Discrete Mathematics 62 (1986) 225–243
North-Holland

FORBIDDEN SUBMATRICES

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Received 19 February 1985
Revised 10 February 1986

We consider a $m \times n$ (0, 1)-matrix A, no repeated columns, which has no $k \times l$ submatrix F. We may deduce bounds on n, polynomial in m, depending on F. The best general bound is $O(m^{2k-1})$. We improve this and provide best possible bounds for $k \times 1$ F's and certain $k \times 2$ F's. In the case that all columns of F are the same, good bounds are obtained which are best possible for $l = 2$ and some other cases. Good bounds for $1 \times l$ F's are provided, namely $n \leq (l-1)m + 1$, which are shown to be best possible for $F = [1010\ldots10]$. The paper finishes with a study of the 14 different 3×2 possibilities for F, solving all but 3.

1. Introduction

Our results concern the following general situation. Let a matrix be *simple* if it is a (0, 1)-matrix with no repeated columns. Let A be a simple $m \times n$ matrix and let F be a $k \times l$ (0, 1)-matrix. Assume A does not have a submatrix F (i.e., F is a 'forbidden' submatrix). A number of questions can be posed. In this paper, we consider how n is bounded in terms of m. The bound will be a polynomial in m, depending on F. Sauer's bound is the starting point [8, 9]. Let P_k be a $k \times 2^k$ (0, 1)-matrix consisting of all possible columns on k rows.

Theorem 1.1 ([8, 9]). *Let A be an $m \times n$ simple matrix with no submatrix being a row (and column) permutation of P_k. Then*

$$n \leq \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}. \tag{1.1}$$

One can obtain a polynomial bound (in m) on n in the case of forbidden submatrices using Sauer's bound [1]. It was noted in [3] that the pigeonhole principle and Sauer's bound yield the following, our best general bound.

* Support provided by NSERC Grant U0236. The bulk of the research was done while the first author was at the University of Waterloo.

0012-365X/86/$3.50

Theorem 1.2. *Let A be an $m\times n$ simple matrix and F a $k\times l$ $(0,1)$-matrix. Assume A does not have a submatrix F. Then*

$$n\leqslant\left((l-1)\binom{m}{k}+1\right)\left(\binom{m}{k-1}+\binom{m}{k-2}+\cdots+\binom{m}{0}+1\right)-1. \tag{1.2}$$

i.e., n is $O(m^{2k-1})$.

Proof. If n exceeds the bound of (1.2), then A contains $(l-1)\binom{m}{k}+1$ copies of P_k, each successive one entirely to the right of its predecessors. But then, by the pigeonhole principle, there are at least l such copies of P_k in the same set of l rows. From the ith copy, we can select the ith column of F and so produce the forbidden submatrix. □

We conjecture, on the basis of our experience, that for each $k\times l$ matrix F, there is a constant c_F so that (1.2) may be replaced by

$$n\leqslant c_F m^k. \tag{1.3}$$

This paper endeavors to improve on the bound (1.2) as well as finding some best possible bounds for some F. In particular, Section 2 provides best possible bound for a $k\times 1$ matrix F, namely the Sauer bound for P_k. For a large class of $k\times 2$ matrices F, we show that the Sauer bound for P_{k+1} is best possible. The constructions are the critical contribution.

Section 3 studies the case of F being a $k\times l$ matrix $[\alpha\alpha\ldots\alpha]$ $(l\geqslant 2)$, i.e., all columns of F are identical. If α is all 1's, then

$$n\leqslant\binom{m}{k-1}+\binom{m}{k-2}+\cdots+\binom{m}{0}+\binom{m}{k}+\frac{l-2}{k+1}\binom{m}{k}, \tag{1.4}$$

and this is shown to be asymptotically best possible, for $l\leqslant k+2$, using a result of Rödl. In the case of arbitrary α, we need to determine t, the number of blocks of size 2 when α^{T} is decomposed into a minimum number of blocks chosen from $\{0,1,01,10\}$. Then the bound is

$$n\leqslant\binom{m}{k-1}+\binom{m}{k-2}+\cdots+\binom{m}{0}+(l-1)\binom{m-t}{k-t}. \tag{1.5}$$

This is shown to be best possible for $l=2$ and for all l when $\alpha^{\mathrm{T}}=(10)$. Section 4 studies the case that F is a $1\times l$ matrix. Then

$$n\leqslant(l-1)m+1, \tag{1.6}$$

and this bound is shown to be best possible (or within 1) for $F=[1010\ldots]$. Section 5 applies our methods to 3×2 forbidden submatrices. A result of Frankl, Füredi and Pach [3], is used for two cases.

We have only concentrated on one question in looking at matrices with forbidden submatrices. We wish to point out a tantalizing direction for further research. Let A be a simple $m \times n$ matrix with no $k \times l$ submatrix F. Let A be the constraint matrix of a linear programme (LP). Then the ellipsoid algorthm would solve the LP in polynomial time in m and the number of bits of the objective function and the right hand side. The question is how to use the forbidden submatrix F directly to derive a polynomial algorithm. The case for

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.7}$$

has been solved by Farber [2], Hoffman, Kolen and Sakarovitch [6].

We will use a bar to denote (0, 1)-complementation, i.e., $\bar{0} = 1$, $\bar{1} = 0$.

2. Some best possible constructions

The following construction technique proves useful. Let $(A_i: i = 1, 2, \ldots)$ be a family of simple matrices where A_i has i rows (for $i = 1, 2, \ldots.$). We construct a new family of matrices $(f(A)_i: i = 1, 2, \ldots)$ as follows. Let $f(A)_1 = [0, 1]$ and $m > 1$ let

$$f(A)_m = \left[\begin{array}{cccccc} 0 & 0 & & & & 1\ 1 \ldots 1 \\ 0 & 0 & & 0 & \cdot^{\cdot^{\cdot}} & \\ \vdots & \vdots & & & & A_{m-1} \\ & & & 1\ 1 \ldots 1 & & \\ 0 & 0 & 1\ 1 & A_2 & & \\ 0 & 1 & A_1 & & & \end{array}\right] \tag{2.1}$$

Lemma 2.1. *The matrix $f(A)_m$ is simple.*

The construction can now be repeated on the new family of matrices. There are two easy ways to introduce forbidden matrices into the construction.

Lemma 2.2. *Let $(A_i: i = 1, 2, \ldots)$ be a family of simple matrices where A_i has i rows and has no submatrix $\left[\begin{smallmatrix} 0 \\ \alpha \end{smallmatrix}\right]$ where α is a $k \times 1$ (0, 1)-column. Then $f(A)_m$ has no submatrices*

$$B_\beta = \begin{bmatrix} 1 & 0 \\ \beta & \alpha \end{bmatrix}, \qquad \gamma = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix}, \tag{2.2}$$

where β is any (0, 1)-column of k rows.

Proof. Let m be the smallest index for which $f(A)_m$ has B_β as a submatrix. We note that B_β cannot be chosen using the first row in (2.1) since that row has no submatrix [10]. Also, by hypothesis, A_{m-1} has no submatrix $[{0 \atop \alpha}]$ and so B_β does not use the final columns of (2.1) containing A_{m-1}. But then B_β is contained in $f(A)_{m-1}$, the submatrix of $f(A)_m$ consisting of all but the first row and the final columns containing A_{m-1}. This contradicts our choice of m, so $f(A)_m$ does not have B_β as a submatrix.

Let m be the smallest index for which $f(A)_m$ has γ as a submatrix. Certainly γ does not appear using the first row or final columns containing A_{m-1}. As above, γ is then contained in $f(A)_{m-1}$, in contradiction to our choice of m. Thus $f(A)_m$ does not have γ as a submatrix. □

Lemma 2.3. *Let* $(A_i\colon i = 1, 2, \ldots)$ *be a family of simple* (0, 1)*-matrices, where* A_i *has* i *rows and no submatrix* $[C\alpha]$ *or* $[{1 \atop \alpha}]$ *where* α *is a* $k \times 1$ (0, 1)*-column and* C *is a* (0, 1)*-matrix of* k *rows. Then* $f(A)_m$ *has no submatrices*

$$B = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ & C & & \alpha \end{bmatrix}, \qquad \delta = \begin{bmatrix} 1 \\ 1 \\ \alpha \end{bmatrix}. \tag{2.3}$$

Proof. Let m be the smallest index for which $f(A)_m$ has B as a submatrix. Then since A_{m-1} has no submatrix $[C\alpha]$ (all but the first row of B) we deduce that B does not use the first row or final columns containing A_{m-1} in (2.1). Thus B appears in $f(A)_{m-1}$, contradicting our choice of m, and so $f(A)_m$ does not have a submatrix B.

Let m be the smallest index for which $f(A)_m$ has δ as a submatrix. Since A_{m-1} has no submatrix $[{1 \atop \alpha}]$ (all but the first row of δ) we deduce that δ does not use the final columns containing A_{m-1} in (2.1). Since δ starts with a 1, it does not use the first row. Thus δ appears in $f(A)_{m-1}$, contradicting our choice of m, and so $f(A)_m$ does not have a submatrix δ. □

We may apply these constructions to certain forbidden submatrices.

Theorem 2.4. *Let* A *be a simple* $m \times n$ *matrix with no* $k \times 1$ *submatrix* α. *Then*

$$n \leqslant \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}, \tag{2.4}$$

and there are matrices for which equality in (2.4) *holds.*

Proof. Note that a row permutation of P_k (defined in the introduction) is merely a column permutation. Thus α is a submatrix of any row and column permutation of P_k and so A has no submatrix which is a row and column permutation of P_k. Thus Sauer's bound yields (2.4).

To construct a matrix with equality holding in (2.4), let $1(A)_m$ denote $f(A_m)$

and let $0(A)_m$ denote the matrix derived from (2.1) by interchanging the roles of 0 and 1 (leaving the A_i's fixed). Let $\alpha^T = (a_1 a_2 \ldots a_k)$. Let A_i be the $i \times 1$ matrix of all $\bar{a}_k$'s. Then A_i has no submatrix $[a_k]$. We deduce, using Lemma 2.3, that the matrix

$$a_1(a_2(a_3(\ldots a_{k-1}(A)\ldots)))_m \tag{2.5}$$

is a simple $m \times n$ matrix with no submatrix α. Equality holds in (2.4) as desired using the formulas (2.10) and (2.11). □

A matrix, for which equality holds in (2.4), is seen to consist of all columns with no submatrix α and so is unique up to a column permutation.

Certain $k \times 2$ matrices are covered by our constructions. Define $B(p, q, \beta)$ as

$$B(p, q, \beta) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ \vdots & \vdots \\ \beta & \beta \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\ \vdots\\1\end{matrix}}\right\} p \\ \left.\vphantom{\begin{matrix}1\\0\\1\\ \vdots\end{matrix}}\right\} q \\ \} r \end{matrix}, \tag{2.6}$$

where β is a $r \times 1$ (0, 1)-column (possibly $r = 0$), $p + q + r = k$, and we require $q > 0$.

Theorem 2.5. *Let a be a simple $m \times n$ matrix with no $k \times 2$ submatrix $B(p, q, \beta)$ for given p, q, β. Then*

$$n \leq \binom{m}{k} + \binom{m}{k-1} + \cdots + \binom{m}{0}, \tag{2.7}$$

and there are matrices for which equality in (2.7) *holds.*

Proof. Consider the $(k+1) \times 2$ matrix C obtained from $B(p, q, \beta)$ by inserting a row [01] after the first p rows. Any column permutation of C contains $B(p, q, \beta)$ as a submatrix and so any row or column permutation of P_{k+1} contains $B(p, q, \beta)$ as a submatrix. Thus Sauer's bound yields (2.7).

Let α be the $(k-p-1) \times 1$ submatrix of $B(p, q, \beta)$ consisting of the second column and all but the first $p+1$ rows. By Theorem 2.4, we know there is a family of simple matrices $(A_i: i = 1, 2, \ldots)$ where A_i is of size i by

$$\binom{i}{k-p-1} + \binom{i}{k-p-2} + \cdots + \binom{i}{0}, \tag{2.8}$$

and does not have $\left[\begin{smallmatrix}0\\ \alpha\end{smallmatrix}\right]$ as a submatrix. Then by Lemma 2.2, $f(A)_m$ does not have $B(0, q, \beta)$ or

$$\begin{bmatrix} 1 \\ 0 \\ \alpha \end{bmatrix} \tag{2.9}$$

as submatrices. The number of columns of $f(A)_m$ is

$$2 + \sum_{i=1}^{m-1} \left(\binom{i}{k-p-1} + \binom{i}{k-p-2} + \cdots + \binom{i}{0} \right)$$
$$= \binom{m}{k-p} + \binom{m}{k-p-1} + \cdots + \binom{m}{0}, \tag{2.10}$$

using the identity

$$\sum_{i=j}^{m-1} \binom{i}{j} = \binom{m}{j+1}. \tag{2.11}$$

We now apply Lemma 2.3 repeatedly to deduce that $f^{p+1}(A)_m$ has no submatrix $B(p, q, \beta)$ and computations similar to (2.10) verify that $f^{p+1}(A)_m$ is a simple $m \times n$ matrix with equality holding in (2.7). □

Taking (0, 1)-complements, reversing the row or column order, and/or specializing the values of p, q, β yield a host of forbidden submatrix theorems. Note that we require $q > 0$. The case $q = 0$ is handled in the next section in Theorem 3.5.

3. Forbidden submatrix of repeated columns

Let α be a $k \times 1$ (0, 1)-column with $\alpha^T = (a_1 a_2 \ldots a_k)$. Let A be a simple $m \times n$ matrix with no $k \times l$ submatrix

$$F = [\alpha\alpha \ldots \alpha]. \tag{3.1}$$

Note that any column permutation of A will not have a submatrix F, so column order is unimportant.

Let us start with the simplest case that α is all 1's. The problem is a design 'packing' problem essentially solved by Rödl [7]. The case $l = 1$ has already been solved. We easily deduce, that for $l \geq 2$, we may asssume that A has all possible columns of column sum at most k if we are trying to maximize n.

Consider the remaining columns as subsets of an m-set where the rows index the elements of the m-set.

Lemma 3.1. *Let A be a simple $m \times n$ matrix with no $k \times l$ submatrix of* 1's ($l \geq 2$).

Then

$$n \leq \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} + \binom{m}{k} + \frac{l-2}{k+1}\binom{m}{k} \tag{3.2}$$

Proof. The bound follows from a set of size more than k having at least $k+1$ subsets of size k. Thus if there are more than $\binom{m}{k}(l-2)/(k+1)$ such sets, then there is at least one k-set contained in $l-1$ sets of size more than k. The k-set itself completes the forbidden submatrix. □

Let l be fixed. Restrict ourselves to $l \leq k+3$. Let $S(m, k, t)$ be the maximum size of a family F of $k+1$ subsets of an m-set such that any k-subset of the m-set is included in at most t members of F. Rödl's theorem guarantees that $S(m, k, t)$ will be close to the obvious bound.

Theorem 3.2 (Rödl [7]). *Let k, b be fixed. Then there exists a family F of b-subsets of an m-set X so that each k-subsets of X is included in at most one member of F and*

$$|F| \geq \frac{\binom{m}{k}}{\binom{b}{k}}(1 - o(1)). \tag{3.3}$$

We deduce that $S(m, k, 1) \geq \binom{m}{k}(1-o(1))/(k+1)$ and we may extend this to $S(m, k, t)$.

Theorem 3.3. *Let l, k be fixed. For $2 \leq l \leq k+3$, there exists a simple $m \times n$ matrix A with no $k \times l$ submatrix of* 1*'s, where*

$$n = \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} + \binom{m}{k} + \frac{l-2}{k+1}\binom{m}{k}(1 - o(1)). \tag{3.4}$$

Proof. An easy consequence of Rödl's theorem is that for every fixed k, b and t, there exists a family F of b-subsets of an m-set X so that each k-subset of X is included in at most l members of F and

$$|F| \geq \frac{l\binom{m}{k}}{\binom{b}{k}}(1 - o(1)),$$

whenever $m \to \infty$. (This is not claimed in [7] but follows from Rödl's methods.

Proof. Let F_1 be a family of b-subsets of X as defined in (3.3). Let $\pi_1, \pi_2, \ldots, \pi_l$

be random permutations of X and let $F = \{\pi_i F_1 \mid 1 \leq i \leq l\}$ where $\pi_i F_1$ is the set of images of the sets of F_1 under π_i. Because $|F_1| < \binom{m}{k}$, we have that the mean value

$$E(\pi_i F_1 \cap \pi_j F_1) \leq \frac{\binom{m}{k}^2}{\binom{m}{b}} < \binom{m}{k-1}.$$

This gives $|F| \geq l\,|F_1| - \binom{l}{2} m^{k-1}$.)

We now use F with l replaced by $l-2$ and b replaced by $k+1$ to form a simple matrix A whose columns are the characteristic vectors of the sets of F and all columns of at most k 1's. Thus A satisfies (3.4). □

The case of arbitrary $\alpha^{\mathrm{T}} = (a_1 a_2 \ldots a_k)$ in (3.1) is less obvious. The sequence of 0's and 1's in α is important . Let $b_1 b_2 \ldots b_{k-t}$ be the *decomposition* of α if it is determined by the following recursive algorithm which will output the blocks b_i where $b_i \in \{0, 1, 01, 10\}$ and $b_1 b_2 \ldots b_{k-t} = \alpha^{\mathrm{T}}$. The algorithm is a greedy approach to obtain the most number of blocks of size 2. Our bound, improving on (1.2), will involve t, the number of blocks of size 2.

Algorithm Decompose (α^{T}, k)

Input $\alpha^{\mathrm{T}} = a_1 a_2 \ldots a_k$.
If $k = 1$, then output 'a_1', STOP.
If $k > 1$ and $a_1 = a_2$, then output 'a_1' and Decompose $((a_2 a_3 \ldots a_k), k-1)$.
If $k = 2$ and $a_1 \neq a_2$, then output '$a_1 a_2$', STOP.
If $k > 2$ and $a_1 \neq a_2$, then output '$a_1 a_2$' and Decompose $((a_3 a_4 \ldots a_k), k-2)$.
STOP.

Theorem 3.4. *Let A be a simple $m \times n$ matrix with no $k \times l$ submatrix $F = [\alpha\alpha \ldots \alpha]$, where the decomposition of α is $b_1 b_2 \ldots b_{k-t}$ (i.e., t blocks of size 2). Then*

$$n \leq \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} + (l-1)\binom{m-t}{k-t}. \tag{3.5}$$

Proof. Theorem 2.4 ensures that there are precisely $\binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}$ columns with no submatrix α. We say that γ has α as a *special* submatrix if the entries of a block of size 2 of α (from its decomposition) are chosen from consecutive rows of γ. Note that if γ has a submatrix α then it has a special submatrix α. This follows from the simple observation that if $(c_1 c_2 \ldots c_i)$ is a $(0,1)$-row with $c_1 = 0$ and $c_i = 1$, then there is an index j $(1 \leq j \leq i)$ with $c_j = 0$ and $c_{j+1} = 1$. There are only $\binom{m-t}{k-t}$ choices for the k rows of α to be a special submatrix of an $m \times 1$ column. Thus if B is a $m \times n$ matrix with each column having a submatrix α and $n > (l-1)\binom{m-t}{k-t}$, then by the pigeonhole principle, B has F as a submatrix. This proves (3.5). $\square$

Theorems 3.3 and 3.4 provide some evidence for the conjectured bound (1.3). The bound (3.5) need not be best possible as Theorem 3.3 points out. The nicest result of this section is that we have the exact answer for $l = 2$.

Theorem 3.5. *Let A be a simple $m \times n$ matrix with no $k \times 2$ submatrix $[\alpha\alpha]$, where the decomposition of α has t blocks of size 2. Then*

$$n \leq \binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0} + \binom{m-t}{k-t}, \tag{3.6}$$

and there exist simple matrices A for which equality in (3.6) *holds.*

Proof. The bound is from Theorem 3.4. We will provide an inductive construction of some matrices achieving equality in (3.6), with the induction on the number of blocks of α.

Our inductive hypothesis for α, where α is $i \times 1$ and the decomposition of α has $i - s$ blocks of which s are blocks of size 2, is the existence of simple $j \times \binom{j-s}{i-s}$ matrices A_j for $j \geq i$, so that each column of A_j has a submatrix α and yet A_j has no submatrix $[\alpha\alpha]$. In addition, whenever α appears as a submatrix of a column of A_j, the entries above the first entry of α are all $\bar{a}_1$'s where a_1 is the first entry of α (i.e., the first entry of α is chosen as the first available a_1). We do not concern ourselves with special submatrices.

The base of induction is easy to verify. For $\alpha^{\mathrm{T}} = b_1 = 1$, let A_j be the $j \times \binom{j}{1}$ simple matrix

$$A_j = \begin{bmatrix} 0 & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ 1 & & & 0 \end{bmatrix} \tag{3.7}$$

and for $\alpha^{\mathrm{T}} = b_1 = [10]$, let A_j be the $j \times \binom{j-1}{1}$ simple matrix

$$A_j = \begin{bmatrix} 0 & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & 0 \\ & 1 & \cdot^{\cdot^{\cdot}} & \\ 1 & 0 & & \\ 0 & & & 0 \end{bmatrix} \tag{3.8}$$

The remaining two cases are handled by taking (0, 1)-complements.

Assume the inductive hypothesis works for columns α with fewer than $k - t$

blocks. Let $\alpha^{\mathrm{T}} = (a_1 a_2 \ldots a_k)$ have a decomposition with $k-t$ blocks $b_1 b_2 \ldots b_{k-t}$.

Case 1. $b_1 = a_1$ (thus $a_1 = a_2$).

Let $\beta = (a_2 a_3 \ldots a_k)$. The decomposition of β is $b_2 b_3 \ldots b_{k-t}$. By our inductive hypothesis, there are simple $j \times \binom{j-t}{k-1-t}$ matrices B_j as described, each column of which has a submatrix β. For $m \geqslant k$, let

$$A_m = \left[\begin{array}{ccc} & \bar{a}_1\text{'s} & \cdot\ a_1 a_1 \ldots a_1 \\ & a_1 a_1 \ldots a_1 \ \cdot & \\ a_1 a_1 \ldots a_1 & & B_{m-1} \\ B_{k-1} & B_k & \end{array}\right] \tag{3.9}$$

We deduce that A_m is a simple $m \times \binom{m-t}{k-t}$ matrix using the formula of (2.11) to see that

$$\sum_{i=k-1}^{m-1} \binom{i-t}{k-1-t} = \binom{m-t}{k-t}. \tag{3.10}$$

We find that every column of A_m has a submatrix α. Assume α occurs as a submatrix of some column in A_m for which the first entry of α is not chosen as the first available a_1. Then some column of a B_i has a submatrix α. But then β appears as a submatrix of that column not using the first available a_2, the first entry of β. This contradicts our inductive hypothesis.

If $[\alpha\alpha]$ is a submatrix of A_m, then we deduce that $[\beta\beta]$ is a submatrix of B_i, for some i, contradicting our inductive hypothesis. Thus A_m has the desired properties.

Case 2. $b_1 = a_1 a_2$ (thus $a_1 \neq a_2$).

Let $\beta = (a_3 a_4 \ldots a_k)$. The decomposition of β has $k-1-t$ blocks ($t-1$ of size 2). By our inductive hypothesis, there are simple $j \times \binom{j-(t-1)}{(k-2)-(t-1)}$ matrices B_j as described. Let

$$A_m = \left[\begin{array}{ccc} & \bar{a}_1\text{'s} & \cdot\ a_1 a_1 \ldots a_1 \\ & a_1 a_1 \ldots a_1 \ \cdot & a_2 a_2 \ldots a_2 \\ a_1 a_1 \ldots a_1 & a_2 a_2 \ldots a_2 \ \cdot & \\ a_2 a_2 \ldots a_2 & & B_{m-2} \\ B_{k-2} & B_{k-1} & \end{array}\right] \tag{3.12}$$

As before, using (2.11), we deduce that A_m is a simple $m \times \binom{m-t}{k-t}$ matrix with every column having a submatrix α. Assume α occurs as a submatrix of some column in which the first entry of α, namely a_1, is not the first available a_1. Then α is a submatrix of some column of some B_i. Now β can be chosen as a submatrix of that column (from the rows chosen for α) not using the first available a_3

whether it be a_1 or a_2. This contradicts our hypothesis. As before, this implies that A_m has no submatrix $[\alpha\alpha]$ and so A_m has the desired properties.

By induction, the result is proved. □

The bounds of Theorem 3.4 can sometimes be shown to be exact for larger l. The following result is the best along these lines.

Theorem 3.6. *Let A be a simple $m \times n$ matrix with no $2 \times l$ submatrix $[\alpha\alpha \ldots \alpha]$, where $\alpha^{\mathrm{T}} = (10)$, $l > 1$. Then*

$$n \leqslant lm - (l - 2), \tag{3.12}$$

and there exists a matrix A for which equality in (3.12) *holds for $l \leqslant m$.*

Proof. The bound follows from Theorem 3.4. We will construct a matrix A as follows. Let B_k be the simple $m \times (m - 1)$ matrix

$$B_k = \left[\begin{array}{ccccccccc}
1 & & & & & & & & \\
0 & 1 & \ddots & & & 0 & & & \\
0 & 0 & & & & & & & \\
\cdot & 0 & \ddots & & & & & & \\
\cdot & \cdot & & \ddots & & & & & \\
\cdot & \cdot & & & & & & & \\
0 & \cdot & & 1 & 1 & 1 & \cdots & 1 & 1 \\
 & 0 & & 0 & 1 & 1 & \cdots & 1 & 1 \\
 & & & 0 & 0 & 1 & & 1 & 1 \\
 & & & 0 & 1 & 0 & & 1 & 1 \\
 & & \ddots & \cdot & \cdot & \cdot & & \cdot & \cdot \\
 & 1 & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
 & & & \cdot & \cdot & \cdot & & \cdot & \cdot \\
 & & & 0 & 1 & 1 & & 0 & 1 \\
 & & & 0 & 1 & 1 & & 1 & 0
\end{array}\right]\left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\}k \tag{3.13}$$

It has $m - k + 1$ columns with a 1 followed by $k - 1$ 0's and with 0's above the 1 and with 1's below the $k - 1$ 0's. B_k also has $k - 2$ columns which have 0's in the first $m - k$ rows, 1's elsewhere with the exception of a single 0 in row j for $j > m - k + 2$. We may check that no column of B_k is a column of B_s for $2 \leqslant k < s \leqslant l$. Also, every column of B_k has a submatrix α. Let $B_k(i, j)$ be the number of columns in B_k with a 1 in row i, a 0 in row j. Then, for $i < j$, we have

$$B_k(i, j) = \begin{cases} 0 & \text{for } j - i \geqslant k, \\ 1 & \text{for } 0 < j - i < k,\ i \neq m - k - 1, \\ 2 & \text{for } i = m - k - 1 < j \end{cases} \tag{3.14}$$

Let C be the $m \times (m + 1)$ simple matrix of all columns without a sumbatrix α

(Theorem 2.4). Let

$$A = [B_2B_3 \dots B_lC]. \tag{3.15}$$

Then A is the desired $m \times n$ matrix with equality holding in (3.12) and without the $k \times l$ submatrix $[a\alpha \dots \alpha]$. □

4. Forbidden rows

Let $\alpha = (a_1a_2 \dots a_l)$ be a $1 \times l$ (0, 1)-row. Let A be a simple $m \times n$ matrix with no submatrix α. Our rough bound of (1.2) becomes

$$n \leq 2(l-1)m + 1. \tag{4.1}$$

To improve this bound, we use more fully the fact that A has distinct columns.

For a given row $\beta = (b_1b_2 \dots b_n)$ of A, we say that there is a *change in position* $(i, i+1)$ of β if $b_i \neq b_{i+1}$. Two changes in A are *adjacent* if they occur in the same row and in positions $(i, i+1)$, $(i+1, i+2)$ for some i. Note that then the entries in the row in columns i and $i+2$ are equal.

Lemma 4.1. *Let A be a simple $m \times n$ matrix. Then A must have $n-1$ nonadjacent changes.*

Proof. We will show something stronger, that the $n-1$ changes can be chosen in different positions. Consider a multigraph G in which the vertices represent the n columns and the edges correspond to changes and are labelled with the row in which they occur. We will show there is a path from 1 to n in which no two incident edges have the same label.

Note that there are no edges (i, j) with $|i-j| \neq 1$ and that there is some edge $(i, i+1)$, for each i, since columns are different. The proof of our claim is by induction on n and is clearly seen to be true for $n = 1, 2$.

Assume the result is true for any matrix having fewer than n columns. Assume that for some i that $(i, i+1)$ has at least 3 edges. Then, by induction, there are paths from 1 to i and from $i+1$ to n as desired. One of the three edges $(i, i+1)$ can be chosen to complete the path from 1 to n as desired.

Thus we may assume G has at most 2 edges joining $(i, i+1)$ for each $i = 1, 2, \dots, n-1$. If, for all i, there are precisely 2, then one can easily find the desired path from 1 to n, in a greedy fashion. Thus the only impediment to finding our path occurs as follows where possibly $i+1 = j$.

Vertices	Labels of edges joining vertices
$i, i+1$	a_1
$i+1, i+2$	a_1, a_2
$i+2, i+3$	a_2, a_3
$\vdots$	$\vdots$
$j-1, j$	a_{j-i}, a_{j-i+1}
$j, j+1$	a_{j-i+1}

(4.2)

We find that each row has an even number of changes in the interval i to $j+1$, which forces columns i and $j+1$ to be equal. This contradiction proves the result. □

Lemma 4.2. *Let A be a $1 \times n$ (0, 1)-row $(b_1 b_2 \ldots b_n)$ with p nonadjacent changes. Then any $1 \times p$ (0, 1)-row α is a submatrix of A.*

Proof. The p nonadjacent changes correspond to $2p$ distinct column indices (i_1, i_1+1), $(i_2, i_2+1), \ldots, (i_p, i_p+1)$ where $i_1+1<i_2$, $i_2+1<i_3, \ldots, i_{p-1}+1 < i_p$, and

$$\{b_{i_1}, b_{i_1+1}\} = \{b_{i_2}, b_{i_2+1}\} = \cdots = \{b_{i_p}, b_{i_p+1}\} = \{0, 1\}. \tag{4.3}$$

Thus if $\alpha = (a_1 a_2 \ldots a_p)$ then a_k occurs either in column i_k or i_k+1, for $k = 1, 2, \ldots, p$, and so α is a submatrix of A. □

Theorem 4.3. *Let A be a simple $m \times n$ matrix with no $1 \times l$ submatrix $\alpha = (a_1 a_2 \ldots a_l)$. Then*

$$n \leqslant (l-1)m + 1. \tag{4.4}$$

Proof. By Lemma 4.1, A has at least $n-1$ nonadjacent changes. For $n > (l-1)m+1$, at least l of these changes will occur in the same row. By Lemma 4.2, that row, and hence A, will have α as a submatrix. This contradiction proves (4.4.). □

We know that the bound (4.4) need not be best possible for certain α. For a $1 \times l$ row $\alpha = [11 \ldots 1]$, Theorem 3.3 shows that for $l \geqslant 2$

$$n \leqslant \tfrac{1}{2} lm + 1. \tag{4.5}$$

The following result shows that (4.4) is best possible sometimes.

Theorem 4.4. *Let A be a simple $m \times n$ matrix with no $1 \times l$ submatrix* [1010...]. *Assume $l > 1$. Then*

$$n \leqslant \begin{cases} (l-1)m + 1 & \text{for } l \text{ even}, \\ (l-1)m & \text{for } l \text{ odd}, \end{cases} \tag{4.6}$$

and there exist matrices for which equality in (4.6) *holds, when $m \geqslant \frac{1}{2}(l+3)$.*

Proof. The bound for l even follows from Theorem 4.3. For l odd, assume that $n = (l-1)m+1$. Then each row of A would have $l-1$ nonadjacent changes. Since each row cannot have the $1 \times l$ submatrix [1010...], then each row can be written

$$0^*(01)1^*(10)0^*(01)1^* \ldots 1^*(10)0^*, \tag{4.7}$$

where a^* denotes any number of a's, possibly none. The parity of l forces the first and last entry to be equal and thus the first and last column of A would be equal. This contradiction proves (4.6).

Using Lemma 4.5 (which follows), we need only construct small examples, for which equality in (4.6) holds, to verify existence for larger m. The 2×4 matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \tag{4.8}$$

has no submatrix [101] and so Lemma 4.5 shows that there exist $m \times 2m$ simple matrices with no submatrix [101] for $m \geq 2$. The 4×16 matrix

$$A_4 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{4.9}$$

has no submatrix [10101] and so Lemma 4.5 shows that there exist $m \times 4m$ simple matrices with no submatrix [10101] for $m \geq 4$. The first 13 columns of A_4 show that there exist $m \times (3m+1)$ simple matrices with no sumbatrix [1010] for $m \leq 4$. After constructing this matrix, we were told by Griggs of its use in the interval number of a graph [5, p. 47].

The theorem is proven if we can show that there exists a $t \times (t(2t-4))$ simple matrix A_t with no $1 \times (2t-3)$ submatrix [1010...1], where all but the last $t-1$ columns is a $t \times (t(2t-5)+1)$ simple matrix with no $1 \times (2t-4)$ submatrix [1010...10]. The matrix A_t will begin with a column of 0's followed by a column with one 1 in the first row and A_t will end with k columns of decreasing column sum from k to 1. We inductively construct A_t, starting from A_4, as follows. Assume A_{k-1} has been constructed. For each column α in A_{k-1}, add an initial zero:

$$[\alpha] \rightarrow \begin{bmatrix} 0 \\ \alpha \end{bmatrix}. \tag{4.10}$$

with the exception that for certain chosen pairs of adjacent columns α_i, α_{i+1} we insert additional columns:

$$[\alpha_i \alpha_{i+1}] \rightarrow \begin{bmatrix} 0 & 1 & 1 & 0 \\ \alpha_i & \alpha_i & \alpha_{i+1} & \alpha_{i+1} \end{bmatrix}. \tag{4.11}$$

A total of $k-3$ disjoint pairs are chosen, one pair consisting of the column of all 1's in A_{k-1} and the preceding column (columns k and $k-1$ from the end) and $k-4$ additional pairs of columns, each column beginning with two 0's and not chosen from the last $k-2$ columns. For $k=5$ the chosen columns are 8, 9, 12, 13. For larger k the existence of these pairs is readily verified in view of (4.10) and the rest of the construction. From this resulting matrix we form A_k by inserting between the first and second column, the $k \times 2k$ matrix

$$\left[\begin{array}{cccccc|ccccccc|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & & \cdot & \cdot & \cdot \\
 & & & & & & & 1 & 1 & 1 & & \cdot & \cdot & \cdot \\
 & & & & & & & & & 1 & & \cdot & \cdot & \cdot \\
 & & & 0 & & & & & 0 & & \ddots & 1 & 0 & 0 \\
 & & & & & & & & & & & 1 & 1 & 0
\end{array}\right] \tag{4.12}$$

We must verify that this yields a simple matrix. Columns in (4.12) begin with a 1 and so we need only check overlap with columns generated in (4.11). The first pair of columns chosen for (4.11) have column sums k, $k-1$ and so do not arise in 4.12 (exept for $k=5$ which may be checked directly). The remaining columns generated in (4.11) have zeros in rows 2 and 3 thus the only possible overlap is with columns 1 and 5 of (4.12). Column 1 is not generated since column 2 of A_{k-1} has a 1 in row 1. Column 5 will not be generated for $k=5$ using the pairs indicated above. For larger k, A_{k-1} has 4 columns with one 1, the second, the last and two other columns with their 1's in the last two rows. Thus column 5 is not generated in (4.11). Hence A_k is a $k \times (k(2k-4))$ simple matrix with no $1 \times 2k-3$ submatrix [1010 . . . 1], where each row has the form (4.7) except the third from last row ending in a 1. We then see that if the last $k-1$ columns are deleted, the resulting $k \times (k(2k-5)+1)$ matrix has no $1 \times (2k-4)$ submatrix [1010 . . . 10]. □

The bound on m is clearly not best possible but a general construction rule for matrices such as A_4 is elusive.

Lemma 4.5. *Let A be a simple $m \times n$ matrix with no $1 \times l$ submatrix* [1010 . . .]. *For $n \geq l$ and any $p \geq 1$, there is a simple matrix B of size $(m+p) \times (n+p(l-1))$ with no $1 \times l$ submatrix* [1010 . . .].

Proof. A construction goes as follows. Let U, D be two $p \times (p+1)$ matrices

$$U = \begin{bmatrix} 0 & & 0 & & 1 \\ 0 & & & \cdot^{\cdot^{\cdot}} & \\ \vdots & & 1 & & \\ 0 & 1 & & & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ & 1 & & 1 & 0 \end{bmatrix} \tag{4.13}$$

Let α_i be the ith column of A. For l even, let

$$B = \left[\begin{array}{c|c|c|c|c|c} U & D & U & \cdots & U & \mathbf{1} \\ \alpha_1\alpha_1\ldots\alpha_1 & \alpha_2\alpha_2\ldots\alpha_2 & \alpha_3\alpha_3\ldots\alpha_3 & & \alpha_{l-1}\alpha_{l-1}\ldots\alpha_{l-1} & \alpha_l\alpha_{l+1}\ldots\alpha_n \end{array}\right], \tag{4.14}$$

($\mathbf{1}$ denotes a block of 1's) and for l odd, let

$$B = \left[\begin{array}{c|c|c|c|c|c} U & D & U & \cdots & D & \mathbf{0} \\ \alpha_1\alpha_1\ldots\alpha_1 & \alpha_2\alpha_2\ldots\alpha_2 & \alpha_3\alpha_3\ldots\alpha_3 & & \alpha_{l-1}\alpha_{l-1}\ldots\alpha_{l-1} & \alpha_l\alpha_{l+1}\ldots\alpha_n \end{array}\right] \tag{4.15}$$

($\mathbf{0}$ denotes a block of 0's). □

5. The 3 × 2 forbidden submatrices

As an example of our results, we consider the possible 3×2 forbidden sumbatrices. There are 14 essentially different 3×2 (0, 1)-matrices (different up to taking (0, 1)-complements and reversing row and/or column order):

$$F_1=\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix},\quad F_2=\begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix},\quad F_3=\begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix},\quad F_4=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix},$$

$$F_5=\begin{bmatrix}1&0\\1&1\\0&0\end{bmatrix},\quad F_6=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix},\quad F_7=\begin{bmatrix}1&1\\1&0\\0&0\end{bmatrix},\quad F_8=\begin{bmatrix}1&1\\0&0\\0&0\end{bmatrix},$$

$$F_9=\begin{bmatrix}0&0\\1&1\\0&0\end{bmatrix},\quad F_{10}=\begin{bmatrix}1&0\\1&0\\1&0\end{bmatrix},\quad F_{11}=\begin{bmatrix}1&0\\1&0\\0&0\end{bmatrix},\quad F_{12}=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix},$$

$$F_{13}=\begin{bmatrix}1&0\\0&0\\1&0\end{bmatrix},\quad F_{14}=\begin{bmatrix}1&0\\0&1\\0&1\end{bmatrix}. \tag{5.1}$$

Theorem 5.1. *Let F be one of F_1, F_2, F_3, F_4, F_5, F_6, F_7. Let A be a simple $m\times n$ with no submatrix F. Then*

$$n \le \binom{m}{3}+\binom{m}{2}+\binom{m}{1}+\binom{m}{0}, \tag{5.2}$$

and there exists a matrix A with equality in (5.2) holding.

Proof. For F_1, this follows from Theorem 3.3, and for F_2, F_3, F_4, F_5, F_6, F_7, this follows from Theorem 2.5. □

Theorem 5.2. *Let F be one of F_8, F_9. Let A be a simple $m \times n$ matrix with no submatrix F. Then*

$$n \leqslant \binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m-1}{2}, \tag{5.3}$$

and there exists a matrix A with equality in (5.3) *holding.*

Proof. This follows from Theorem 3.5. □

A theorem of Frankl, Füredi, and Pach is required for F_{10}, F_{11}.

Theorem 5.3 (Frankl, Füredi, Pach [3]). *Let A be a simple $m \times n$ matrix with no $k \times 2$ submatrix of a column of* 1*'s followed by a column of* 0*'s. For $k = 2$*

$$n \leqslant \binom{m}{2} + \binom{m}{1} + \binom{m}{0} + m - 2 + \delta_{1m}, \tag{5.4}$$

(*where $\delta_{1m} = 1$ for $m = 1$ and* 0 *otherwise*) *and there exist matrices for which equality holds. For $k > 2$*

$$n \leqslant \binom{m}{k} + O(m^{k-1}), \tag{5.5}$$

and there exist matrices A with

$$n = 2\left(\binom{m}{k-1} + \binom{m}{k-2} + \cdots + \binom{m}{0}\right) + \binom{m}{k} - \binom{2k+1}{k}. \tag{5.6}$$

Thus Theorem 5.3 handles F_{10}. For F_{11}, we need the following construction.

Proposition 5.4. *Let F be a $k \times l$ matrix. Assume that there is a constant c_F so that if A is a simple $m \times n$ matrix with no submatrix F, then*

$$n \leqslant c_F m^p. \tag{5.7}$$

Then there is a constant c_H so that if B is a simple $m \times n$ matrix with no $(k+1) \times l$ submatrix

$$H = \begin{bmatrix} F \\ 00\ldots0 \end{bmatrix}, \tag{5.8}$$

then

$$n \leqslant c_H m^{p+1} \tag{5.9}$$

Proof. Consider all columns of B with a 0 in row i and 1's in rows $i+1$, $i+2, \ldots, m$. Then the submatrix formed by these columns and the first $i-1$

rows will have no submatrix F and so there will be at most $c_F(i-1)^p$ such columns for $i>1$. But every column in B is uniquely described as such a column, for some $i>1$, with the exception of a column of 1's or a column of 1's with a 0 in the first row. Thus

$$n \leqslant c_F(m-1)^p + c_F(m-2)^p + \cdots + c_F 1^p + 1 + 1. \tag{5.10}$$

Now the bound of (5.9) follows readily. □

Note that this supports our conjectured general bound (1.3) when $k=p$. Neither the bound (5.9) nor indeed the bound (5.10) need be best possible.

Remark 5.5. If the bound of (5.7) is replaced by the Sauer bound for P_{p+1}, then the bound (5.9) becomes the Sauer bound for P_{p+2}.

Theorem 5.6. *Let A be a simple $m \times n$ matrix with no submatrix F_{11}. Then*

$$n \leqslant \binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0} + \binom{m-2}{2}, \tag{5.11}$$

and there exists a matrix A with equality in (5.11) *holding.*

Proof. Using the bound of Theorem 5.3 for $k=2$ and applying this as in Proposition 5.4 in (5.10), we deduce

$$n \leqslant \sum_{i=1}^{m-1} \left(\binom{i}{2} + \binom{i}{1} + \binom{i}{0} + i - 2 + \delta_{12} \right) + 2, \tag{5.12}$$

which yields (5.11).

A construction which meets this bound is as follows. Let C be the $m \times (\binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0})$ simple matrix obtained by the construction of Lemma 2.2 where $\alpha = (000)^{\mathrm{T}}$ and A_i is as determined in Theorem 2.4. Thus C has no F_{11} or no column $(1000)^{\mathrm{T}}$. Let B be a $m \times \binom{m-2}{2}$ matrix consisting of all columns with no submatrix $(110)^{\mathrm{T}}$ and yet containing $(1000)^{\mathrm{T}}$. Then $A = [BC]$ meets the bound of (5.11) and has no submatrix F_{11}. □

Theorem 5.7. *Let F be one of F_{12}, F_{13}, F_{14}. Let a be a simple $m \times n$ matrix with no submatrix F. Then*

$$n \leqslant \binom{m}{4} + \binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}, \tag{5.13}$$

and there exist matrices A with

$$n = \binom{m}{3} + \binom{m}{2} + \binom{m}{1} + \binom{m}{0}. \tag{5.14}$$

Proof. The bound of (5.13) follows from Sauer's bound. For example, any column permutation of the 5×2 matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

has F_{12} as a submatrix and so for $F = F_{12}$ we deduce A has no P_5.

The existence of the matrices with equality in (5.14) follows from the constructions of Lemma 2.2 applied to matrices without a certain 3×1 matrix obtained in Theorem 2.4. □

Thus we do not have very good solutions for F_{12}, F_{13}, F_{14}, and it may be possible that one of them violates our conjectured bound (1.3).

Acknowledgments

We thank Noga Alon for fruitful discussions concerning Theorem 3.6.

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