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FAMILIES OF FINITE SETS WITH MINIMUM SHADOWS

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The following problem was answered by a theorem of Kruskal, Katona, and Lindström about 20 years ago: Given a family of k-element sets $\mathcal{F}$, $|\mathcal{F}|=m$, at least how many $(k-d)$-element subsets are contained in the members of $\mathcal{F}$? This paper deals with the extremal families, e.g., they are completely described for infinitely many values of m.

1. Introduction

Let X be a finite set, $k\geqq g$ positive integers. Denote by $\binom{X}{k}$ the family of all k-subsets of X, and 2^X is the power-set. For a family $\mathcal{F}$ we denote by $d_{\mathcal{F}}(x)$ the degree of the element x, i.e., $d_{\mathcal{F}}(x)=|\{F\in\mathcal{F}: x\in F\}|$. Let $\mathcal{F}(x)$ and $\mathcal{F}(\neg x)$ denote the subfamilies of $\mathcal{F}$ as follows: $\mathcal{F}(x)=\{F: x\in F\in\mathcal{F}\}$, $\mathcal{F}(\neg x)=\mathcal{F}-\mathcal{F}(x)$. For integers u and v, $\binom{u}{v}$ denotes the binomial coefficient, where $\binom{u}{v}=0$ except if $u\geqq v\geqq 0$. The *g-shadow* $\Delta_g\mathcal{F}$ of the family $\mathcal{F}$ is defined as $\Delta_g\mathcal{F}=:\{G: |G|=g, \exists F\in\mathcal{F}: G\subset F\}$.

Kruskal [4], Katona [3], and Lindström [5], solved in the middle of the sixties the following problem: For a given family of k-element sets $\mathcal{F}$, $|\mathcal{F}|=m$, at least how many $(k-d)$-element subsets are contained in the members of $\mathcal{F}$? Denote by $K(m, k, g)=\min\{|\Delta_g\mathcal{F}|: |\mathcal{F}|=m, \mathcal{F}$ is a family of k-subsets$\}$. They proved that if $m=\binom{a_k}{k}+\binom{a_{k-1}}{k-1}+\ldots+\binom{a_s}{s}$ where $a_k>a_{k-1}>\ldots>a_s\geqq s\geqq 1$ are integers (the representation of m in this so called *k-cascade form* is unique), then $K(m, k, k-d)=\binom{a_k}{k-d}+\binom{a_{k-1}}{k-1-d}+\ldots+\binom{a_s}{s-d}$. Call the family of k-subsets $\mathcal{F}$ *(k, g)-extremal* if $|\Delta_g\mathcal{F}|$ is minimal.

Define the antilexicographical order of all k-subsets of positive integers as follows: $A<B$ if considering the symmetric difference $A\triangle B$ we have $\max(A\triangle B)\in B$. (E.g., $\{1, 2\}<\{1, 3\}<\{2, 3\}<\{1, 4\}<\ldots$). Consider the first m members of this

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ordering of k-sets. We obtain a family $\mathcal{F}=\mathcal{F}(m,k)$ with

$$|\Delta_{k-d}\mathcal{F}| = \sum_{i\leq k}\binom{a_i}{i-d}.$$

$\Big($This can easily be seen because $\mathcal{F}(m,k)=\binom{\{1,2,\ldots,a_k\}}{k}\cup\left(\mathcal{F}\left(m-\binom{a_k}{k},k-1\right)+\{a_k+1\}\right)$ where $\mathcal{H}+\{x\}$ denotes the family of sets of the form $H\cup\{x\}$, $H\in\mathcal{H}$.$\Big)$ Hence, by the Kruskal—Katona—Lindström theorem, $\mathcal{F}(m,k)$ is a (k,g)-extremal family for all $g<k$.

The aim of this paper is to investigate the extremal families. Especially, we prove that up to isomorphisms $\mathcal{F}(m,k)$ is the only (k,g)-extremal family for infinitely many values of m. Later we will see other extremal families.

We have to mention a not so strong but much more applicable form of Kruskal—Katona—Lindström theorem due to Lovász [6] (Problem 13.31). Write m in the form $\binom{x}{k}$, where $x\geq k$ is a *real number* $\left(\binom{x}{k}=x(x-1)\ldots(x-k+1)/k!\right)$. Then $K(m,k,g)\geq\binom{x}{g}$. Recently, Frankl [2] gave a short, common proof for the original Kruskal—Katona—Lindström theorem and for the Lovász version.

2. Results

Theorem 2.1. *Let $\mathcal{F}$ be a family of k-sets such that its g-shadow is minimal. Then its $(g-1)$-shadow is minimal as well.*

An obvious consequence of this theorem is that a (k,g)-extremal family is $(k,1)$-extremal:

Corollary 2.2. *Let $\mathcal{F}$ be a (k,g)-extremal family, $|\mathcal{F}|\leq\binom{n}{k}$. Then $|\cup\mathcal{F}|\leq n$.* ∎

This implies that for $m=\binom{n}{k}$ the only (k,g)-extremal families are the complete hypergraphs. Moreover the extremal hypergraph is unique in the cases $m\leq k+1$ and $m=\binom{n}{k}-1$. The function $K(m,k,g)$ is monotone increasing in m for fixed k and g. Call m a *jumping number* (more exavtly a (k,g)-jumping number) if $K(m+1,k,g)>K(m,k,g)$. E.g., the value $m=\binom{n}{k}$ is a jumping number.

Proposition 2.3. *Write m in k-cascade form, $m=\binom{a_k}{k}+\binom{a_{k-1}}{k-1}+\ldots+\binom{a_s}{s}$. Then m is (k,g)-jumping iff $s>k-g$.*

Example 2.4. Let m,k,g $(k>g)$ be given and let m' be the largest integer satisfying $K(m',k,g)=K(m,k,g)$. (This $m'\geq m$ is always a jumping number.) Delete $(m'-m)$ edges from $\mathcal{F}(m',k)$ arbitrarily. We obtain a (k,g)-extremal family $\mathcal{F}$, because $K(m,k,g)\leq|\Delta_g\mathcal{F}|\leq|\Delta_g\mathcal{F}(m',k)|=K(m',k,g)=K(m,k,g)$.

Proposition 2.5. *Let* $m>k+1$, $k\geqq 2$, $m\neq\binom{n}{k}-1$. *If* m *is not a jumping number then Example* 2.4 *gives more than one non-isomorphic extremal families.*

Theorem 2.6. *Let* m *be a* (k,g)*-jumping number. Then the only* (k,g)*-extremal family is* $\mathcal{F}(m,k)$.

3. Proof of Theorem 2.1

Let m be a positive integer and write

$$m=\sum_{s\leqq i\leqq k}\binom{a_i}{i}\quad\text{where}\quad a_k>a_{k-1}>\ldots>a_s\geqq s\geqq 0.$$

We call this a *near-k-cascade* form of m. The following simple Lemma says that instead of cascades we can use near-cascades for characterizing the (k,g)-extremal families (although the near-cascade form is not unique).

Lemma 3.1. *Let* $m=\sum_{i\leqq k}\binom{a_i}{i}$ *be a near-k-cascade form of* m. *Then* $K(m,k,k-d)=$ $=\sum_{i\leqq k}\binom{a_i}{i-d}$. ▮

Let $\mathcal{H}\subset\binom{X}{k}$ where $X=\{1,2,\ldots,n\}$. We recall the definition of the left-shifting operation $S_{ij}\colon 2^X\to 2^X$, where $1\leqq i<j\leqq n$, which was introduced by Erdős, Ko and Rado [1]: For $H\in\mathcal{H}$ set

$$S_{ij}(H)=\begin{cases}H-\{j\}\cup\{i\} & \text{if}\quad i\notin\mathcal{H},\ j\in H\ \text{ and }\ H-\{j\}\cup\{i\}\notin\mathcal{H},\\ H & \text{otherwise.}\end{cases}$$

Finally, let $S_{ij}(\mathcal{H})=\{S_{ij}(H)\colon H\in\mathcal{H}\}$. This is well-known (cf., e.g., [3]):

Proposition 3.2. $|S_{ij}(\mathcal{H})|=|\mathcal{H}|$, *and* $\Delta_g S_{ij}(\mathcal{H})\subset S_{ij}(\Delta_g\mathcal{H})$ *(i.e.,* $|\Delta_g S_{ij}(\mathcal{H})|\leqq$ $\leqq|\Delta_g(\mathcal{H})|$*).* ▮

Call the family $\mathcal{H}$ 1-*shifted* if $S_{1j}(\mathcal{H})=\mathcal{H}$ for all $2\leqq j\leqq n$. In other words

(1) $$H\in\mathcal{H},\ 1\notin H,\ j\in H\ \text{ implies }\ H-\{j\}\cup\{1\}\in\mathcal{H}.$$

The idea of the following Proposition comes from the short proof of Frankl [2].

Proposition 3.3. *Let* $\mathcal{H}$ *be a* (k,g)*-extremal,* 1*-shifted family,* $|\mathcal{H}|=m=\binom{a_k}{k}+\ldots$ $\ldots+\binom{a_s}{s}$ *(cascade form,* $s\geqq 1$*). Define* $\mathcal{H}_1=\{H-\{1\}\colon 1\in H\in\mathcal{H}\}$. *Then*

(2) $$|\Delta_t\mathcal{H}|=|\Delta_t\mathcal{H}_1|+|\Delta_{t-1}\mathcal{H}_1|\quad\textit{for}\quad 1\leqq t<k,$$

(3) $$|\mathcal{H}_1|\geqq\binom{a_k-1}{k-1}+\ldots+\binom{a_s-1}{s-1},$$

(4) $$|\Delta_g\mathcal{H}_1|=\binom{a_k-1}{g}+\ldots+\binom{a_s-1}{s-(k-g)},$$

(5) $$|\Delta_{g-1}\mathcal{H}_1|=\binom{a_k-1}{g-1}+\ldots+\binom{a_s-1}{s-(k-g+1)}.$$

Proof. For any family $\mathscr{F}$, element x, and $G\in\Delta_g\mathscr{F}$ we have either $x\in G$ or not, hence

$$(6)\qquad |\Delta_t\mathscr{F}| = |\Delta_t\mathscr{F}(\neg x)\cup\Delta_t(\mathscr{F}(x)-\{x\})|+|\Delta_{t-1}(\mathscr{F}(x)-\{x\})|.$$

Now consider the 1-shifted family $\mathscr{H}$, and let $\mathscr{H}_0=\mathscr{H}(\neg 1)$, $\mathscr{H}_1=\mathscr{H}(1)-\{1\}$. Then (1) implies $\mathscr{H}_1\supset\Delta_{k-1}\mathscr{H}_0$, hence

$$(7)\qquad \Delta_t\mathscr{H}_1\supset\Delta_t\mathscr{H}_0$$

holds for $t\leqq k-1$. Using (6) we obtain (2).

To prove (3) suppose $|\mathscr{H}_1|<\sum_{i\leqq k}\binom{a_i-1}{i-1}$. Then

$$(8)\qquad |\mathscr{H}_0| = |\mathscr{H}|-|\mathscr{H}_1| > \sum_{i\leqq k}\binom{a_i}{i}-\sum_{i\leqq k}\binom{a_i-1}{i-1} = \sum_{i\leqq k}\binom{a_i-1}{i}.$$

Here, although the right-hand side of (8) is not in cascade form, it implies

$$|\Delta_{k-1}\mathscr{H}_0| \geqq \sum_{i\leqq k}\binom{a_i-1}{i-1}.$$

Because $|\mathscr{H}_1|\geqq|\Delta_{k-1}\mathscr{H}_0|$, by (7), it is a contradiction.

Now, Proposition 3.1 and (3) implies that

$$|\Delta_t\mathscr{H}_1| \geqq \sum_{i\leqq k}\binom{a_i-1}{i-(k-t)}$$

holds for $t\leqq k-1$. To prove equality here for $t=g$, $g-1$ use the Kruskal—Katona—Lindström theorem, (2), and the (k,g)-extremality of $\mathscr{H}$:

$$K(m,k,g) = \sum_{i\leqq k}\binom{a_i}{i-(k-g)} = |\Delta_g\mathscr{H}| = |\Delta_g\mathscr{H}_1|+|\Delta_{g-1}\mathscr{H}_1|$$

$$\geqq \sum_{i\leqq k}\binom{a_i-1}{i-(k-g)}+\sum_{i\leqq k}\binom{a_i-1}{i-(k-g+1)} = \sum_{i\leqq k}\binom{a_i}{i-(k-g)}. \quad \blacksquare$$

We need one more Proposition.

Proposition 3.4. *Let* $\mathscr{G}\subset\binom{S}{k}$, $|\mathscr{G}|>1$, $|S|=s$. *Suppose* $x\in S$ *has minimum degree, and denote* $\mathscr{A}=\mathscr{G}(\neg x)$, $\mathscr{B}=\mathscr{G}(x)-\{x\}$. *Then* $|\Delta_{k-1}\mathscr{A}|\geqq|\mathscr{B}|$. *Equality holds only if* $\mathscr{G}=\binom{S}{k}$.

Proof. Clearly,

$$|\mathscr{B}| = \min_y d_{\mathscr{G}}(y) \leqq \frac{1}{s}\sum_{y\in S} d_{\mathscr{G}}(y) = \frac{k}{s}|\mathscr{G}|,$$

i.e.,

$$(9)\qquad |\mathscr{B}| \leqq (k/s)|\mathscr{G}|,$$

and

$$(10)\qquad |\mathscr{A}| \geqq ((s-k)/s)|\mathscr{G}|.$$

Count the pairs (A', A), where $A \in \mathscr{A}$, $A' \subset A$, $|A'| = k-1$. We get

$$k|\mathscr{A}| = \#(A', A) \leqq \big((s-1)-(k-1)\big)|\Delta_{k-1}\mathscr{A}|.$$

This implies (using (10) and (9)):

$$|\Delta_{k-1}\mathscr{A}| \geqq \frac{k}{s-k}|\mathscr{A}| \geqq \frac{k}{s-k}\frac{s-k}{s}|\mathscr{G}| = \frac{k}{s}|\mathscr{G}| \geqq |\mathscr{B}|.$$

Finally, the equality $|\Delta_{k-1}\mathscr{A}| = |\mathscr{B}|$ implies $k|\mathscr{A}| = (s-k)|\Delta_{k-1}\mathscr{A}|$ from which $\mathscr{A} = \binom{S-\{x\}}{k}$ easily follows. ∎

Now we are ready to prove Theorem 2.1. We use induction on m and k. The case $m=1$ is trivial. Let $\mathscr{F}$ be a (k, g)-extremal family, $m, g \geqq 2$. Let $S = \cup \mathscr{F}$, and $\mathscr{A}$ and $\mathscr{B}$ defined by Proposition 3.4. (Clearly $\mathscr{A}, \mathscr{B} \neq \emptyset$). Replace $\mathscr{A}$ with a copy of $\mathscr{F}(|\mathscr{A}|, k)$ and $\mathscr{B}$ with $\mathscr{F}(|\mathscr{B}|, k-1)$. More exactly let $\overline{\mathscr{F}}$ be a family of k-subsets of positive integers $\overline{\mathscr{F}} = \overline{\mathscr{A}} \cup (\overline{\mathscr{B}} + \{z\})$, where $\overline{\mathscr{A}} = \mathscr{F}(|\mathscr{A}|, k)$, $\overline{\mathscr{B}} = \mathscr{F}(|\mathscr{B}|, k-1)$, and $z = |\overline{\mathscr{A}} \cup \overline{\mathscr{B}}| + 1$. By the definition of the antilexicographical ordering $\Delta_g \mathscr{F}(m, k) = \mathscr{F}(|\Delta_g \mathscr{F}(m, k)|, g)$. Hence

$$\Delta_g \overline{\mathscr{A}} \supset \Delta_g \overline{\mathscr{B}} \tag{11}$$

because $|\Delta_{k-1}\mathscr{A}| \geqq |\mathscr{B}|$ (by Proposition 3.4) and both $\Delta_{k-1}\overline{\mathscr{A}}$ and $\overline{\mathscr{B}}$ are initial segments of the antilexicographical ordering of $(k-1)$-element sets of positive numbers. Now apply the Kruskal—Katona—Lindström theorem, (11), (6), the Kruskal—Katona—Lindström theorem, and (6) again. We get

(12)

$$\begin{aligned} |\Delta_g \mathscr{A}| + |\Delta_{g-1}\mathscr{B}| &\geqq K(|\mathscr{A}|, k, g) + K(|\mathscr{B}|, k-1, g-1) = |\Delta_g \overline{\mathscr{A}}| + |\Delta_{g-1}\overline{\mathscr{B}}| = \\ &= |\Delta_g \overline{\mathscr{A}} \cup \Delta_g \overline{\mathscr{B}}| + |\Delta_{g-1}\overline{\mathscr{B}}| = |\Delta_g \overline{\mathscr{F}}| \geqq K(|\overline{\mathscr{F}}|, k, g) = |\Delta_g \mathscr{F}| = \\ &= |\Delta_g \mathscr{A} \cup \Delta_g \overline{\mathscr{B}}| + |\Delta_{g-1}\overline{\mathscr{B}}| \geqq |\Delta_g \mathscr{A}| + |\Delta_{g-1}\mathscr{B}|. \end{aligned}$$

Hence equality holds throughout in (12), which yields:

$$|\Delta_g \mathscr{A}| = K(|\mathscr{A}|, k, g), \tag{13}$$

i.e., $\mathscr{A}$ is (k, g)-extremal,

$$|\Delta_{g-1}\mathscr{B}| = K(|\mathscr{B}|, k-1, g-1), \tag{14}$$

i.e., $\mathscr{B}$ is $(k-1, g-1)$-extremal,

$$\Delta_g \mathscr{A} \supset \Delta_g \mathscr{B}, \tag{15}$$

$$|\Delta_g \overline{\mathscr{F}}| = |\Delta_g \mathscr{F}|, \tag{16}$$

i.e., $\overline{\mathscr{F}}$ is (k, g)-extremal.

Use the induction hypothesis for $\mathscr{A}$ and $\mathscr{B}$. We get that $\mathscr{A}$ is $(k, g-1)$- and $\mathscr{B}$ is $(k-1, g-2)$-extremal. Equality in (12) yields that $\Delta_g \mathscr{A} \supset \Delta_g \mathscr{B}$, so that

$\Delta_{g-1}\mathscr{A}\supset\Delta_{g-1}\mathscr{B}$. Hence by (6) we have

(16a) $$|\Delta_{g-1}\mathscr{F}| = |\Delta_{g-1}\mathscr{A}\cup\Delta_{g-1}\mathscr{B}|+|\Delta_{g-2}\mathscr{B}| = |\Delta_{g-1}\mathscr{A}|+|\Delta_{g-2}\mathscr{B}| =$$
$$= |\Delta_{g-1}\overline{\mathscr{A}}|+|\Delta_{g-2}\overline{\mathscr{B}}| = |\Delta_{g-1}\overline{\mathscr{A}}\cup\Delta_{g-1}\overline{\mathscr{B}}|+|\Delta_{g-2}\overline{\mathscr{B}}| =$$
$$= |\Delta_{g-1}\overline{\mathscr{F}}|.$$

Thus we are done if we prove that $\overline{\mathscr{F}}$ is $(k, g-1)$-extremal. Note that $\overline{\mathscr{F}}$ is 1-shifted. Denote $\overline{\mathscr{F}}_0=\overline{\mathscr{F}}(\neg 1)$, $\overline{\mathscr{F}}_1=\overline{\mathscr{F}}(1)-\{1\}$. Let $|\overline{\mathscr{F}}|=m=\sum_{i\leq k}\binom{a_i}{i}$ in k-cascade form. Then by (4) and (5) we get that the family $\Delta_g\overline{\mathscr{F}}_1$ is $(g, g-1)$-extremal. Use the induction hypothesis for $\Delta_g\overline{\mathscr{F}}_1$ $(g<k)$. We obtain that it is $(g, g-2)$-extremal, i.e., $|\Delta_{g-2}\overline{\mathscr{F}}_1|=\sum_{i\leq k-2}\binom{a_i-1}{i-(k-g+2)}$. Now applying (2) we have

$$|\Delta_{g-1}\overline{\mathscr{F}}| = |\Delta_{g-1}\overline{\mathscr{F}}_1|+|\Delta_{g-2}\overline{\mathscr{F}}_1|$$
$$= \sum_{i\leq k}\binom{a_i-1}{i-(k-g+1)}+\sum_{i\leq k}\binom{a_i-1}{i-(k-g+2)}$$
$$= \sum_{i\leq k}\binom{a_i}{i-(k-g+1)}$$
$$= K(m, k, g-1).$$

Hence $\overline{\mathscr{F}}$ is a $(k, g-1)$-extremal family. ▮

4. Proofs of Propositions 2.3 and 2.5

Proof of 2.3. Trivial. ▮

Proof of 2.5. Consider the hypergraphs $\mathscr{F}(m+1, k)-\{E\}$ where E varies over all $E\in\mathscr{F}(m+1, k)$. We claim that there exist some two of them which are non-isomorphic.

Lemma 4.1. *Suppose* $\mathscr{H}=\{E_1, \ldots, E_m\}$, $E_i\subset S$, $|E_i|=k$ *for all* $1\leq i\leq m$ *and let* $\mathscr{H}_i=\{E_j\in\mathscr{H}: 1\leq j\leq k,\ j\neq i\}$. *Suppose that* $\mathscr{H}_i\cong\mathscr{H}_j$ *for all* $i\neq j$. *Then there exists a partition* $k=n_1+n_2+\ldots+n_t$ *(for all* $n_i\geq 1$*),* $S=S_1\cup S_2\cup\ldots\cup S_t$ *such that* $|E\cap S_i|=n_i$ *for all* i *and* $E\in\mathscr{H}$, *and* $d_{\mathscr{H}}(x)=d_{\mathscr{H}}(y)$ *for all* $x, y\in S_i$.

Proof. Let $d_1<d_2<\ldots<d_t$ be the different values of $d_{\mathscr{H}}(x)$, and $S_i=\{x\in S: d_{\mathscr{H}}(x)=d_i\}$. The hypergraphs $\mathscr{H}_i$ and $\mathscr{H}_j$ have the same degree-sequence. ▮

Now it is easy to see that the conclusions of Lemma 4.1 hold for $\mathscr{H}=\mathscr{F}(m, k)$ only in the cases $m\leq k+1$ and $m=\binom{\eta}{k}$. ▮

5. Proof of Theorem 2.6

Let $\mathcal{G}$ be a family of g-subsets of the n-element set S, and denote by $\Delta^k\mathcal{G}=\{F\subset S: |F|=k, \exists G\in\mathcal{G} \text{ such that } G\subset F\}$. Set $f(m,n,k,g)=\min\{|\Delta^k\mathcal{G}|: \mathcal{G}\subset\binom{S}{g}, |\mathcal{G}|=m, |S|=n\}$. We call the family $\mathcal{G}$ *minimal* if $|\Delta^k\mathcal{G}|=f(m,n,k,g)$. Now we give a reformulation of Corollary 2.2 with this terminology:

Corollary 2.2′. *Suppose* $\mathcal{G}\subset\binom{S}{g}$, $|S|=n$, $|\mathcal{G}|=m\leqq\binom{n-1}{g-1}$ *a minimal family. Then* $\bigcap\mathcal{G}\neq\emptyset$.

Proof of 2.2′. For $\mathcal{H}\subset\binom{S}{g}$ denote by $\mathcal{H}^c$ the family of complements, $\mathcal{H}^c=\{S-H: H\in\mathcal{H}\}$. By definition $(\Delta^k\mathcal{H})^c=\Delta_{n-k}(\mathcal{H}^c)$. Hence for the minimal family $\mathcal{G}$,

$$f(m,n,k,g)=|\Delta^k\mathcal{G}|=|(\Delta^k\mathcal{G})^c|=|\Delta_{n-k}(\mathcal{G}^c)|\geqq K(m,n-g,n-k). \tag{17}$$

Indeed, in (17) equality holds because $\mathcal{G}^c$ is an $(n-g,n-k)$-extremal family on n points, i.e., $\mathcal{G}\subset\binom{S}{g}$ is minimal iff $\mathcal{G}^c\subset\binom{S}{n-g}$ is $(n-g,n-k)$-extremal.

Now $|\mathcal{G}^c|\leqq\binom{n-1}{g-1}=\binom{n-1}{n-g}$, and $\mathcal{G}^c$ is $(n-g,n-k)$-extremal, hence by Corollary 2.2 we have $|\bigcup\mathcal{G}^c|\leqq n-1$. This implies $\bigcap\mathcal{G}\neq\emptyset$. ∎

Now we are ready to prove Theorem 2.6. Let $m=\sum_{s\leqq i\leqq k}\binom{a_i}{i}$. Here $s>k-g$ by Proposition 2.3. Consider a (k,g)-extremal family $\mathcal{F}$. We are going to prove that $\mathcal{F}\cong\mathcal{F}(m,k)$. We will use induction on $k-s$. If $k-s=0$ then $m=\binom{a}{k}$ and Corollary 2.2 implies $|\bigcup\mathcal{F}|\leqq a$, i.e., $\mathcal{F}\cong\mathcal{F}\left(\binom{a}{k},k\right)$. From now on we suppose that $k-s\geqq 1$.

By Corollary 2.2 we have $|\bigcup\mathcal{F}|=a_k+1$. Define $S=\bigcup\mathcal{F}$, $\mathcal{G}=\left\{G\in\binom{S}{g}: G\not\subset F \text{ for every } F\in\mathcal{F}\right\}$, i.e., $\mathcal{G}=\binom{S}{g}-\Delta_g\mathcal{F}$. Clearly $\Delta^k\mathcal{G}\cap\mathcal{F}=\emptyset$. We claim

Proposition 5.1. *$\mathcal{G}$ is minimal.*

Proof. Suppose on the contrary that there exists an $\mathcal{H}\subset\binom{S}{g}$, $|\mathcal{H}|=|\mathcal{G}|$, $|\Delta^k\mathcal{H}|<|\Delta^k\mathcal{G}|$. Then define $\mathcal{F}'=\binom{S}{k}-\Delta^k\mathcal{H}$. We have

$$|\Delta_g\mathcal{F}'|\leqq\left|\binom{S}{g}-\mathcal{H}\right|=\left|\binom{S}{g}-\mathcal{G}\right|=|\Delta_g\mathcal{F}|, \tag{18}$$

$$|\mathcal{F}'|=\left|\binom{S}{k}\right|-|\Delta^k\mathcal{H}|>\left|\binom{S}{k}\right|-|\Delta^k\mathcal{G}|\geqq|\mathcal{F}|. \tag{19}$$

Hence $|\mathcal{F}'|>|\mathcal{F}|$ and its g-shadow is not larger. This is a contradiction, for m is a (k, g)-jumping number. ∎

Since m is a (k, g)-jumping number, the argument above yields that $\Delta^k\mathcal{G}\cup\cup\mathcal{F}=\binom{S}{k}$. Now we apply Corollary 2.2′ to $\mathcal{G}$. As $|\Delta_g\mathcal{F}|>\binom{a_k}{g}$, we have $|\mathcal{G}|<<\binom{a_k+1}{g}-\binom{a_k}{g}=\binom{a_k}{g-1}=\binom{|S|-1}{g-1}$. We obtain that there exists a point $x\in S$, such that $x\in G$ for all $G\in\mathcal{G}$. The equality $\mathcal{F}=\binom{S}{k}-\Delta^k\mathcal{G}$ implies that $\binom{S-\{x\}}{k}\subset\subset\mathcal{F}$. Split $\mathcal{F}$ into two parts, $\mathcal{F}=\mathcal{F}(x)\cup\mathcal{F}(\neg x)$, let $\mathcal{B}=\mathcal{F}(x)-\{x\}$. Then

$$|\mathcal{B}| = \binom{a_{k-1}}{k-1}+\ldots+\binom{a_s}{s} \tag{20}$$

and

$$|\Delta_g\mathcal{F}| = \binom{a_k}{g}+\ldots+\binom{a_s}{s-(k-g)} = |\Delta_g\mathcal{F}(\neg x)|+|\Delta_{g-1}\mathcal{B}| = \binom{a_k}{g}+|\Delta_{g-1}\mathcal{B}|. \tag{21}$$

Hence (20) and (21) imply that $\mathcal{B}$ is a $(k-1, g-1)$-extremal family and $|\mathcal{B}|$ is a $(k-1, g-1)$-jumping number. Applying the induction hypothesis for $\mathcal{B}$ we get $\mathcal{B}\cong\mathcal{F}(|\mathcal{B}|, k-1)$, hence $\mathcal{F}\cong\mathcal{F}(m, k)$. ∎

6. Remarks, problems

It seems that the simplest open problem is to describe those $(k, k-1)$-extremal families which are left-shifted.

Proposition 6.1. *If* $m=\sum_{i\leq k}\binom{a_i}{i}$ *(cascade form) and* $\mathcal{H}$ *is a* 1*-shifted* $(k, k-1)$*-extremal family, then* $|\mathcal{H}_0|=\sum_{i\leq k}\binom{a_i-1}{i}$, $|\mathcal{H}_1|=\sum_{i\leq k}\binom{a_i-1}{i-1}$ *and* $\mathcal{H}_1$ *is a* $(k-1, k-2)$*-extremal family.*

Proof. (4) implies this trivially, because in this case $\Delta_g\mathcal{H}_1=\Delta_{k-1}\mathcal{H}_1=\mathcal{H}_1$. ∎

Example 6.2. Let $|S|=s+3$ $(s\geqq 4)$, $S_0\subset S$, $|S_0|=s$. Define $\mathcal{F}=\left\{F\in\binom{S}{3}: |S\cap S_0|\geqq\geqq 2\right\}$. Then $|\mathcal{F}|=\binom{}{}+3\binom{s}{2}=\binom{s+2}{3}+\binom{s-2}{2}+\binom{s-3}{1}$ and $|\Delta_2\mathcal{F}|=\binom{s}{2}+3\binom{s}{1}==\binom{s+2}{2}+\binom{s-2}{1}+1$. Hence $\mathcal{F}$ is a $(3, 2)$-extremal family and it is not given by Example 2.4.

We recently learned that M. Mörs [7] has independently discovered results covered here by 2.2, 2.5, and 2.6. These are deduced from more general results so the arguments there are longer and more complicated than ours. Our Theorem 2.1 here is entirely new.

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