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Colouring Finite Incidence Structures

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Abstract. One of our results: Let $\mathbb{P}$ denote a finite projective plane of order n. Colour its vertices by χ colours. If no colour appears more than twice on any of the lines, then $\chi \geq n + 1$. Explicit constructions show that this bound is sharp when $\mathbb{P}$ is desarguesian.

1. Introduction and Notations

A *finite incidence structure* (also known as *finite linear space*, *finite geometry*, *general 2-design*, etc.) is an ordered pair $(P, \mathbb{L})$ where P is a finite set, $|P| \geq 2$, and $\mathbb{L}$ is a collection of subsets, called lines, of P satisfying the following condition: Each line has at least two points, and each pair of two distinct points belongs to precisely one line. Classical examples of finite geometries are finite projective and affine planes.

In a *projective plane of order n* there are $n^2 + n + 1$ points, $n^2 + n + 1$ lines, with $n + 1$ points on each line. The (desarguesian) projective plane over the finite field $\mathbf{F}_q$ is denoted by $PG(2, q)$. The points of this plane are the equivalence classes in $\mathbf{F}_q^3 - \{(0,0,0)\}$ of the relation "$\sim$" defined by $(x, y, z) \sim (x', y', z')$ if there exists $c \neq 0$ in $\mathbf{F}_q$ such that $(x', y', z') = (cx, cy, cz)$. The lines of $PG(2, q)$ have equations of the form $ax + by + cz = 0$ in $\mathbf{F}_q$ with $(a, b, c) \neq (0, 0, 0)$.

For an integer n the set $L \subset \{0, 1, 2, \ldots, n^2 + n\}$ is called a *difference set modulo* $M, M = n^2 + n + 1$, if $|L| = n + 1$ and for all distinct pairs $\{x, y\}, \{u, v\} \subset L$ we have $x - y \not\equiv u - v \pmod M$. Singer proved [11] that $PG(2, q)$ is *cyclic* for all q, that is, $PG(2, q)$ has an automorphism of order $q^2 + q + 1$. That is, $PG(2, q)$ can be represented as $(P, \mathbb{L})$ with $P = \{0, 1, \ldots, q^2 + q\}$ and $\mathbb{L} = \{L + i: 0 \leq i \leq M - 1\}$ where $M = q^2 + q + 1$, L is a difference set $(\mathrm{mod}\, M)$, and $L + i = \{l + i (\mathrm{mod}\, M): l \in L\}$.

An *arc* in a projective plane $\mathbb{P}$ is a set of points, no three of which are collinear. It is well-known (Bose [1]) that the number of points of an arc in $\mathbb{P}$ of order n is at most $n + 1$ or $n + 2$ according as n is odd or even. These bounds are attained in desuarguesian projective planes.

If $f: \mathbf{F}_q^3 \to \mathbf{F}_q$ is homogeneous we define the *zero set* of f as $Z(f) =: \{(x, y, z): f(x, y, z) = 0\} \subset PG(2, q)$. The zero set $Z(f)$ is called a *conic* provided that f is quadratic, $|Z(f)| = q + 1$, and $Z(f)$ is not a line.

In an *affine plane* of order n there are $n^2 + n$ lines of n points each and n^2 points. The desarguesian affine plane is denoted by $AG(2, q)$.

2. The Basic Colouring Lemma

A χ-*colouring* of the incidence structure $I = (P, \mathbb{L})$ is a map from P to $\{1, 2, \ldots, \chi\}$ which may also be viewed as a χ-partition $(P_1, P_2, \ldots, P_\chi)$ of P. The *multiplicity* of colour i on line L is $|L \cap P_i|$, that is, the number of occurrences of colour i on L. We let m denote the maximum multiplicity:

$$m = \max\{|L \cap P_i|: 1 \le i \le \chi, L \in \mathbb{L}\}.$$

The *deficiency*, $d(L)$, of a line is the difference between the length of L and the number of colours appearing on L, that is,

$$d(L) = |L| - |\{i: L \cap P_i \ne \varnothing\}|.$$

We have

$$d(L) = \sum_{\substack{i \\ L \cap P_i \ne \varnothing}} (|L \cap P_i| - 1).$$

We let $\tilde{d}$ and d, resp., denote the average deficiency and maximum deficiency. So we have

$$\tilde{d}(\chi) = \sum_L d(L)/|\mathbb{L}|,$$

and

$$d(\chi) = \max_L d(L).$$

The inequalities

$$m \le d + 1, \tag{1}$$

$$d(L) \le \frac{m-1}{m}|L|, \tag{2}$$

are easy consequences of the definitions.

2.1 Lemma. *For every χ-colouring of the incidence structure $(P, \mathbb{L})$*

$$|P| \le \chi\left(1 + m\tilde{d}\frac{|\mathbb{L}|}{|P|}\right).$$

Our lemma supplies lower bounds for the number of colours needed in the applications that follow. These lower bounds are surprisingly good and are best possible in a variety of cases.

Proof of the Lemma. As the lines cover each pair of points exactly once,

$$\binom{|P_i|}{2} = \sum_L \binom{|L \cap P_i|}{2} \tag{3}$$

for every i. By Jensen's inequality

$$\frac{1}{\chi}|P|(|P| - \chi) = \chi\frac{|P|}{\chi}\left(\frac{|P|}{\chi} - 1\right) \le \sum_i |P_i|(|P_i| - 1). \tag{4}$$

On the other hand (3) yields

$$\begin{aligned}\sum_i |P_i|(|P_i| - 1) &= \sum_i \sum_L |L \cap P_i|(|L \cap P_i| - 1)\\ &\le \sum_i \sum_{\substack{L\\ L\cap P_i \neq \varnothing}} m(|L \cap P_i| - 1)\\ &= m \sum_L \sum_{\substack{i\\ L\cap P_i \neq \varnothing}} (|L \cap P_i| - 1) = m \sum_L d(L).\end{aligned} \tag{5}$$

Combining (4) and (5) we obtain

$$\frac{1}{\chi}|P|(|P| - \chi) \le m \sum_L d(L),$$

and this completes the proof. □

3. Colourings with Small Deficiency

In this section we consider χ-colourings of projective planes of order n such that the maximum deficiency is 1.

Theorem 3.1. *Suppose that $\mathbb{P}$ is a projective plane of order n and $\chi < (n^2 + n + 1)/3$. Then any χ-colouring of $\mathbb{P}$ results in at least one line having $n - 1$ or fewer colours.*

Proof. Assume that $d = 1$. Then, from (1) we get $m \le 2$, and the lemma yields $|P| \le 3\chi$, a contradiction. □

The above Theorem is a significant improvement over Kabell's result [8] which says $\chi \le n$ implies $d > 1$.

Construction 3.2. *Suppose that $n \equiv 1 \pmod 3$ and $\mathbb{P}$ is a cyclic projective plane of order n. Then $\mathbb{P}$ has an $(n^2 + n + 1)/3$-colouring with $d = 1$.*

Proof. Let $M = n^2 + n + 1$. We may assume that $\mathbb{P} = (P, \mathbb{L})$ where $P = \{0, 1, \ldots, n^2 + n\}$ and the lines are difference sets $(\operatorname{mod} M)$. Consider the $M/3$-colouring $(P_1, P_2, \ldots, P_{M/3})$ where $P_i = \left\{i, i + \frac{M}{3}, i + \frac{2M}{3}\right\}$. Now if $L \in \mathbb{L}$ then $P_i \not\subseteq L$. Indeed $P_i \subset L$ would imply that $M/3$ has at least two representations as a difference of elements of L, contradicting the fact that L is a difference set. So we must have $|L \cap P_i| \le 2$.

To complete the proof it suffices to show that equality holds for at most one i. Suppose now that $|L \cap P_i| = |L \cap P_j| = 2$. Then $M/3 \equiv x_i - x_i' \pmod M$ and $M/3 \equiv x_j - x_j' \pmod M$ with x_i, $x_i' \in L \cap P_i$ and x_j, $x_j' \in L \cap P_j$. Since x_i, x_i', x_j, x_j' are all elements of the difference set L, they cannot be all distinct and we must have $i = j$. □

The first theorem of this section easily generalizes to.

Theorem 3.3. *Suppose that a χ-colouring of $\mathbb{P}$ of order n has deficiency d. Then $\chi \ge (n^2 + n + 1)/(d^2 + d + 1)$.* □

It will be shown in Theorem 4.2 that this inequality is sharp, even for d as high as $\sqrt{n}$.

4. Colourful Colourings

In this section $\mathbb{P}$ denotes a finite projective plane of order n. A colourful colouring of $\mathbb{P}$ is a colouring such that all colours are present on each line. The maximum number of colours in a colourful colouring is denoted by $c(\mathbb{P})$. Erdös and T. Sós [6] proved that

$$c(\mathbb{P}) \geq n/2 \log n.$$

Clearly,

$$c(\mathbb{P}) \leq n + 1.$$

The following theorem gives a better upper bound.

Theorem 4.1. $c(\mathbb{P}) \leq n - \sqrt{n} + 1$.

Proof. In a colourful χ-colouring we must have $d(L) \leq n + 1 - \chi$, so Theorem 3.3 implies

$$n^2 + n + 1 \leq \chi(d^2 + d + 1) \leq (n + 1 - d)(d^2 + d + 1).$$

Then $d \geq \sqrt{n}$. □

Theorem 4.2. *Suppose that n is a square of a primpower. Then there exists an* $(n - \sqrt{n} + 1)$*-colouring of* $\mathbb{P} = PG(2, n)$ *such that on each line* $n - \sqrt{n}$ *colours appear exactly once and one colour appears* $\sqrt{n} + 1$ *times.*

Corollary 4.3. *If* $\mathbb{P}$ *is as in Theorem* 4.2 *then* $c(\mathbb{P}) = n - \sqrt{n} + 1$.

Proof of Theorem 4.2. Bruck [3] showed that $\mathbb{P}$ decomposes into $n - \sqrt{n} + 1$ Baer subplanes. Using this decomposition one obtains the following coloring. Let $P = \{0, 1, \ldots, n^2 + n\}$, $M = n^2 + n + 1$, $N = n - \sqrt{n} + 1$. Then $P_i = \{i + \alpha N \pmod{M}: 0 \leq \alpha < M/N = n + \sqrt{n} + 1\}$ is the desired colouring $(1 \leq i \leq N)$. □

Further properties of this Baer-plane decomposition can be found in [4].

5. No Three Monochromatic Points on a Line 1. (Projective Planes)

Theorem 5.1. *Let* $\mathbb{P}$ *be a projective plane of order n. If no colour appears more than twice on any of the lines in a* χ*-colouring of* $\mathbb{P}$, *then* $\chi \geq n + 1$.

Proof. The case when n is odd was proved in [5]. It follows from the fact that if we n-colour the $n^2 + n + 1$ points of $\mathbb{P}$, then some $n + 2$ points must have the same colour. It is well-known [1] that of these points three must be collinear, because no $(n + 2)$-arc exists in $\mathbb{P}$.

If n is even, substituting $m = 2$ in (2) we get $d \leq \lfloor (n + 1)/2 \rfloor = n/2$ and our Lemma 2.1 yields $n^2 + n + 1 \leq \chi(n + 1)$ implying $\chi \geq n + 1$. □

We define a *good-colouring* as one in which no three monochromatic points are collinear, and let $w(\mathbb{P})$ denote the smallest number w such that $\mathbb{P}$ has a good w-colouring.

Theorem 5.2. *If* $\mathbb{P} = PG(2, q)$, *then* $\mathbb{P}$ *can be partitioned into* q *nondegenerate conics and a singleton.*

Corollary 5.3. *If* $\mathbb{P}$ *is as above then* $w(\mathbb{P}) = q + 1$.

This corollary shows that the inequality in Theorem 5.1 is sharp. Before we proceed with the proof of 5.2 we need the following.

Lemma 5.4. *Let* $f(x, y, z)$ *and* $g(x, y, z)$ *be homogeneous quadratic polynomials over* $\mathbf{F}_q$ *such that* $f(0, 0, 1) \neq 0$ *and* $g(x, y, z) = 0$ *if and only if* $x = y = 0$. *Then the family of zero sets* $\{Z(f + cg): c \in \mathbf{F}_q\}$ *is a partition of* $PG(2, q) - \{(0, 0, 1)\}$.

Observe that the lemma does not immediately imply the theorem, as a degenerate conic may result if f and g are not choosen carefully.

Proof of 5.4. Trivially, $(x, y, z) \in Z(f + cg)$ if and only if $(x, y, z) \neq (0, 0, 1)$ and $c = -f(x, y, z)/g(x, y, z)$. □

Proof of 5.2. We distinguish between three cases.
(i) q is odd and -1 is a square in $\mathbf{F}_q$,
(ii) q is odd and -1 is not a square,
(iii) q is even.
In the first two cases (q odd) we can use the well-known fact (see, e.g., in [9]) that if $h(x, y, z) = Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fxz$ then

$$Z(h) \text{ is nondegenerate if and only if}$$

$$\det(h) = \det\begin{pmatrix} A & D & F \\ D & B & E \\ F & E & C \end{pmatrix} \neq 0.$$

We now proceed to prove each of the three cases separately.

Case (i). (q odd, -1 a square). Choose $b \in \mathbf{F}_q$ such that b is not a square. Then $-b$ is not a square either. Let

$$f(x, y, z) = 2xy + z^2,$$
$$g(x, y, z) = x^2 + by^2.$$

Then

$$\det(f + cg) = \begin{vmatrix} c & 1 & 0 \\ 1 & cb & 0 \\ 0 & 0 & 1 \end{vmatrix} = c^2 b - 1 \neq 0,$$

so $Z(f + cg)$ is a nondegenerate conic for all $c \in \mathbf{F}_q$. By Lemma 5.4 these conics partition $PG(2, q) - \{(0, 0, 1)\}$. □

Case (ii). (q odd, -1 is not a square). First we observe that in this case there exists $r \in \mathbf{F}_q$ such that r is a square and $r + 1$ is a non-square. If this were not the case the set of $(q-1)/2$ non-zero squares would decompose into orbits of the form $\{r+k: 0 \le k < p\}$ where p is the characteristic of $\mathbf{F}_q$. This would then lead to the contradiction $p \,\Big|\, \frac{q-1}{2}$.

Having choosen $r = (2a)^2$ such that $r+1$ is a non-square we define

$$f(x,y,z) = 2axy + 2xz + z^2,$$

and

$$g(x,y,z) = x^2 + y^2.$$

The conditions of the partition lemma (Lemma 5.4) are readily fulfilled. Moreover

$$\det(f+cg) = \begin{vmatrix} c & a & 1 \\ a & c & 0 \\ 1 & 0 & 1 \end{vmatrix} = c^2 - c - a^2.$$

This determinant is never zero. For $c^2 - c = a^2$ would imply $(2c-1)^2 = 4a^2 + 1 = r+1$, a contradiction. So $\{Z(f+cg): c \in \mathbf{F}_q\}$ is a conicpartition of $PG(2,q) - \{(0,0,1)\}$. □

Case (iii). (q is a power of 2). In this case $x^2 = a$ has a unique solution, $\sqrt{a}$, for every $a \in \mathbf{F}_q$. It is well-known for every finite field $\mathbf{F}$ that

for every $A, B \in \mathbf{F}$, $(A \neq 0, B \neq 0)$ there exists a $C \in \mathbf{F}$

such that $Ax^2 + Bx + C \neq 0$ for all $x \in \mathbf{F}$.

Indeed, $p(x) = x(Ax+B)$ vanishes twice so p cannot be a permutation polynomial. For a given A and B we can then choose a C such that $-C$ is not in the range of p.

Now we proceed with the proof of case (iii). First we choose a such that $x^2 + x + a^2$ never vanishes. Then we fix a and choose b such that $x^2 + ax + b$ never vanishes. Defining f and g as

$$f(x,y,z) = x^2 + yz + z^2,$$
$$g(x,y,z) = x^2 + axy + by^2$$

guarantees that the conditions in the partition lemma are fullfilled. All that is left is to show that $Z(f+cg)$ is a nondegenerate conic for all c. Each of the three points $(1, 0, \sqrt{c+1})$, $(\sqrt{cb}, \sqrt{1+c+c^2a^2}, ac\sqrt{cb})$, and $(\sqrt{cb}, \sqrt{1+c+c^2a^2}, \sqrt{1+c+c^2a^2} + ac\sqrt{cb})$ belongs to $Z(f+cg)$. These three points are noncollinear as their determinant

$$\det\begin{pmatrix} 1 & 0 & \sqrt{c+1} \\ \sqrt{cb} & \sqrt{1+c+a^2c^2} & ac\sqrt{cb} \\ \sqrt{cb} & \sqrt{1+c+a^2c^2} & \sqrt{1+c+a^2c^2} + ac\sqrt{cb} \end{pmatrix}$$
$$= 1 + c + a^2c^2 \neq 0.$$

Since $Z(f+cg)$ has at least 3 points it must have at least $q+1$. But

$$q^2 + q = |PG(2,q) - \{(0,0,1)\}| = \sum_c |Z(f+cg)| \ge q(q+1).$$

Equality is forced and hence every zero-set has exactly $q + 1$ points. With three non-collinear points each they must all be non-degenerate conics. □

6. No Three Monochromatic Points on a Line 2. (Affine Planes)

Theorem 6.1. *Let A be an affine plane of order n. If no colour appears more than twice on any of the lines in a χ-colouring $(P_1, \ldots, P_\chi)$ of A, then*

$$\chi \geq \begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. For all i either $|P_i| \leq n + 1$ or $|P_i| \leq n + 2$ according as n is odd or even by Bose's theorem. Consequently, when n is odd, $\chi \geq n^2/(n + 1) = n - 1 + 1/(n + 1)$, implying $\chi \geq n$ as shown in [5], while $\chi \geq \lceil n^2/(n + 2) \rceil = n - 1$ when n is even. □

The next two constructions illustrate that Theorem 6.1 is best possible for affine coordinate planes.

Theorem 6.2. *If $A = AG(2, q)$ then A can be partitioned into q conics.*

Proof. Let $P_i = \{(x, y): y = x^2 + i\}$. Then $\{P_1, \ldots, P_q\}$ is a satisfactory partition. □

We remark that if we partition $PG(2, q)$ as in the proof of Theorem 5.2, then the removal of any line incident with $(0, 0, 1)$ also induces a good partition of $AG(2, q)$.

Theorem 6.3. *If q is even then the affine plane $AG(2, q)$ decomposes into $q - 1$ arcs.*

Proof. Let C be a (proper) conic of $PG(2, q)$. As q is even, it is well-known [1] that the tangents of C are concurrent. This common point p of the tangents is called the nucleus of C. Then $C \cup \{p\}$ is an arc with $q + 2$ points. To prove 6.3 it suffices to demonstrate the existence of a partition $\{P_0, P_1, \ldots, P_{q-1}\}$ of $PG(2, q)$ such that P_0 is a conic with its nucleus, P_1 is a line and $P_2, \ldots, P_{q-1}$ are nondegenerate conics.

We now proceed with the construction in $PG(2, q)$. This is similar to the case of q even (case (iii)) in 5.3. We again use the decomposition lemma (5.4). Let

$$f(x, y, z) = z^2 + xy$$

and

$$g(x, y, z) = x^2 + xy + by^2$$

where $b \in \mathbf{F}_q$ is such that

$$x^2 + x + b \neq 0$$

in $\mathbf{F}_q$. By Lemma 5.4 $\{Z(f + cg): c \in \mathbf{F}_q\}$ is a partition of $PG(2, q) - \{(0, 0, 1)\}$.

We claim that the three points $(0, 1, \sqrt{cb})$, $(1, 0, \sqrt{c})$, and $(1, 1, \sqrt{1 + cb})$ are noncollinear points of $Z(f + cg)$ except when $c = 1$. Direct substitution verifies that these points indeed belong to $Z(f + cg)$. We find that

$$\det \begin{pmatrix} 0 & 1 & \sqrt{cb} \\ 1 & 0 & \sqrt{c} \\ 1 & 1 & \sqrt{1 + cb} \end{pmatrix} = \sqrt{c} + \sqrt{cb} + \sqrt{1 + cb} = \sqrt{c} + 1.$$

Hence $\sqrt{c} + 1 \neq 0$ unless $c = 1$. Looking at this case we find $f(x, y, z) + g(x, y, z) = z^2 + xy + x^2 + xy + by^2 = (x + y + \sqrt{b}z)^2$, so that $Z(f + g)$ is a line. As in 5.3 (iii) $|Z(f + cg)| \geq q + 1$ and equality holds. It follows that $Z(f + cg)$ is a non-degenerate conic whenever $c \neq 1$. Finally observe that the lines $x = 0$ and $y = 0$ are tangents to $Z(f)$ at $(0, 1, 0)$ and $(1, 0, 0)$, respectively, and intersect at the point $(0, 0, 1)$. This point then is the nucleus of the conic $Z(f)$ and we can let $P_0 = Z(f) \cup \{(0, 0, 1)\}$. □

7. Related Results, Open Problems

The following theorem was proved in [7]: Let $P = P_1 \cup \cdots \cup P_\chi$ be a coloration of the nontrivial (i.e., $|\mathbb{L}| > 1$) incidence structure $(P, \mathbb{L})$, $|P_1| \leq |P_2| \leq \cdots \leq |P_\chi|$. Then the number of multicolored lines (i.e., the lines having at least two colours) is at least $|P_1| + |P_2| + \cdots + |P_{\chi-1}|$. This is a generalization of the deBruijn-Erdös theorem ($|P_1| = \cdots = |P_\chi| = 1$) [2]. The case $|P_1| = \cdots = |P_\chi|$ was proved by Meshulam [10].

Almost all of our results are not known for non-desarguesian planes. Even more generally, we can introduce the function $\chi(I, m, d) =: \max\{\chi$: there exists a χ-colouring of the incidence structure I with multiplicity $\leq m$, and deficiency $\leq d\}$.

Can we say something non-trivial (more then Lemma 2.1) about $\chi(I, m, d)$?

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References

1. Bose, R.C.: Mathematical theory of the symmetrical factorial design. Sankhyā **8**, 107–166 (1947)
2. deBruijin, N.G., Erdös, P.: On a combinatorial problem. Indagationes Math. **10**, 421–423 (1948)
3. Bruck, R.H.: Quadric extensions of cyclic planes. Proc. Symp. Appl. Math. **X**, 15–44. (1960)
4. Boros, E., Szönyi, T.: On the sharpness of a theorem of B. Segre, Combinatorica (to appear)
5. Csima, J.: Colouring finite planes of odd order. Discrete Math. **49**, 309 (1984)
6. Erdös, P., Sós, V.T.: unpublished.
7. Füredi, Z., Rosenberg, I.G.: Multicolored lines in a finite geometry. Discrete Math., submitted.
8. Kabell, J.A.: A note on colorings of finite planes. Discrete Math. **44**, 319–320 (1983)
9. Lidl, R., Niederreider, H.: Finite Fields. pp. 278–289. Reading: Addison-Wesley 1983
10. Meshulam, R.: On multicolored lines in a linear space. J. Comb. Theory (A) **40**, 150–155 (1985)
11. Singer, J.: A theorem in finite projective geometry and some applications to number theory. Trans. Amer. Math. Soc. **43**, 377–385 (1938)

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