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FINITE PROJECTIVE SPACES AND INTERSECTING HYPERGRAPHS

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Let $\mathscr{F}$ be a family of k-subsets on an n-set X and c be a real number $0<c<1$. Suppose that any t members of $\mathscr{F}$ have a common element $(t\geqq 2)$ and every element of X is contained in at most $c|\mathscr{F}|$ members of $\mathscr{F}$. One of the results in this paper is (Theorem 2.9): If

$$c = (q^{t-1}+\ldots+q+1)/(q^t+\ldots+q+1)$$

where q is a prime power and n is sufficiently large, $(n>n(k,c))$ then

$$\max|\mathscr{F}| = \binom{n-q^t-\ldots-q-1}{k-q^{t-1}-\ldots-q-1}(q^t+\ldots+q+1).$$

The corresponding lower bound is given by the following construction. Let Y be a $(q^t+\ldots+q+1)$-subset of X and $H_1, H_2, \ldots, H_{|Y|}$ the hyperplanes of the t-dimensional projective space of order q on Y. Let $\mathscr{F}$ consist of those k-subsets which intersect Y in a hyperplane, i.e., $\mathscr{F}=\left\{F\in\binom{X}{k}\right.$: there exists an i, $1\leqq i\leqq|Y|$, such that $Y\cap F=H_i\}$.

1. Introduction, notations

Let $n, k, t\geqq 2$ be positive integers, X an n-element set. Let 2^X denote the power set of X, $\binom{X}{k}$ the family of all k-element subsets of X. A family of sets is *t-wise intersecting* if any t members of it have a common element. The 2-wise intersecting families are briefly called intersecting. The signs $\lceil\ \rceil$, $\lfloor\ \rfloor$ mean upper and lower integer part, respectively.

Erdős, Ko and Rado [6] proved that if $\mathscr{F}\subset\binom{X}{k}$ is an intersecting family and $|X|=n\geqq 2k$ then $|\mathscr{F}|\leqq\binom{n-1}{k-1}$. Equality holds in the case $n>2k$ only if the members of $\mathscr{F}$ have a common element. Hilton and Milner [23] proved that if we exclude this family, i.e., if we make the additional assumption $\cap\mathscr{F}=\emptyset$ then we have $|\mathscr{F}|\leqq\binom{n-1}{k-1}-\binom{n-k-1}{k-1}+1$. Here equality holds for $k\geqq 4$ if and only

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if for some $x\in X$, $D\subset X$, $|D|=k$, $x\notin D$ we have $\mathcal{F}=\left\{F\in\binom{X}{k}: x\in F,\ D\cap F\neq\emptyset\right\}\cup\{D\}$. (In the case of $k=3$ there exists another extremal family, see Theorem 2.2.)

Let c be a real number, $0<c\leqq 1$. The degree of an element x in the set-system $\mathcal{F}$ is denoted by $d_{\mathcal{F}}(x)$ or simply by $d(x):=|\{F\in\mathcal{F}: x\in F\}|$. Erdős, Rothschild and Szemerédi [9] raised the following question: How large can an intersecting set-system $\mathcal{F}\subset\binom{X}{k}$ be if each point has degree at most $c|\mathcal{F}|$. The class of these systems is denoted by $\mathcal{F}(n, k, c)$; $f(n, k, c)$ is the maximum size of such an $\mathcal{F}$. Generally,

$\mathcal{F}^t(n, k, c) := \Big\{\mathcal{F}\subset\binom{X}{k}$: $\mathcal{F}$ is t-wise intersecting and $d_{\mathcal{F}}(x)\leqq c|\mathcal{F}|$ holds for all $x\in X\Big\}$,

$$f^t(n, k, c) = \max\{|\mathcal{F}|: \mathcal{F}\in\mathcal{F}^t(n, k, c)\}.$$

The above mentioned theorems imply

(1) [6] $$f(n, k, 1) = \binom{n-1}{k-1} \quad \text{if} \quad n > n_0(k).$$

(2) [23] $$f(n, k, <1) = k\binom{n-2}{k-2}+O(n^{k-3}).$$

For a family $\mathcal{H}\subset 2^X$ we define $\mathcal{F}(\mathcal{H})=:\Big\{F\in\binom{X}{k}$: there exists an $H\in\mathcal{H}$ such that $H\subset F\Big\}$, with the notation $Y=\cup\mathcal{H}$ we set $\mathcal{F}_0(\mathcal{H}):=\left\{F\in\binom{X}{k}: F\cap Y\in\mathcal{H}\right\}$. Obviously, $\mathcal{F}_0(\mathcal{H})\subseteq\mathcal{F}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$, $\mathcal{F}_0(\mathcal{H})$ are t-wise intersecting families whenever $\mathcal{H}$ is t-wise intersecting.

Erdős, Rothschild and Szemerédi [9] solved the case $c=2/3$, by proving

(3) $$f(n, k, 2/3) = 3\binom{n-3}{k-2} \quad \text{if} \quad n > n_0(k).$$

This result was extended by Frankl [11]

(4) $$f(n, k, c) = 3\binom{n-3}{k-2}+\binom{n-3}{k-3}, \quad \text{if} \quad 1>c>2/3, \quad n>n_0(k, c).$$

Moreover the only extremal family if $\mathcal{F}(K_2^3)$, where K_v^u denotes the complete v-hypergraph on u element $(u\geqq v)$, i.e., $K_v^u\simeq\binom{U}{v}$ for some U, $|U|=u$.

In [5] it was conjectured and Frankl ([11] for $r\leqq 3$) and Füredi [15] proved the following

(5) If $r\geqq 1$ and there exists an r-uniform finite projective plane $\mathcal{P}(2, r-1)$ on an r^2-r+1 element set Y $(Y\subset X)$ and $1/(r-1)>c>r/(r^2-r+1)$, $n>n_0(k, c)$ then

$$f(n, k, c) = (r^2-r+1)\binom{n}{k-r}+O(n^{k-r-1}).$$

Here the extremal family is $\mathcal{F}(\mathcal{P}(2, r-1))$.

The only remaining case which is solved is $1/2<c<2/3$, see [11] or [17]. (In the case $3/5<c<2/3$ there are 6 non-isomorphic extremal families.) Finally, [15] contains the following general result.

(6) There exists an infinite sequence $1=c_1>c_2>\ldots>c_r>\ldots>0$ tending to 0, and a positive function $f(c)$ such that

$$f(n,k,c)=f(c)\binom{n}{k-r}+O(n^{k-r-1})$$

holds for $c_r\leqq c<c_{r-1}$. Moreover the function f is piecewise either constant or a rectangular hyperbola arc. The aim of this paper is to extend these results for t-wise intersecting families.

We note that there is a strong connection between intersecting hypergraphs and graphs of diameter 2. See Pach and Surányi [26].

2. Results

The earlier known cases

Theorem 2.1 (Frankl [10]). $f^t(n,k,1)=\binom{n-1}{k-1}$ *whenever* $n\geqq kt/(t-1)$. *In the case* $n>kt/(t-1)$ *the only extremal family is* $\mathcal{F}(\{x\})$. ∎

Clearly, $f^t(n,k,1)=\binom{n-1}{k-1}$ for $n\geqq 2k$ follows from (1). Theorem 2.1 says that this holds for $2k>n\geqq kt/(t-1)$ as well. For $n<kt/(t-1)$ obviously we have $f^t(n,k,1)=\binom{n}{k}$.

Let $k\geqq t\geqq 2$, define $\mathcal{F}_{D_1,D_2}$ as follows: $|D_1|=t-1$, $|D_2|=k-t+2$, $D_1\cap D_2=\emptyset$, $D_1\cup D_2\subset X$ and $\mathcal{F}_{D_1,D_2}:=\left\{F\in\binom{X}{k}: D_1\subset F,\ D_2\cap F\neq\emptyset\right\}\cup\{D_1\cup D_2-\{x\}: x\in D_1\}$.

Theorem 2.2. *For* $n>n_0(k)$ *we have*

$$f^t(n,k,<1)=\begin{cases}|\mathcal{F}_{D_1,D_2}| & \text{if } k>2t-1, \text{ and in the case } k=3,\ t=2,\\ |\mathcal{F}(K_t^{t+1})| & \text{if } t\leqq k\leqq 2t-1.\end{cases}$$

Moreover the only maximal families are the above mentioned two families. ∎

Hence in all cases we have one extremum except $k=3$, $t=2$. This theorem is an easy consequence of a theorem of Frankl [11] about set-systems $\mathcal{F}\subset\binom{X}{k}$ satisfying $|F\cap F'|\geqq t-1$ for all $F,F'\in\mathcal{F}$. Indeed, if $\mathcal{F}$ is t-wise intersecting, and $\cap\mathcal{F}=\emptyset$ then $|F\cap F'|\geqq t-1$ holds. However, our method yields a comparatively good estimation for $n_0(k)$, $n_0(k)<h(t)k$ (see Theorem 8.1). For $t>k$ there is no $\mathcal{F}\subset\binom{X}{k}$, $\cap\mathcal{F}=\emptyset$ which is t-wise intersecting. From now on we always suppose, to avoid trivialities, that $k>t$.

Theorem 2.3. (Gronau [20]). *Suppose* $t/(t+1)<c<1$. *Then for* $n>n^t(k, c)$

$$f^t(n, k, c) = (t+1)\binom{n-t-1}{k-t}+\binom{n-t-1}{k-t-1},$$

and the only extremal family is $\mathcal{F}(K_t^{t+1})$. *Moreover for* $n>n^t(k)$ *we have*

$$f^t(n, k, t/(t+1)) = (t+1)\binom{n-t-1}{k-t}$$

and the extremal family is $\mathcal{F}_0(K_t^{t+1})$. ▮

This is also implied by a theorem of Frankl [11] on $(t-1)$-intersecting families. The present paper is selfcontained, Theorem 2.2 and 2.3 come as byproducts of the main lemma (Theorems 6.6—7). Moreover our new proof gives linear upper bound for $n^t(k, c)$ (i.e., $n^t(k, c)<kh^t(c)$ where $h^t(c)$ depends only on t and c)

The main results

Theorem 2.4. *Let* $n>k>t>1$ *be integers and* c *a real,* $0<c<1$. *There exists an infinite sequence* $1=c_1^t\geqq c_2^t\geqq\ldots\geqq c_r^t\ldots>0$, *tending to* O *and a positive valued function* $f^t(c)$: $(0, 1)\to\mathbf{R}$ *with the following properties: If* $c_r^t\leqq c<c_{r-1}^t$ *then*

$$f^t(n, k, c) = f^t(c)\binom{n}{k-r}+O(n^{k-r-1}).$$

Let us remark, that the calculation of $f^t(c)$ (and r) is a finite problem in the following sence: Let $\mathcal{F}\in\mathcal{F}^t(n, k, c)$ be maximal. Then there exists a family of $\leqq r$-sets $\mathcal{H}$ over the $\leqq 4^r$ elements set Y such that $|\mathcal{F}-\mathcal{F}(\mathcal{H})|=O(n^{k-r-1})$, i.e., the main part of $\mathcal{F}$ belongs to $\mathcal{F}(\mathcal{H})$. About the computation of $f^t(c)$ and the determination of such $\mathcal{H}$ we know much more. See Chapter 3. Roughly speaking it is enough to consider finitely many linear programming problems.

We have $c_1^t=c_2^t=\ldots=c_{t-1}^t=1$, and $c_t^t=t/(t+1)$ by Theorem 2.3 and the following theorem:

Theorem 2.5. *Let* $t<r\leqq(3/2)t-1$, *and* $1-2/(3t-r+2)\leqq c<1-2/(3t-r+3)$. *Then for* $n>n^t(k, c)$ *we have*

$$f^t(n, k, c) = \left\lfloor\frac{1}{1-c}\binom{n-r-1}{k-r}\right\rfloor.$$

One of the extremal families is given by Example 2.7 (see later). This is the only maximum in the first part of the interval $[c_r^t, c_{r-1}^t)$ (whenever $c_r^t= =1-2/(3t-r+2)\leqq c<1-2/(3t-r+2.5)$). Another extremal families can be obtained (in the interval $[1-2/(3t-r+8/3), 1-2/(3t-r+3))$) from the nucleus $\mathcal{H}'$ (see Figure 2) or $\mathcal{H}''$ (Figure 3).

Denote by $\mathcal{H}^i$ the hypergraph over $3i$ vertices whose edges are the complements of the edges of i disjoint triangles.

Theorem 2.6. *Let t be an odd integer $(t \geqq 3)$ and denote $(3t-1)/2$ by r. Suppose $(3t-1)/(3t+3) < c < (3t+3)/(3t+7)$ then for $n > n^t(k, c)$ we have*

$$f^t(n, k, c) = (r+2)\binom{n-r-1}{k-r} + \binom{n-r-2}{k-r-2},$$

and one of the extremal families is $\mathcal{F}(\mathcal{H}^{(t+1)/2})$. Moreover

$$f^t\left(n, k, \frac{3t-1}{3t+3}\right) = (r+2)\binom{n-r-2}{k-r}$$

and the only extremum is $\mathcal{F}_0(\mathcal{H}^{(t+1)/2})$.

Example 2.7. Let $Y = D_0 \cup D_1 \cup \ldots \cup D_{r-t}$ be an $(r+1)$-set, $|D_0| = 3t-2r+1$ $(\geqq 3)$ and $|D_1| = \ldots = |D_{r-t}| = 3$ are pairwise disjoint. Define the set-system $\mathcal{H}$ as follows: $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ where $\mathcal{H}_1 = \{H \subset Y : |H| = r,\ D_0 \not\subset H\}$, $\mathcal{H}_2 = \{H \subset Y : |H| = r-1,\ \exists i\ (1 \leqq i \leqq r-t)$ such that $|D_i \cap H| = 1\}$ (see Fig. 1). Let $\mathcal{H}_2 = \{H_1, H_2, \ldots, H_{3(r-t)}\}$ and choose the family of k-sets $\mathcal{F}_i$ such a way that all $F \in \mathcal{F}_i$ contain H_i, $|\mathcal{F}_i| < \binom{n-r-1}{k-r}$, at least $\frac{1}{2}\binom{n-r-1}{k-r}$ of them intersect Y in H_i precisely, finally $|\cup \mathcal{F}_i| = \left\lfloor \frac{1}{1-c}\binom{n-r-1}{k-r} \right\rfloor - |\mathcal{H}_0|\binom{n-r-1}{k-r}$. (Remark that the difference between the two coefficients of the binomial factor is less than $\frac{3}{2}(t-r)+\frac{1}{2}$ and greater than $\frac{3}{2}(t-r)$.) The parts of the members of $\mathcal{F}_i$ outside Y can be located arbitrarily. Finally, let $\mathcal{F} = \bigcup_{1 \leqq i \leqq 3(r-t)} \mathcal{F}_i \cup \{F \in \mathcal{F} : F \cap Y \in \mathcal{H}_1\}$ (i.e., $\mathcal{F} \subset \mathcal{F}(\mathcal{H})$).

The above results cover the cases when c is close to 1. The next chapter sheds light on the asymptotic behavior of c_r^t.

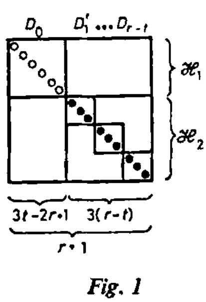

Fig. 1

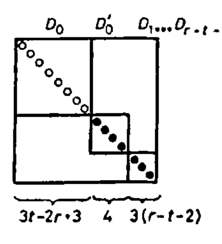

Fig. 2

Families with strong constraints on the maximal degree

Let $\mathcal{P}_Y(t, q)$ denote the family of hyperplanes of a projective geometry of order q and dimension t with point set Y. If it causes no confusion, we simply write $\mathcal{P}$. Recall that $|\mathcal{P}| = |Y| = (q^{t+1}-1)/(q-1)$. Note that for $t \geqq 3$, q must be a prime power while for $t = 2$ it is a famous open problem whether planes of non-prime power order exist.

Theorem 2.8. *Suppose $q \geqq 2$ is a prime power and $(q^{t-1}+\ldots+q+1)/(q^t+\ldots+q+1) < c < (q^{t-1}+\ldots+q+1)/(q^t+\ldots+q+1)+1/q^{2t}$. Then*

$$f^t(n,k,c) = \frac{q^{t+1}-1}{q-1}\binom{n-(q^{t+1}-1)/(q-1)}{k-(q^t-1)/(q-1)}\left(1+O\left(\frac{1}{n}\right)\right)$$

holds. Moreover if $\mathcal{F} \in \mathcal{F}^t(n,k,c)$ is maximal then for $n > n^t(k,c)$ there is a $\mathcal{P}_Y(t,q)$ with $Y \subset X$ such that $\mathcal{F} \supset \mathcal{F}(\mathcal{P})$. Finally, $|\mathcal{F}-\mathcal{F}(\mathcal{P})| = O(|\mathcal{F}(\mathcal{P})|/n^{q^t-1.5})$.

Theorem 2.9. *If q is a prime power then for $n > n^t(k,c)$*

$$f^t\left(n, k, \frac{q^t-1}{q^{t+1}-1}\right) = |\mathcal{F}_0(\mathcal{P})|$$

and $\mathcal{F}_0(\mathcal{P})$ is the only extremal family.

Corollary 2.10. *Let c_r^t be as in Theorem 2.4. Then for fixed t*

$$\lim_{r\to\infty} c_r^t / \sqrt[t-1]{r} = 1.$$

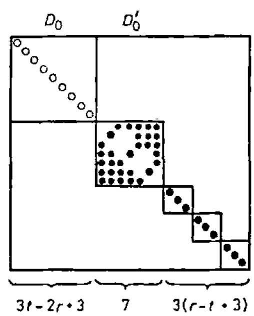

Fig. 3. $1-2/(3t+r+2.75) \leqq c < 1-2(3t-r+3)$

More on the case $t=3$

Theorems 2.3 and 2.6 state that

$$f^3(n,k,c) = |\mathcal{F}(\mathcal{H}^{(1)})| = 4\binom{n-4}{k-3}+\binom{n-4}{k-4}$$

for $0.75 < c < 1$,

$$f^3(n,k,0.75) = |\mathcal{F}_0(\mathcal{H}^{(1)})| = 4\binom{n-4}{k-3},$$

$$f^3(n,k,c) = |\mathcal{F}(\mathcal{H}^{(2)})| = 6\binom{n-6}{k-4}+O(n^{k-5})$$

for $2/3 < c < 0.75$ and

$$f^3(n,k,2/3) = |\mathcal{F}_0(\mathcal{H}^{(2)})| = 6\binom{n-6}{k-4},$$

holds for $n > n^3(k,c)$. Moreover the extremal families are obtained from $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ (see Fig 4). We can solve the following cases as well.

Theorem 2.11. *Suppose* $5/8<c<2/3$ *and let* $\mathcal{H}^{(3)}$ *be the hypergraph on* 8 *points given by Fig. 4. Then*

$$f^3(n,k,c)=|\mathcal{F}(\mathcal{H}^{(3)})|=8\binom{n-8}{k-5}+O(n^{k-6})$$

holds for $n>n^3(k,c)$. *The only extremal family is* $\mathcal{F}(\mathcal{H}^{(3)})$.

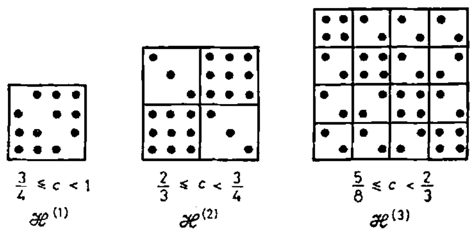

Fig. 4. $t=3$, $5/8\le c<1$

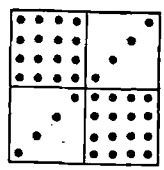

Fig. 5. $t=3$, $c=5/8$.

Theorem 2.12. *Let* $\mathcal{H}^{(4)}$ *be the hypergraph on* 8 *elements given by Fig. 5. Then*

$$f^3\left(n,k,\frac{5}{8}\right)=|\mathcal{F}_0(\mathcal{H}^{(3)})|=|\mathcal{F}_0(\mathcal{H}^{(4)})|=8\binom{n-8}{k-5}.$$

The only extremal families are $\mathcal{F}_0(\mathcal{H}^{(3)})$ *and* $\mathcal{F}_0(\mathcal{H}^{(4)})$. *Moreover for* $c<5/8$ *we have* $f^3(n,k,c)=O(n^{k-6})$, *i.e.,* $c_1^3=c_2^3=1$, $c_3^3=3/4$, $c_4^3=2/3$, $c_5^3=5/8$.

3. Further generalizations

We call the family $\mathcal{F}$ *t-wise s-intersecting* if $|F_1\cap F_2\cap\ldots\cap F_t|\ge s$ holds whenever $F_1,\ldots,F_t\in\mathcal{F}$. Assume that $n\ge k>s$, $t\ge 1$, $t\ge 2$, $0<c\le 1$ is a real and define $\mathcal{F}^{t,s}(n,k,c)=\left\{\mathcal{F}\subset\binom{X}{k}: |X|=n, \mathcal{F}\right.$ is t-wise s-intersecting, $d_{\mathcal{F}}(x)\le c|\mathcal{F}|$ for all $\left.x\in X\right\}$. Set $f^{t,s}(n,k,c)=\max\{|\mathcal{F}|: \mathcal{F}\in\mathcal{F}^{t,s}(n,k,c)\}$. Clearly, $f^{t,1}(n,k,c)=f^t(n,k,c)$. The results of Chapter 2 can be extended to t-wise s-intersecting families.

Theorem 3.1. *There exists a sequence* $1=c_1^{t,s}\ge c_2^{t,s}\ge\ldots\ge\ldots>0$ *and a positive valued function* $f^{t,s}(c)\colon(0,1)\to\mathbf{R}$ *with the following properties: If* $c_r^{t,s}\le c<c_{r-1}^{t,s}$ *then*

$$f^{t,s}(n,k,c)=f^{t,s}(c)\binom{n}{k-r}+O(n^{k-r-1}).$$

Theorem 3.2. *Suppose* $s>(t-1)l(l-1)$ *and* $(lt+s-l)/(lt+s)<c<\big((lt+s-l)/(lt+s)\big)+$ $+\big(1/(l+s)(lt+s)\big)$. *Then*

$$f^{t,s}(n,k,c) = |\mathcal{F}(K^{lt+s}_{lt+s-l})|.$$

The only extremal family is $\mathcal{F}(K^{lt+s}_{lt+s-l})$.

The above theorem was proved in [11] for $t=2$. The following is an extension of Theorem 2.8. Define $[q^i]=q^i+q^{i-1}+\ldots+q+1$.

Theorem 3.3. *Let* q *be a prime power,* $t\geqq 2$, $l\geqq 0$ *integers. Suppose* $[q^{t+l-1}]/[q^{t+l}]<$ $<c<[q^{t+l-1}]/[q^{t+l}]+1/q^{2t+2l+2}$. *Then for* $n>n(k,c)$

$$f^{t,[q^t]}(n,k,c) = [q^{t+l}]\binom{n-[q^{t+l}]}{k-[q^{t+l-1}]}\cdot\big(1+O(1/n)\big).$$

The lower bound is given by $\mathcal{F}(\mathcal{P}(t+l,q))$. Moreover

Theorem 3.4. $f^{t,[q^t]}(n,k,[q^{t+l-1}]/[q^{t+l}])=\big|\mathcal{F}_0(\mathcal{P}(t+l,q))\big|$ *holds for* $n>n(k)$.

The main theorem (Theorem 6.6) gives further results. E.g.: define the biplane $\mathcal{B}_6$ as follows. Let $X=\{(i,j)\colon 1\leqq i,\ j\leqq 4\}$, $\mathcal{B}_6\subset\binom{X}{6}$, $\forall B, B'\in\mathcal{B}_6$ we have $|B\cap B'|=$ $=2$, $|\mathcal{B}_6|=16$. ($\mathcal{B}_6$ is representable as the set $\{B(i,j)\colon 1\leqq i,\ j\leqq 4\}$ where $B(i,j)=$ $=\{(u,v)\colon 1\leqq u, v\leqq 4,\ (u,v)\neq(i,j)$ and either $i=u$ or $j=v\}$. See Fig. 6.) See [21].

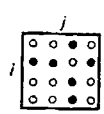

Fig. 6.

Theorem 3.5. *Let* $\mathcal{F}\subset\binom{X}{k}$ *be a family of finite sets with property* $|F\cap F'|\geqq 2$ *for* $F, F'\in\mathcal{F}$, $n\geqq k>6$. *Suppose* $d_{\mathcal{F}}(x)\leqq(3/8)|\mathcal{F}|$ *holds for* $x\in X$. *Then for* $n>n_0(k)$

$$|\mathcal{F}|\leqq|\mathcal{F}_0(\mathcal{B}_6)| = 16\binom{n-16}{k-6}$$

where $\mathcal{B}_6$ *is the biplane of order* 6.

Finally we remark that our $n^t(k,c)$ $\big(n^{t,s}(k,c)\big)$ is polynomial (in the cases of Theorems 2.2, 2.3, 2.5, 2.6, 2.9, 2.11, 2.12, 3.2, 3.4, 3.5 actually linear) in k. Cf. Theorem 8.1.

4. Finding the kernel of the extremal families I. critical hypergraphs

Define an *edge-contraction* as the following operation on a family $\mathcal{H}$: we substitute an edge $E\in\mathcal{H}$ by a smaller, non-empty $E'\subsetneqq E$, and thus we get the set-system $(\mathcal{H}-\{E\})\cup\{E'\}$. A set-system having property P is *P-critical* if it has no multiple edges and the hypergraph obtained by contracting any of its edges does not have property P. We can get a P-critical family from any $\mathcal{H}$ having property

P by contracting its edges as far as possible and deleting all but one copy of the appearing multiple edges. The obtained (smaller) family, $\mathcal{K}$ is called the *P-kernel* of $\mathcal{H}$. (Of course, $\mathcal{K}$ is not necessarily unique, but this is not important for us.)

We are interested in t-wise s-intersecting families, $t \geqq 2$, $s \geqq 1$. Call a hypergraph (t, s)*-critical* if it is critical t-wise s-intersecting. The *rank* of a family $\mathcal{H}$ is $\max_{H \in \mathcal{H}} |H|$. One more definition: The family $F_1, F_2, \dots, F_t$ is called a *t-star* with kernel N if, for every $1 \leqq i < j \leqq t$, we have $F_i \cap F_j = N$. The well-known Erdős—Rado theorem [8] says: If the set-system $\mathcal{H}$ of rank r does not contain a t-star, then $|\mathcal{H}| \leqq (t-1)^r r!$

Lemma 4.1. *If $\mathcal{H}$ is (t, s)-critical of rank r then $|\mathcal{H}| \leqq r^{2r}$.*

Proof. It is clear, that a (t, s)-critical $\mathcal{H}$ does not contain an $(r+1)$-star. Hence $|\mathcal{H}| \leqq r^r r! \leqq r^{2r}$ by the Erdős—Rado theorem. ∎

Remark 4.2. Denote by $E^{t,s}(r)$ and $V^{t,s}(r)$ the maximum number of edges (vertices) of a (t, s)-critical hypergraph of rank r. $V^{2,1}(r) < \infty$ was proved by Calczynska—Karlowicz [4]. His bounds were improved by Erdős and Lovász [7] who showed $\lfloor r!(e-1) \rfloor \leqq E^{2,1}(r) \leqq r^r$, $\frac{1}{8}\binom{2}{r} < V^{2,1}(r) \leqq \frac{r}{4}\binom{2}{r}$. The current best result is due to Tuza [29], $V^{2,1}(r) \leqq \frac{1}{2}\binom{2r}{r}$. Lovász conjectures $E^{2,1}(r) = \lfloor r!(e-1) \rfloor$. (It was proved for $r \leqq 4$ by Hanson and Toft [22].) The best bounds in the general case are due to Alon [1]:

$$3^{(k-s)/3t} < V^{t,s}(r) < t^{(k-s)/t}.$$

Considering the complete hypergraph K_{tl+s-t}^{tl+s} one can see that $E^{t,s}(r)$ is exponential in r for all fixed t and s.

Looking for an extremal hypergraph $\mathcal{F} \in \mathcal{F}^{t,s}(n, k, c)$ we use the following operation

Definition 4.3. (Erdős, Ko and Rado [6]). *Suppose $\mathcal{F} \subset 2^X$, $X = \{1, 2, \dots, n\}$, $i < j$. Define the (left) shifting S_{ij} on $\mathcal{F}$ as follows:*

$$S_{ij}(F) = \begin{cases} F - \{j\} \cup \{i\} & \text{if } j \in F,\ i \notin F,\ F - \{j\} \cup \{i\} \notin \mathcal{F}, \\ F & \text{otherwise,} \end{cases}$$

$S_{ij}(\mathcal{F}) = \{S_{ij}(F) \colon F \in \mathcal{F}\}$.

Lemma 4.4. (Frankl [13]). *Suppose $\mathcal{F} \subset \binom{X}{k}$ is a t-wise s-intersecting family. Then $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$ and $S_{ij}(\mathcal{F})$ is also t-wise s-intersecting.*

Now let $\mathcal{F} \in \mathcal{F}^{t,s}(n, k, c)$. Apply repeatedly the operation S_{ij} to $\mathcal{F}$ until we obtain either a family $\mathcal{H}$ such that $d_{\mathcal{H}}(x) > c|\mathcal{H}|$ $(= c|\mathcal{F}|)$ or a family $\mathcal{G}$ which is stable, i.e., $S_{ij}(\mathcal{G}) = \mathcal{G}$ holds for all $1 \leqq i < j \leqq n$.

To avoid the first possibility we apply S_{ij} only for those pairs where $d_{\mathcal{H}}(i)$, $d_{\mathcal{H}}(j) \leqq (c/2)|\mathcal{H}|$. Consider the family finally obtained $\mathcal{G} \in \mathcal{F}^{t,s}(n, k, c)$, and suppose $d_{\mathcal{G}}(1) \geqq d_{\mathcal{G}}(2) \geqq \dots \geqq d_{\mathcal{G}}(n)$.

Proposition 4.5. *Suppose* $Y=\{1,2,\ldots,\lfloor(2/c)k\rfloor+2k\}$. *Then the family* $\mathcal{G}|Y:=$ $:=\{G\cap Y\colon G\in\mathcal{G}\}$ *is t-wise s-intersecting.*

Proof. We use the obvious fact that $\mathcal{G}$ contains at most $(2/c)k$ points with degree at least $(c/2)|\mathcal{G}|$. Hence the family $\mathcal{G}$ is stable on the points $\{\lfloor(2/c)k\rfloor+1,\ldots,n\}$. The following argument is similar to the proof of Lemma 2.2 in [14]. Suppose on the contrary that $G_1,G_2,\ldots,G_t\in\mathcal{G}$, $|G_1\cap\ldots\cap G_t\cap Y|<s$, and $|G_1\cap\ldots\cap G_t|$ is minimal subject to this constraint. Then there exists a point $j\in G_1\cap\ldots\cap G_t$, $j>|Y|$ and a point $i\in Y$, $i>\lfloor(2/c)k\rfloor$ which is not covered by at least two G_α's, say G_1 and G_2. Then $G_1'=G_1-\{j\}\cup\{i\}\in\mathcal{G}$ as well, and $|G_1'\cap G_2\cap\ldots\cap G_t|<|G_1\cap\ldots\cap G_t|$, a contradiction. ∎

Summarizing the results of this chapter we obtain

Lemma 4.6. *There exists an* $\mathcal{F}\in\mathcal{F}^{t,s}(n,k,c)$ *with maximum cardinality* $(|\mathcal{F}|=$ $=f^{t,s}(n,k,c))$ *such that it has a* (t,s)*-critical kernel* $\mathcal{K}$ *with* $|\bigcup\mathcal{K}|\leq(2/c)k+2k$. *Moreover* $\mathcal{F}$ *is partially stable, i.e.,* $1\leq i<j\leq n$, $F\in\mathcal{F}$, $i\notin F$, $j\in F$, $d_{\mathcal{F}}(i)$, $d_{\mathcal{F}}(j)\leq$ $\leq(c/2)|\mathcal{F}|$ *imply* $F-\{j\}\cup\{i\}\in\mathcal{F}$. ∎

5. Finding the kernel of the extremal families II. fractional matchings of hypergraphs

Definitions

Let $\mathcal{H}$ denote a family of sets on a ground-set X. Let $\nu(\mathcal{H})$ or briefly ν denote the *matching number* of $\mathcal{H}$, i.e., the maximum number of pairwise disjoint edges in $\mathcal{H}$, i.e., $\nu(\mathcal{H})=\max\{w\colon \exists E_1,E_2,\ldots,E_w\in\mathcal{H},\ E_i\cap E_j=\emptyset\}$. Let $\tau(\mathcal{H})$ or briefly τ denote the *covering number* of $\mathcal{H}$, i.e., the minimum cardinality of a transversal, i.e., $\tau(\mathcal{H})=\min\{|T|\colon T\cap E\neq\emptyset$ for all $E\in\mathcal{H}\}$. Clearly, $\tau\leq\nu r$, where r denotes the rank of $\mathcal{H}$.

A *fractional matching* of $\mathcal{H}$ is a function $w\colon\mathcal{H}\to\mathbf{R}$ satisfying $w(E)\geq 0$ for every edge $E\in\mathcal{H}$ and

$$\sum\{w(E)\colon x\in E\in\mathcal{H}\}\leq 1 \quad \text{for every} \quad x\in X.$$

The value of the fractional matching w is $|w|=\sum_{E\in\mathcal{H}}w(E)$. The maximum of $|w|$ when w ranges over all fractional matchings is called the *fractional matching number* and is denoted by

$$\nu^*(\mathcal{H})=\max\{|w|\colon w \text{ is a fractional matching of } \mathcal{H}\}.$$

Similarly, the fractional covering number is the minimum value of fractional covers of $\mathcal{H}$, i.e.,

$$\tau^*(\mathcal{H})=\min\Big\{\sum_{x\in X}t(x)\colon\ t\colon X\to\mathbf{R},\ t(x)\geq 0,\ \forall E\in\mathcal{H} \text{ we have } \sum_{x\in E}t(x)\geq 1\Big\}.$$

Clearly, to determine the fractional matching and covering number is a linear programming problem. This is a dual pair, so by the Duality Principle of linear programming we have $\tau^*(\mathcal{H})=\nu^*(\mathcal{H})$ for every hypergraph $\mathcal{H}$.

In view of the fact that $w(E)\equiv 1/D$ (where $D=\max_{x\in X} d_{\mathscr{H}}(x)$), and $t(x)\equiv \equiv 1/\min |E|$ are fractional matching, covering resp., we have

$$(7) \qquad \frac{|\mathscr{H}|}{D(\mathscr{H})} \leqq v^*(\mathscr{H}) \leqq \frac{|X|}{\min\{|E|:\ E\in\mathscr{H}\}}.$$

This implies, e.g., $v^*(\mathscr{P}(t, q))=[q^t]/[q^{t-1}]$. (Recall that $[q^i]$ stands for $q^i+q^{i-1}+\dots +1$.)

Hypergraphs with maximum fractional matchings

A family $\mathscr{F}$ is called *v^*-critical* if $v^*(\mathscr{F}')<v^*(\mathscr{F})$ holds for each subfamily $\mathscr{F}'\subsetneqq\mathscr{F}$, We are going to use the following results of the second author.

Lemma 5.1 [16]). *If the family $\mathscr{F}$ is v^*-critical then $|\mathscr{F}|\leqq|\cup\mathscr{F}|$.* ∎

Lemma 5.2 [19]. *If the family $\mathscr{F}$ of rank r is v^*-critical then $|\mathscr{F}|\leqq rv^*$.* ∎

Define $v^*(r, t, s)=\sup\{v^*(\mathscr{H}): \mathscr{H}$ is t-wise s-intersecting of rank $r\}$. $(t\geqq 2)$. In this section $\mathscr{H}$ will always denote a t-wise s-intersecting hypergraph of rank r.

Proposition 5.3. *There exists a t-wise s-intersecting hypergraph $\mathscr{H}$ of rank r such that $v^*(\mathscr{H})=v^*(r, t, s)$.*

Proof. To determine $v^*(r, t, s)$ it is enough to consider the v^*-critical hypergraphs $\mathscr{H}$. By Lemma 5.2 its cardinality is at most $rv^*(r, t, s)\leqq r^2$, hence we have finitely many possibilities. ∎

Theorem 5.4. *Let $\mathscr{F}$ be a (2-wise) s-intersecting family of rank r. Then either*

(a) $\qquad v^*(\mathscr{F}) = (r-1)/s+(1/r)$ *and $\mathscr{F}$ is an (r, s)-design, or*

(b) $\qquad v^*(\mathscr{F}) \leqq (r-1)/s+(1/r)-(r-s)/r(r-1)s.$

We recall that an (r, s)-design is a pair $(X, \mathscr{B})$ where X is a set of $r(r-1)/s+1$ elements, $\mathscr{B}$ is a family of r-sets of X, $|\mathscr{B}|=|X|$ and any two members of $\mathscr{B}$ intersect in exactly s elements. (Hence any pair $\{x, y\}\subset X$ is contained in exactly s members of $\mathscr{B}$.) The best-known example is $\mathscr{P}(a, q)$, it is a $([q^{a-1}], [q^{a-2}])$-design.

Proof. The case $s=1$ was conjectured by Lovász [25] and proved in [16]. The proof given here is simpler. We invoke a lemma.

Lemma 5.5 (see [28]). *An (r, s)-design $\mathscr{B}$ is a maximal s-intersecting family of rank r. I.e., if T is an r-set and $|T\cap B|\geqq s$ for all $B\in\mathscr{B}$ then $T\in\mathscr{B}$.* ∎

First consider an s-intersecting family $\mathscr{F}$ of rank r. By Lemma 5.1 one can choose a subfamily $\mathscr{G}\subset\mathscr{F}$ such that $v^*(\mathscr{G})=v^*(\mathscr{F})$ and $|\mathscr{G}|\leqq|\cup\mathscr{G}|$. Denote $\cup\mathscr{G}$ by X. Then

$$(8) \qquad \frac{1}{|X|}\sum_{x\in X} d_{\mathscr{G}}(x) = \frac{1}{|X|}\sum_{G\in\mathscr{G}} |G| \leqq r\frac{|\mathscr{G}|}{|X|} \leqq r.$$

Hence $\min_{x\in X} d_{\mathscr{G}}(x) = d \leqq r$. Let x be a vertex with minimum degree d and $G_1, G_2, \ldots, G_d$ be the edges of $\mathscr{G}$ through x. Let $w: \mathscr{G}\to\mathbf{R}$ be a fractional matching, $\sum_{1\leqq i\leqq d} w(G_i)=q$. $(q\leqq 1)$. Then the following holds for all fixed i

$$(9)\qquad q+r-1 \geqq \sum_{x\in G_i}\Big(\sum_{x\in E\in\mathscr{G}} w(E)\Big) = \sum_{E\in\mathscr{G}} w(E)|E\cap G_i|$$
$$\geqq s\sum_{E\in\mathscr{G}} w(E)+(r-s)\,w(G_i).$$

Summing up (9) for all i we get $dq+d(r-1)\geqq sd|w|+(r-s)q$ which yields

$$(10)\qquad (r-1)/s+(d-r+s)\,q/sd \geqq |w|.$$

The second term of the left hand side is at most $(1/r)-(r-s)/(r-1)rs$ if $d<r$. Hence we have obtained that (b) holds in this case.

Next consider the case $d=r$. (8) implies that $\mathscr{G}$ is r-regular and r-uniform (i.e., $\forall x\, d_{\mathscr{G}}(x)=r$, $\forall G\in\mathscr{G}$ $|G|=r$.) Then by (7) we have

$$(11)\qquad \nu^*(\mathscr{G}) = |\mathscr{G}|/r = |X|/r.$$

Consider an arbitrary edge $G_0\in\mathscr{G}$. We have

$$(12)\qquad r^2 = \sum_{x\in G_0} d_{\mathscr{G}}(x) = \sum_{G\in\mathscr{G}} |G\cap G_0| \geqq r-s+s|\mathscr{G}|$$

i.e., $|\mathscr{G}|\leqq(r^2-r+s)/s$. If $|\mathscr{G}|\leqq(r^2-r+s-1)/s$, then (b) holds by (11). If $|\mathscr{G}|= =(r^2-r+s)/s$ then equality holds in (12). Hence $\mathscr{G}$ is an (r, s) design.

Finally, by Lemma 5.5, we have that in the latter case $\mathscr{G}=\mathscr{F}$. ∎

Theorem 5.6. *Suppose $\mathscr{H}$ is a t-wise $[q^l]$-intersecting family of rank $[q^{t+l-1}]$. Then either*

(a) *$\mathscr{H}$ is isomorphic to $\mathscr{P}(t+l, q)$ and then $\nu^*(\mathscr{H})=[q^{t+l}]/[q^{t+l-1}]$, or*

(b) *$\nu^*(\mathscr{H})<[q^{t+l}]/[q^{t+l-1}]-1/q^{2t+2l}$.*

Proof. We are going to use Theorem 5.4. If $\mathscr{H}$ is $[q^{t+l-2}]$-intersecting then we are done. Suppose that there exist $H_1, H_2\in\mathscr{H}$ with $|H_1\cap H_2|<[q^{t+l-2}]$. Let a be maximal such that $\exists F_1, F_2, \ldots, F_a\in\mathscr{H}$ with $|F_1\cap F_2\cap\ldots\cap F_a|<[q^{t+l-a}]$. We have $a\geqq 2$. On the other hand $a<t$ as $\mathscr{H}$ is t-wise $[q^l]$-intersecting. Hence every $H\in\mathscr{H}$ intersects $Y=F_1\cap\ldots\cap F_a$ in at least $[q^{t+l-a-1}]$ elements. Thus the function $t: Y\to\mathbf{R}$, $t(y)\equiv 1/[q^{t+l-a-1}]$ is a fractional cover of $\mathscr{H}$. This means

$$\nu^*(\mathscr{H}) \leqq |Y|/[q^{t+l-a-1}] \leqq (q^{t+l-a}+\ldots+q^2+q)/(q^{t+l-a-1}+\ldots+q+1) = q$$
$$< [q^{t+l}]/[q^{t+l-1}]-1/q^{2t+2l}. \quad ∎$$

Lemma 5.7. *Suppose $\mathscr{F}$ is t-wise s-intersecting of rank $lt+s-l$ where $t\geqq 2$, $l\geqq 1$. Suppose $s>(t-1)l/(l-1)$, then either*

(a) *$\mathscr{F}$ is $(lt+s-2l)$-intersecting (e.g., $\mathscr{F}\approx K^{lt+s}_{lt+s-l}$), or*

(b) *$\nu^*(\mathscr{F})\leqq(s+l-1)/s$ $\big(<(lt+s)/(lt+s-l)\big)$.*

Proof. The case $t=2$ was proved in [11]. We proceed as in the proof of Theorem 5.6. Suppose on the contrary that there exist $H_1, H_2\in\mathscr{F}$, $|H_1\cap H_2|<lt+s-2l$. Let a be maximal such that there exist $F_1, \ldots, F_a\in\mathscr{F}$ with $|Y|=|H_1\cap\ldots\cap H_a|<$

$<lt+s-al$, we have $2 \leqq a < t$. Then for each $H \in \mathcal{F}$ we have $|H \cap Y| \geqq lt+s-(a+1)l$ yielding $v^*(\mathcal{F}) \leqq (lt+s-al-1)/(lt+s-al-a) \leqq (s+l-1)/s$. ∎

We need the following lemmas to prove Theorems 2.5—2.6. From now on $\mathcal{H}$ will denote t-wise intersecting hypergraph of rank r.

Lemma 5.8 [18]. *Suppose* $2 \leqq t \leqq r \leqq 3t/2-1$. *Then* $v^*(r, t, 1) = 1+2/(3t-r)$.
(a) *Suppose* $v^*(\mathcal{H}) > v^*(r-1, t, 1) = \big(=1+2/(3t-r+1)$ *or* $1\big)$ *then there exists an element* x *with* $d_{\mathcal{H}}(x) = |\mathcal{H}|-1$, *and the only edge* E *not containing* x *has* r *elements.*
(b) *Suppose* $v^*(\mathcal{H}) > 1+2/(3t-r+0.5)$ *then* $\mathcal{H}$ *is isomorphic to Example 2.7. (Hence* $|\mathcal{H}| = |\cup \mathcal{H}| = r+1$). ∎

Lemma 5.9 [18]. *Suppose* $r = (3t-1)/2$. *Then* $v^*(r, t, 1) = 1+2/r$.
(a) *Suppose* $v^*(\mathcal{H}) > v^*(r-1, t, 1)$ $\big(=1+2/(r+2)\big)$. *Then there exists an element* x *with* $d_{\mathcal{H}}(x) \geqq |\mathcal{H}|-2$, *and the edge(s) avoiding* x *has (have)* r *elements.*
(b) *Suppose* $v^*(\mathcal{H}) > 1+2/(r+1)$ *then* $\mathcal{H}$ *is isomorphic to* $\mathcal{H}^{(t+1)/2}$ *(see Theorem 2.6).* ∎

The following lemma exhibits how v^* can be used to derive lower bounds for the maximum degree:

Lemma 5.10. *Let* $a\colon \mathcal{F} \to \mathbf{R}$ *be any non-negative real-valued function on the edges of* $\mathcal{F}$, $\mathcal{F} \subset 2^X$. *Then*

$$\max_{x \in X} \Big(\sum_{x \in F \in \mathcal{F}} a(F) \Big) \geqq \frac{1}{v^*(\mathcal{F})} \Big(\sum_{F \in \mathcal{F}} a(F) \Big) = \frac{|a|}{v^*(\mathcal{F})}.$$

($|a|$ denotes the sum $\sum a(F)$). This lemma is an extension of the well-known inequality $\max_{x \in X} d_{\mathcal{F}}(x) \geqq |\mathcal{F}|/v^*(\mathcal{F})$ (see (7)).

Proof. Let $M = \max_{x \in X} \sum_{x \in F \in \mathcal{F}} a(F)$. Then the function a/M is a fractional matching of $\mathcal{F}$, thus $|a/M| = |a|/M \leqq v^*(\mathcal{F})$. ∎

6. Finding the kernel of the extremal families III. The main theorem, linear programming and intersecting hypergraphs

The determination of the order the magnitude of $f^{t,s}(n, k, c)$

Lemma 6.1. *Suppose that* $c \geqq 1/v^*(r, t, s)$. *Then*

$$f^{t,s}(n, k, c) > \binom{n}{k-r} - O(n^{k-r-1}).$$

Proof. This was proved in [15] for $t=2$, $s=1$. Now we can do it in the same way. Let $\mathcal{H}$ be a t-wise s-intersecting hypergraph of rank r on a $Y \subset X$ such that $v^*(\mathcal{H}) = v^*(r, t, s)$ (such an $\mathcal{H}$ exists by Proposition 5.3), and let $w\colon \mathcal{H} \to \mathbf{R}$ be an optimal fractional matching. We can suppose that $w(H)$ is rational, e.g., $w(H)N$ is an integer for some integer N, for all $H \in \mathcal{H}$. Moreover we can suppose that $\mathcal{H}$ is r-uniform (otherwise we add new vertices). Let $Y\subset X$, $m=N\left[\binom{n-|Y|}{k-r}\Big/N\right]$ $\left(m>\binom{n-|Y|}{k-r}-N\right)$. Define $\mathcal{F}(H)$ as follows: $\mathcal{F}(H)\subset\left\{F\in\binom{X}{k}: F\cap Y=H\right\}$, $|\mathcal{F}(H)|=w(H)m$. Then for $\mathcal{F}=\bigcup_{H\in\mathcal{F}}\mathcal{F}(H)$ we have $\mathcal{F}\in\mathcal{F}^{t,s}(n,k,c)$ and $|\mathcal{F}|\geqq v^*(\mathcal{H})\binom{n-|Y|}{k-r}-$ $-v^*(\mathcal{H})N$. ∎

Lemma 6.2. *Suppose that* $1/v^*(r,t,s)\leqq c<1/v^*(r-1,t,s)$. *Then*

$$f^{t,s}(n,k,c)\leqq\frac{r^{2r}}{1-cv^*(r-1,t,s)}\binom{n}{k-r}+O(n^{k-r-1}).$$

Corollary 6.3. *The value of* $c_r^{t,s}$ *(defined in Theorem* 2.4 *and* 3.1*) is* $1/v^*(r,t,s)$. ∎

Proof of 6.2. Let $\mathcal{F}\in\mathcal{F}^{t,s}(n,k,c)$ be maximal. By Lemma 6.1 we can suppose $|\mathcal{F}|>(1-o(1))\binom{n}{k-r}$. Let $Y\subset X$ be a subset having at most $(2/c)k+2k$ elements defined by Lemma 4.6, and $\mathcal{H}_i=\{H\subset Y: H=Y\cap F$ for some $F\in\mathcal{F}$, $|H|=i\}$. Let $\mathcal{F}(\mathcal{H}_{>r})=\{F\in\mathcal{F}: |F\cap Y|>r\}$. Clearly

$$|\mathcal{F}(\mathcal{H}_{>r})|\leqq\binom{|Y|}{r+1}\binom{n}{k-r-1}. \tag{13}$$

Let $\mathcal{B}$ be a (t,s)-*critical kernel* of $\mathcal{H}_{\leqq r}$, i.e., (one of) the (t,s)-critical hypergraph obtained from $\mathcal{H}_{\leqq r}$ contracting its edges as long as it is possible. Then by Lemma 4.1 we have $|\mathcal{B}|\leqq r^{2r}$.

Definition 6.4. Call $\mathcal{B}$ as above, a (t,s)-*critical kernel of rank r* of $\mathcal{F}$.

Define $\mathcal{B}_{<r}:=\{B\in\mathcal{B}: |B|<r\}$. Now we are ready to define a weight function on the edges of $\mathcal{B}$. Choose a $B_F\subset F$, $B_F\in\mathcal{B}$ for every $F\in\mathcal{F}-\mathcal{F}(\mathcal{H}_{>r})$, and let $\mathcal{F}_B=\{F\in\mathcal{F}: B\subset F, B=B_F\}$ $(B\in\mathcal{B})$. Now let

$$w(B)=|\mathcal{F}_B|\Big/\binom{n-|Y|}{k-r}. \tag{14}$$

Obviously,

$$w(B)\leqq 1\quad\text{if}\quad B\in\mathcal{B},\quad |B|=r. \tag{15}$$

We are going to use Lemma 5.10 for $\mathcal{B}_{<r}$ and $w:\mathcal{B}_{<r}\to\mathbf{R}$. By this lemma we obtain a point $y\in Y$ such that

$$\sum_{y\in B\in\mathcal{B}_{<r}}w(B)\geqq\frac{1}{v^*(\mathcal{B}_{<r})}\sum_{B\in\mathcal{B}_{<r}}w(B)\geqq\frac{1}{v^*(r-1,t,s)}\sum_{B\in\mathcal{B}_{<r}}w(B). \tag{16}$$

Now (16), (14) and (13) yield

$$c|\mathcal{F}|\geqq d_{\mathcal{F}}(y)\geqq\frac{1}{v^*(r-1,t,s)}\left(|\mathcal{F}|-\Big(\sum_{B\in\mathcal{B}_r}w(B)\Big)\binom{n-|Y|}{k-r}-|\mathcal{F}(\mathcal{H}_{>r})|\right)$$
$$\geqq\frac{1}{v^*(r-1,t,s)}\left(|\mathcal{F}|-r^{2r}\binom{n-|Y|}{k-r}-\binom{|Y|}{r+1}\binom{n}{k-r-1}\right). \tag{17}$$

Rearranging we get Lemma 6.2. ∎

The asymptotic determination of $f^{t,s}(n, k, c)$

Let $\mathscr{B}$ be a family of rank r on the set Y, $0<c<1$, $\mathscr{B}_r=\{B\in\mathscr{B}: |B|=r\}\neq\emptyset$. The optimum value of the linear programming problem (*) is called the *capacity* of $\mathscr{B}$ belonging to c.

$$(*)\qquad \begin{cases} w: \mathscr{B}\to \mathbf{R}, \\ w(B)\geqq 0 \quad \text{for all} \quad B\in\mathscr{B}, \\ w(B)\leqq 1 \quad \text{for all } B\in\mathscr{B}_r, \\ \sum_{y\in B\in\mathscr{B}} w(B)\leqq c\big(\sum_{B\in\mathscr{B}} w(B)\big)=c|w| \quad \text{for} \quad y\in Y. \end{cases}$$

$\mathrm{Cap}_{\mathscr{B}}(c):=\max\{|w|: w$ satisfies (*)$\}$. It may of course occur that $\mathrm{Cap}_{\mathscr{B}}(c)=0$ or $\mathrm{Cap}_{\mathscr{B}}(c)=\infty$.

Lemma 6.5. *If $\mathscr{B}$ is a t-wise s-intersecting family of rank r and*

$$1/v^*(r, t, s)\leqq 1/v^*(\mathscr{B})\leqq c<1/v^*(r-1, t, s)$$

then

$$v^*(\mathscr{B})\leqq \mathrm{Cap}_{\mathscr{B}}(c)\leqq |\mathscr{B}|/(1-cv^*(\mathscr{B}_{<r})).\quad \blacksquare$$

The proof is similar to the proof of 6.2, see [15].

Theorem 6.6. *If $\mathscr{F}\in\mathscr{F}^{t,s}(n, k, c)$ and $\mathscr{B}$ is its (t, s)-critical kernel of rank r then*

$$(18)\qquad |\mathscr{F}|<\mathrm{Cap}_{\mathscr{B}}(c)\binom{n}{k-r}+K(r, c)\binom{n}{k-r-1}.$$

On the other hand if $\mathscr{B}$ is an arbitrary t-wise s-intersecting family of rank r, then there exists an $\mathscr{F}'\in\mathscr{F}^{t,s}(n, k, c)$ such that

$$(19)\qquad |\mathscr{F}'|>\mathrm{Cap}_{\mathscr{B}}(c)\binom{n}{k-r}-K(r, c)\binom{n}{k-r-1}.$$

(In the case $\mathrm{Cap}_{\mathscr{B}}(c)=\infty$, (19) *means* $\sup|\mathscr{F}'|\Big/\binom{n}{k-r}=\infty$ *whenever* $n\to\infty$.*)*

Proof. (Sketch, a detailed proof can be found in [15] for the case $t=2$, $s=1$.)

Upper bound. Suppose $\mathscr{F}\in\mathscr{F}^{t,s}(n, k, c)$ with $|\mathscr{F}|>\binom{n}{k-r}(1-o(1))$ (by Lemma 6.1). Consider the weight function $w: \mathscr{B}\to\mathbf{R}$ defined in the proof of 6.2 We have using (13) that for $y\in Y$

$$(20)\qquad \big(\sum_{y\in B} w(B)\big)\leqq \frac{\deg_{\mathscr{F}}(y)}{\binom{n-|Y|}{k-r}}\leqq \frac{c|\mathscr{F}|}{\binom{n-|Y|}{k-r}}=c\left(\sum_B w(B)+O\left(\frac{1}{n}\right)\right)$$
$$\leqq (c+o(1))\big(\sum_B w(B)\big).$$

We can see that w satisfies (*) with $(c+\varepsilon)$ for n large enough. Now by definition

$$(21)\qquad \sum_B w(B)\leqq \mathrm{Cap}_{\mathscr{B}}(c+\varepsilon).$$

It is easy to see that the function $\mathrm{Cap}_{\mathscr{B}}$: $\mathbf{R}\to\mathbf{R}$ is continuous from the right hand side even more, it has Lipschitz-property from the right hand side, hence (21) implies that

$$\sum_{B} w(B) \leqq \mathrm{Cap}_{\mathscr{B}}(c)+o(1). \tag{22}$$

Now rearranging (20) and using (22) we obtain (18).

Lower bound. If c is rational, we can proceed exactly in the same way as we did in the proof of Lemma 6.1. Again using the continuity of the function $\mathrm{Cap}_{\mathscr{B}}$ the assertion follows. ∎

Corollary 6.7. *If* $1/v^*(r, t, s)\leqq c<1/v^*(r-1, t, s)$ *then* $f^{t,s}(c)=\max\{\mathrm{Cap}_{\mathscr{B}}(c)\colon \mathscr{B}$ *has rank* r, $\mathscr{B}$ *is t-wise s-intersecting*$\}$. ∎

7. Proofs

Each proof consists of four parts. Suppose $\mathscr{F}\in\mathscr{F}^{t,s}(n, k, c)$, $|\mathscr{F}|$ is maximal. Then we first determine the value of r for which $1/v^*(r, t, s)\leqq c<1/v^*(r-1, t, s)$ holds. As a second step we investigate the (t, s)-critical kernel of $\mathscr{F}$, denoted by $\mathscr{B}$ (see Definition 6.4) and determine its possible maximal capacity $(\mathrm{Cap}_{\mathscr{B}}(c))$. Then we determine $\mathscr{H}_{\leqq r}\subset 2^Y$, ($Y$ is defined by Lemma 4.5 and 4.4), and finally $\max|\mathscr{F}|$.

We illustrate our method in some examples. Here we prove that the theorems holds for n large enough; $n>n^t(k, c)$. In the next chapter we will improve the bounds for $n^t(k, c)$.

Proof of Theorem 2.3. Suppose $\mathscr{F}\in\mathscr{F}^t(n, k, c)$, $t/(t+1)<c<1$. By Lemma 5.8 we have $v^*(r, t, 1)=1$ for $r<t$ and $v^*(t, t, 1)=(t+1)/t$. Moreover the only t-wise intersecting hypergraph $\mathscr{H}$ of rank t with $v^*(\mathscr{H})>1$ is K_t^{t+1}. This implies that the $(t, 1)$-critical kernel of $\mathscr{F}$ is $K_t^{t+1}\approx\mathscr{H}$. Set $\cup\mathscr{H}=Y$. Then for each $F\in\mathscr{F}$ we have $|F\cap Y|\geqq t$, i.e., $\mathscr{F}\subset\mathscr{F}(K_t^{t+1})$. The case $c=t/(t+1)$ is left to the reader. ∎

Proof of Theorem 3.3. Suppose $\mathscr{F}\in\mathscr{F}^{t,[q^l]}(n, k, c)$ where $[q^{t+l-1}]/[q^{t+l}]<c<$ $<[q^{t+l-1}]/[q^{t+l}]+1/q^{2t+2l+2}$. Suppose $|\mathscr{F}|$ is maximal,

$$|\mathscr{F}|\geqq\left|\mathscr{F}_0(\mathscr{P}(t+l, q))\right|=[q^{t+l}]\binom{n-[q^{t+l}]}{k-[q^{t+l-1}]}.$$

Consider the $(t, [q^l])$-critical kernel $\mathscr{B}$ of $\mathscr{F}$ (defined by 6.4). By Lemma 6.5 $v^*(\mathscr{B})>$ $>1/c>[q^{t+l}]/[q^{t+l-1}]-1/q^{2t+2l}$. $\mathscr{B}$ is t-wise $[q^l]$-intersecting of rank $[q^{t+l-1}]$, hence Theorem 5.6 yields $\mathscr{B}\approx\mathscr{P}(t+l, q)$. Now $\mathrm{Cap}_{\mathscr{B}}(c)=[q^{t+l}]$. Theorem 6.6 completes the proof. ∎

Proof of Theorem 3.4. Suppose $\mathscr{F}\in\mathscr{F}^{t,[q^l]}(n, k, [q^{t+l-1}]/[q^{t+t}])$ $|\mathscr{F}|$ is maximal. Then (see the proof of Theorem 3.3) the t-wise $[q^l]$-intersecting kernel of $\mathscr{F}$ is $\mathscr{P}(t+l, q)$. Set $Y=\cup\mathscr{P}(t+l, q)$. For each edge $F\in\mathscr{F}$ and hyperplanes $H_1, \ldots$ $\ldots, H_{t-1}\in\mathscr{P}(t+l, q)$ we have

$$|F\cap H_1\cap\ldots\cap H_{t-1}|\geqq[q^l]. \tag{23}$$

Now use the following theorem of Beutelspacher [2] which is a generalization of an earlier theorem of Pelikán [27] and Bruen [3].

Theorem 7.1. [2] *Suppose that* (23) *holds for the set* F *and every* $H_1, \ldots, H_{t-1} \in \in \mathcal{P}(t+l, q)$. *Then either*

(a) $$F \supset H \text{ for some } H \in \mathcal{P}(t+l, q), \text{ or}$$

(b) $$|F| \geqq [q^{t+l-1}] + \sqrt{q}\, q^{t+l-2}. \quad \blacksquare$$

This implies that every $F \in \mathcal{F}$ intersects Y in at least $[q^{t+l-1}]$ elements. Hence

$$|\mathcal{F}|[q^{t+l-1}] \leqq \sum_{F \in \mathcal{F}} |F \cap Y| = \sum_{y \in Y} d_{\mathcal{F}}(y) \leqq \frac{[q^{t+l-1}]}{[q^{t+l}]} |\mathcal{F}|\,|Y|.$$

Here the left hand side is equal to the right hand side. Since $c = [q^{t+l-1}]/[q^{t+l}]$, $|F \cap Y| = [q^{t+l-1}]$ must hold for all $F \in \mathcal{F}$. Thus Theorem 7.1 implies $F \cap Y \in \mathcal{P}(t+l, q)$, i.e., $\mathcal{F} \subset \mathcal{F}_0(\mathcal{P}(t+l, q))$. $\blacksquare$

The other proofs are equally easy consequences of the corresponding lemmas. That is, Theorem 2.4 and 3.1 are easy consequences of Corollaries 6.3 and 6.7. Theorem 2.5 and 2.6 are implied by Lemma 5.8 and 5.9, Theorem 3.2 is implied by Lemma 5.7 and Theorem 2.11 and 2.12 are special cases of the main results (Theorem 6.6 and Corollary 6.7).

8. Our results are valid for $n > O(k)$

The aim of this chapter is to improve the main results. We will show that $n^{t,s}(k, c) = O(k)$. More exactly we have

Theorem 8.1. *There exists a function* $h^{t,s}(c)$ *such that* $n^{t,s}(k, c)$ *(defined in Theorems* 2.2, 2.3, 2.5, 2.6, 2.8, 2.9, 2.11, 2.12, 3.2, 3.3, 3.4, 3.5) *is less than* $k \cdot h^{t,s}(c)$.

Before the proof we need a lemma.

Lemma 8.2 [12]. *Let* n, b, a, r *be positive integers* $a, b \geqq r$ *and* $n \geqq a+b-r$. *Let* $\mathcal{A} \subset \binom{X}{a}$, $\mathcal{B} \subset \binom{X}{b}$ *be two families of subsets of the n-element set X. Suppose that* $|A \cap B| \geqq r$ *holds for all* $A \in \mathcal{A}$, $B \in \mathcal{B}$. *Then either* $|\mathcal{A}| \leqq \binom{n}{a-r}$ *or* $|\mathcal{B}| \leqq \binom{n}{b-r}$ *holds.* $\blacksquare$

This lemma is a generalization of a theorem of Kleitman [24]. He investigated the case $r=1$ only, but he proved a stronger result $\left(\text{either } |\mathcal{A}| \leqq \binom{n-1}{a-1} \text{ or } |\mathcal{B}| \leqq \binom{n-1}{b-1}\right)$

Proof of Theorem 8.1. Let c, t, s be given, define r as $1/v^*(r, t, s) \leqq c < 1/v^*(r-1, t, s)$. From now on the constants $c_1, c_2, \ldots$ denote positive reals depending only on t, s

and c. Lemma 6.1 yields that for $n>c_1k$ we have

$$f^{t,s}(n,k,c)>\binom{n-c_2}{k-r}. \tag{25}$$

Now we are going to improve Lemma 4.6. Let $\mathcal{F}\in\mathcal{F}^{t,s}(n,k,c)$ and suppose $|\mathcal{F}|$ is maximal. Apply repeatedly the left-shifting operation S_{ij} for $\mathcal{F}$ (see Definition 4.3) if $\deg_{\mathcal{F}}(i),\deg_{\mathcal{F}}(j)\leqq(c/2)|\mathcal{F}|$, $1\leqq i<j\leqq n$. Finally we obtain a family $\mathcal{G}\in\mathcal{F}^{t,s}(n,k,c)$ which is left-stable, i.e.,

$$\text{if}\quad 1\leqq i<j\leqq n,\quad \deg_{\mathcal{G}}(i),\quad \deg_{\mathcal{G}}(j)\leqq(c/2)|\mathcal{G}|\quad\text{and} \tag{26}$$
$$G\in\mathcal{G},\ i\notin G,\ j\in G\quad\text{then}\quad G-\{j\}\cup\{i\}\in\mathcal{G}.$$

We can suppose $\deg_{\mathcal{G}}(1)\geqq\deg_{\mathcal{G}}(2)\geqq\ldots\geqq\deg_{\mathcal{G}}(n)$. Denote by $Y=\{i:\deg_{\mathcal{G}}(i)>(c/2)|\mathcal{G}|\}$. Our first aim is to prove

Proposition 8.3. *$|Y|\leqq 4r/c$ if $n>c_3k$.*

Proof. Let $\mathcal{K}=\{G\in\mathcal{G}: |G\cap Y|\geqq|Y|c/4\}$. Then

$$|\mathcal{K}|>(c/4)|\mathcal{F}|. \tag{27}$$

Indeed, we have

$$(c/2)|Y|\leqq\sum_{i\in Y}\frac{\deg_{\mathcal{G}}(i)}{|\mathcal{G}|}=\frac{\sum_{G\in\mathcal{G}}|G\cap Y|}{|\mathcal{G}|}\leqq\frac{|\mathcal{G}-\mathcal{K}|}{|\mathcal{G}|}\cdot\frac{c}{4}|Y|+\frac{|\mathcal{K}|}{|\mathcal{G}|}\cdot|Y|.$$

Now (25) and (27) give that

$$|\mathcal{K}|>(c/4)\binom{n-c_2}{k-r} \tag{28}$$

holds for $n>c_1k$. On the other hand by definition

$$|\mathcal{K}|\leqq\binom{|Y|}{|Y|(c/4)}\binom{n}{k-|Y|(c/4)}. \tag{29}$$

Finally (28) and (29) give the result. ∎

Let $L=\{1,2,\ldots,\lceil 4r/c\rceil,\ldots,\lceil 4r/c\rceil+tr+r\}$. Let $\mathcal{G}_{>r}:=\{G\in\mathcal{G}: |G\cap L|>r\}$. Proposition 8.3 gives that

$$|\mathcal{G}_{>r}|\leqq\binom{|L|}{r+1}\binom{n-r-1}{k-r-1}<\frac{kc_4}{n}\binom{n-r}{k-r}. \tag{30}$$

In the same way as we did in Proposition 4.5, assumption (26) implies that for $G_i\in\mathcal{G}$ we have

$$\text{if}\quad |G_1\cap\ldots\cap G_t\cap L|<s\quad\text{then}\quad |G_1\cap\ldots\cap G_t\cap(X\setminus L)|\geqq r+1. \tag{31}$$

Now let $\mathcal{H}=\{G\cap Y: G\in\mathcal{G}-\mathcal{G}_{>r}\}$, and define $\mathcal{G}(H)=\{G\in\mathcal{G}: G\cap Y=H\}$ for $H\in\mathcal{H}$. Let $\mathcal{G}_0=\bigcup\left\{\mathcal{G}(H): |G(H)|\leqq\binom{n-r-1}{k-r-1}\right\}$. Clearly,

$$|\mathcal{G}_0|\leqq\left(\binom{|L|}{r}+\ldots+\binom{|L|}{0}\right)\binom{n-r-1}{k-r-1}<\frac{kc_5}{n}\binom{n-r}{k-r}. \tag{32}$$

Define $\mathcal{S}=\{G\cap Y:\ G\in\mathcal{G}\setminus(\mathcal{G}_{>r}\cup\mathcal{G}_0)\}$.

Proposition 8.4. *$\mathcal{S}$ is t-wise s-intersecting family of rank r.*

Proof. Suppose on the contrary. Then we have $S_1, \dots, S_t\in\mathcal{S}$ with $|S_1\cap\dots\cap S_t|<s$. Let $\mathcal{A}_i=\{G-S_i:\ G\in\mathcal{G},\ G\cap Y=S_i\}$. By (31) we have $|A_1\cap\dots\cap A_t|\geqq r+1$ for every $A_i\in\mathcal{A}_i$ $(1\leqq i\leqq t)$. Using Lemma 8.2 we obtain that $\min|\mathcal{A}_i|\leqq\binom{n-|L|}{k-(r+1)}\leqq\binom{n-r-1}{k-r-1}$. This contradicts to the assumption $|\mathcal{G}(S_i)|>\binom{n-r-1}{k-r-1}$. ∎

Finally, using (30) and (32) and Proposition 8.4 we can finish the proof on the same way as in Theorem 6.6. ∎

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