that for some w, all elements in $A_w - A_1$ are Color 1 and in $A_{l+1} - A_w$ are Color 2. This property is stronger than X_l.

This leads to the question: What about families with property X_l? We can provide a partial answer.

Theorem 6. *For* $l = 1, 2$, $f_l(n, 2) = d_l(n, 2) = l\binom{n}{\lfloor\frac{1}{2}n\rfloor}$.
For $l \geq 3$, $f_l(n, 2) = d_l(n, 2) < l\binom{n}{\lfloor\frac{1}{2}n\rfloor}$.

Proof. We first prove that $f_l(n, 2) = d_l(n, 2)$ for $l \geq 1$. Suppose $F \subseteq 2^{[n]}$ attains $f_l(n, 2)$ with respect to a 2-coloring with n_1 and n_2, where $0 \leq n_1 \leq n$, $n_1 + n_2 = n$. By Theorem 2 we may assume that F contains either all or no sets of each possible type. Thus F is described by the $(n_1 + 1) \times (n_2 + 1)$ matrix M with entries $M_{ij} = 1$ whenever F contains the sets of type $(i \mid j)$, and $M_{ij} = 0$ otherwise, $0 \leq i \leq n_1$, $0 \leq j \leq n_2$. By property X_l, all line sums (row and column sums) of M are at most l.

Lemma ([1]). *Let M be a* 0-1 *matrix with line sums at most l. Then there exist* 0-1 *matrices P_r with line sums at most* 1 *such that* $M = \sum_{r=1}^{l} P_r$.

Proof of Lemma. This is a slight generalization of a theorem in Ryser [16, Theorem 2.1, p. 65], and it follows easily from that theorem, which is a consequence of the Gale–Ryser and König–Egerváry Theorems.

Apply the lemma to our matrix M to obtain matrices P_r. For each r the union of the subsets of the types $(i \mid j)$ corresponding to entries $(P_r)_{ij} = 1$ forms a family F_r with property X_1. Hence F is a union of at most l families with property X_1, so that $f_l(n, 2) = |F| \leq d_l(n, 2)$. Then by Theorem 1, $f_l(n, 2) = d_l(n, 2)$.

By definition of $d_l(n, 2)$ we have immediately that $d_l(n, 2) \leq l d_1(n, 2)$, or, by Theorem 4,

$$f_l(n, 2) = d_l(n, 2) \leq l\binom{n}{\lfloor\frac{1}{2}n\rfloor}.$$

Equality holds here for $l = 1$ by Theorem 4. For $l = 2$, equality holds due to this example: Let $[n]$ be given the 2-coloring with $n_1 = 1$, $n_2 = n - 1$. Let $F \subseteq 2^{[n]}$ contain all subsets with either $\lfloor\frac{1}{2}n\rfloor$ or $\lfloor\frac{1}{2}n\rfloor - 1$ elements of Color 2. Then F has property X_2, so that

$$f_2(n, 2) \geq f_2(1 \mid n - 1) \geq |F| = 2\binom{n}{\lfloor\frac{1}{2}n\rfloor}.$$

It remains to prove that for $l \geq 3$, $f_l(n, 2) < l\binom{n}{\lfloor\frac{1}{2}n\rfloor}$. We show that $f_l(n_1 \mid n_2) < l\binom{n}{\lfloor\frac{1}{2}n\rfloor}$. For $n_1 = 0$ or n, $f_l(n_1 \mid n_2) = f_l(n, 1) < l\binom{n}{\lfloor\frac{1}{2}n\rfloor}$ by Theorem 3.

Now assume $1 \leq n_1 \leq n - 1$. Let F attain $f_l(n_1 \mid n_2)$. As in the sketch of the proof of Theorem 5, $2^{[n]}$ can be partitioned into rectangles R which have their middle ranks coinciding with the middle level $\frac{1}{2}n$ of $2^{[n]}$. Consider such a rectangle R which is the product of a chain C_1 of Color 1 and a Chain C_2 of Color 2. For each $A \in C_1$ there are at most l sets $B \in C_2$ such that $A \cup B \in R$, by property X_l. Similarly, for each $B \in C_2$, there are at most l sets $A \in C_1$ such that $A \cup B \in F$. It follows that

$$|F \cap R| \leq \min(|R|, l\,|C_1|, l\,|C_2|) = w(R)\min(l, \max(|C_1|, |C_2|)),$$

where $w(R)$ is the size of the middle level in R, that is, $w(R) = \min(|C_1|, |C_2|)$. Thus, if $l > \max(|C_1|, |C_2|)$, then $|F \cap R| < lw(R)$. This occurs here for some R: the partition of the subsets of a set S into symmetric chains contains $\binom{|S|}{\lfloor \frac{1}{2}|S| \rfloor} - \binom{|S|}{\lfloor \frac{1}{2}|S| \rfloor - 1} > 0$ chains of size 1 (respectively, 2) when $|S|$ is even (respectively, odd). Select chains C_i in color class i of size at most 2. Since $l \geq 3$, we have $l > \max(|C_1|, |C_2|)$. Hence,

$$f_l(n_1 \mid n_2) = |F| = \sum_R |F \cap R| < \sum_R lw(R) = l\binom{n}{\lfloor \frac{1}{2}n \rfloor}. \qquad \square$$

The rectangles method used immediately above is useful in other situations. For instance, it implies that for all even a, $b > 0$, $f_2(a \mid b) < 2\binom{a+b}{\frac{1}{2}(a+b)}$. More exactly if we take into account that the number of chains in the chain decomposition of $2^{[a]}$ with length $<i$ is $\binom{a}{\lfloor \frac{1}{2}a \rfloor} - \binom{a}{\lfloor \frac{1}{2}(a+i) \rfloor}$ then the method above gives the following

Proposition. *Let* $a_i = \binom{a}{\lfloor \frac{1}{2}a \rfloor} - \binom{a}{\lfloor \frac{1}{2}(a+i) \rfloor}$ *and* $b_i = \binom{b}{\lfloor \frac{1}{2}b \rfloor} - \binom{b}{\lfloor \frac{1}{2}(b+i) \rfloor}$ *for* $1 \leq i \leq l$. *Then*

$$f_l(a \mid b) \leq l \cdot \binom{n}{\lfloor \frac{1}{2}n \rfloor} - \sum_{1 \leq i \leq j \leq l} \sum (a_i - a_j)(b_i - b_j).$$

Using $\binom{n}{(\frac{1}{2}n) - x} \sim \binom{n}{\frac{1}{2}n} e^{-2x^2/n}$ (see, e.g., in [21, p. 180]) this implies that there exists a $c > 0$ such that

$$f_l(n, 2) < l\binom{n}{\lfloor \frac{1}{2}n \rfloor}\left(1 - \frac{c \cdot l^4}{n^{5/2}}\right)$$

holds for $l \geq 3$, $n \gg l$.

Combining Theorems 1, 5, and 6 gives us the asymptotic behavior for 2 colors.

Corollary. *For fixed* $l \geq 1$, *as* $n \to \infty$

$$c_l(n, 2), d_l(n, 2), f_l(n, 2) \sim l\binom{n}{\lfloor \frac{1}{2}n \rfloor} = lf_1(n, 2).$$

5. Asymptotic results

In connection with a generalization of the Littlewood–Offord problem (some generalizations and a few exact results can be found, e.g. in [4, 7, 9, 13]), Griggs [8] generalized the Two-Part Sperner Theorem and showed that for arbitrary l, n, k, with $k \geq 2$,

$$f_l(n, k) \leq 2^{k-2} l\binom{n}{\lfloor \frac{1}{2}n \rfloor}. \tag{1}$$

Thus $f_l(n, k)/\binom{n}{\lfloor\frac{1}{2}n\rfloor}$ is at most a constant depending on k and l, independent of n. We have seen that for $l = 1$ and $k = 1, 2$, this constant can be taken to be 1. For $l = 1$ and $k = 3$, (1) is no longer true, e.g., $f_1(3, 3) = 4 > \binom{3}{\lfloor\frac{3}{2}\rfloor}$. Graham [6] asked whether $f_1(n, 3)$ is asymptotic to $\binom{n}{\lfloor\frac{1}{2}n\rfloor}$ as $n \to \infty$, despite being larger for any given n. He also proposed studying the limiting behavior of $f_1(n, k)/\binom{n}{\lfloor\frac{1}{2}n\rfloor}$, as $n \to \infty$ with k fixed, as a function of k. Füredi [5] and Griggs, Odlyzko, and Shearer [10] independently studied these questions. Our intention here is not to restate the arguments from these papers, but to apply the results and methods there to obtain asymptotic results for c_l, d_l and f_l for general l.

We first consider the problem of existence of limits.

Theorem 7. *For all k, $l \geq 1$ there exist constants $\gamma_l(k)$, $\delta_l(k)$, $\varphi_l(k)$ such that as $n \to \infty$*

$$c_l(n, k) \sim \gamma_l(k)\binom{n}{\lfloor\frac{1}{2}n\rfloor}, \qquad d_l(n, k) \sim \delta_l(k)\binom{n}{\lfloor\frac{1}{2}n\rfloor}, \qquad f_l(n, k) \sim \varphi_l(k)\binom{n}{\lfloor\frac{1}{2}n\rfloor}.$$

Further, $\gamma_l(k) \leq \delta_l(k) \leq \varphi_l(k)$ and $\delta_l(k) = l\delta_1(k)$.

The paper [10] proves the existence of the constants $\varphi_1(k)$, and this proof generalizes naturally to prove the existence of all of the constants in the theorem. The inequality for the constants follows from Theorem 1. To prove that $\delta_l(k) = l\delta_1(k)$, first observe that $d_l(n, k) \leq l d_1(n, k)$, so that $\delta_l(k) \leq l\delta_1(k)$. The other direction, $\delta_l(k) \geq l\delta_1(k)$ follows by examination of the exisence proof for $\delta_1(k)$ in [10].

Concerning the actual values of these constants, we already have seen in Theorems 3 and 5 that

$$\gamma_l(k) = \delta_l(k) = \gamma_l(k) = l \quad \text{for } k = 1, 2. \tag{2}$$

For $l = 1$ and $k > 2$ colors the following results were obtained in [5] and [10] in answer to Graham's questions:

$$1.036 < \varphi_1(3) < 1.131, \tag{3}$$

$$\varphi_1(k) \sim \sqrt{\frac{\pi k}{4 \ln k}}, \quad \text{as } k \to \infty. \tag{4}$$

Proofs of (3) and (4) involve obtaining lower and upper bounds on $\varphi_1(k)$. The lower bound proofs are essentially the same in [5] and [10]. The idea is to first partition $[n]$ into k almost equal parts S_i, $|S_i| \sim n/k$. Then for t an integer set

$$F^t = \{A \subseteq [n] \colon \big|\, |A \cap S_i| - \tfrac{1}{2}|S_i| \,\big| < \tfrac{1}{2}t,\ 1 \leq i \leq k\}$$

and

$$F_r^t = \{A \in F^t \colon |A| \equiv r \pmod t\}, \quad 0 \leq r < t.$$

Each family F_r^t has property X_1. Thus $f_1(n, k)$ is at least the average size of the families F_r^t, which is $|F^t|/t$. For large k an appropriate choice for t, which is

$t \sim \sqrt{2n \ln k/k}$ as $n \to \infty$, yields the asymptotic lower bound as $k \to \infty$. To show that $\varphi_1(3) > 1$, Füredi [5] selects $t \sim 1.2\sqrt{n}$ and a specific value of r, which is $r \equiv \lfloor \frac{1}{2}n \rfloor \pmod t$, to obtain $\varphi_1(3) > 1.0189$. In [10] the better value $\varphi_1(3) > 1.036$ follows from an averaging argument which refines the idea above: One selects the sets more carefully, but requires fewer families with property X_1 to cover them all.

The upper bound proof in [10] works by eliminating one color class and using induction on k. The actual details are quite involved. The proof in [5] uses the following 'brick' method related to the proof of Theorem 5. If C_i, $1 \leq i \leq k$, is a chain of subsets of the elements of color i in $[n]$, then the Cartesian product $B = C_1 \times \cdots \times C_k$, ordered componentwise, is called a *brick*. If, say, $\max_i |C_i| = |C_1|$, then B can be partitioned into $|B|/|C_1|$ chains, one for each choice of a set in $C_2 \times \cdots \times C_k$. Now suppose $F \subseteq 2^{[n]}$ has property X_1. Then F intersects each of the $|B|/|C_1|$ chains at most once, so that $|F \cap B| \leq |B|/|C_1|$. If each color class is partitioned into symmetric chains, this induces a partition of $2^{[n]}$ into bricks. Suppose there exists such a brick partition in which every set in F belongs to a brick B with $\max_i |C_i| \geq t$, for some given t. Then adding over such B, we find $|F| \leq (\sum |B|)/t \leq 2^n/t$. To obtain the asymptotic upper bound Füredi [5] actually shows that for large k there exists a brick decomposition of $2^{[n]}$ such that almost all of F is covered by bricks with $\max_i |C_i| \geq t$, where

$$t \sim \sqrt{2n \ln k/k} \quad \text{as } n \to \infty.$$

One can check that the upper bound proofs in both papers extend to $f_l(n, k)$ for arbitrary l. More precisely, the upper bounds U_k on $f_1(n, k)$ in the proofs extend to upper bounds lU_k on $f_l(n, k)$, although we do not yet know whether $f_l(n, k) \leq lf_1(n, k)$ in general. The lower bound on $\varphi_l(k)$ follows from $\varphi_l(k) \geq \delta_l(k) = l\delta_1(k)$ in Theorem 7. This gives us the following extension of (3) and (4) to general l.

Theorem 8. *For* $l \geq 1$,

$$(1.036)l < \delta_l(3) \leq \varphi_l(3) < (1.131)l,$$

$$\delta_l(k),\ \varphi_l(k) \sim l\sqrt{\frac{\pi k}{4 \ln k}} \quad \textit{as } k \to \infty \textit{ with } l \textit{ fixed}.$$

6. Results for symmetric chain orders

Going back as far as Katona [12] most results until recently about k-color Sperner families have been obtained in the more general context of products of symmetric chain orders. A *symmetric chain order* is a finite ranked poset which can be partitioned into chains that are consecutive and symmetric about middle rank. Properties X_l and Y_l can be extended naturally to any product P of k

symmetric chain orders P_i, $P = P_1 \times \cdots \times P_k$. The quantities $f_l(P)$, $d_l(P)$, and $c_l(P)$ may then be defined in the analogous way to $f_l(n, k)$, $d_l(n, k)$, and $c_l(n, k)$. A k-coloring $(n_1 \mid n_2 \mid \cdots \mid n_k)$ of $[n]$ corresponds to considering the poset $2^{[n]}$ as the product $2^{[n_1]} \times 2^{[n_2]} \times \cdots \times 2^{[n_k]}$, where each order $2^{[n_i]}$, a Boolean algebra, is a symmetric chain order [2]. The quantity $\binom{n}{\lfloor \frac{1}{2}n \rfloor}$ in our formulas for $2^{[n]}$ corresponds for general P to the *width*, $w(P)$, which is the size of the largest antichain in P. It also is the size of the largest subset of P with property X_1 when $P = P_1$.

Problems about $P = P_1 \times \cdots \times P_k$ are attacked using the brick decomposition of P induced by the product of the symmetric chain decompositions of the P_i. It was this approach, specialized to $2^{[n]}$, which yielded the general bound (1) on $f_l(n, k)$. Sali [17] improved this bound, and he recently improved it even further [18], obtaining this theorem for products of symmetric chain orders.

Theorem 9 ([18]). *There exists $c_1 > 0$ such that for all k and l for all $P = P_1 \times \cdots \times P_k$, where each P_i is a symmetric chain order, $f_l(P) \leq c_1 l \sqrt{k} w(P)$.*

There exists $c_2 > 0$ such that for all k and l, there exists $P = P_1 \times \cdots \times P_k$, where each P_i is a symmetric chain order, such that $f_l(P) \geq c_2 l \sqrt{k} w(P)$.

Sali shows that the second part of the theorem, that says the bound in the first part is best-possible except for the constant, holds in particular for 'hypercubes' $P = P_1 \times \cdots \times P_k$, where each P_i is a chain of the same length N, and $N \to \infty$ with k, l fixed.

Applying Theorem 9 to $P = 2^{[n]}$ over all possible k-coloring yields

Corollary. *There exists $c_1 > 0$ such that*

$$\varphi_l(k) \leq c_1 l \sqrt{k} \quad \text{for all } k,\ l \geq 1.$$

The constant c_1 here works for all l, so although this bound is not good asymptotically for fixed l as $k \to \infty$, it does say something. It is interesting to compare this bound from the best-possible bound for symmetric chain order products to the asymptotic bound in Theorem 8. Füredi obtained the upper bound by a brick method, which is related to the proof of Theorem 9. The reason he obtained a better bound, Theorem 8, is evidently that the actual brick decomposition selected depends on F: Most of F lies in bricks with a side $|C_i|$ being large.

7. Open problems

(1) Determine $f_l(n, 2)$ for $l \geq 3$.

(2) Although in general $d_l(n, k) \leq f_l(n, k)$ (Theorem 1), it remains open to give

an example with $d_l(n, k) < f_l(n, k)$. Example 1 gives a famly for a 2-coloring that has property X_2 but is not the union of 2 families with property X_1. Nonetheless, for 2-colorings in general $f_l(n_1 \mid n_2) = d_l(n_1 \mid n_2)$ (Theorem 6). For $k \geqslant 3$ the analogue for k of the Lemma in the proof of Theorem 6 is false. Indeed, one can construct families for $k = 3$ that satisfy X_2 and that also satisfy the types condition of Theorem 2 but that are not the union of 2 families with X_1.

(3) It is open whether or not in general $f_l(n, k) \leqslant l f_1(n, k)$, which holds for $k = 1, 2$ (Theorems 3, 6).

(4) The asymptotic version of Problem 3 is open: Is it true that $f_l(n, k) \sim l f_1(n, k)$ as $n \to \infty$ with k, l fixed? Equivalenty, is $\varphi_l(k) = \delta_l(k)$ in general? This is true for $k = 1, 2$, for all l (Theorems 3, 6) and it is true asymptotically for all l: $\varphi_l(k) \sim \delta_l(k)$ as $k \to \infty$ (Theorem 8).

(5) Determine the behavior of $\gamma_l(k)$ *for* $k > 3$. It may well be much less than $\delta_l(k)$ since the union of just two families with property X_1 may contain a long chain $A_1 \subsetneqq \cdots \subsetneqq A_{l+1}$, with all $A_{i+1} - A_i$ monochromatic (but not all the same color) violating Y_l.

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