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On the Dimensions of Ordered Sets of Bounded Degree

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Abstract. Let P be a partially ordered set. Define $k = k(P) = \max_{p \in P} |\{x \in P : p < x \text{ or } p > x\}|$, i.e., every element is comparable with at most k others. Here it is proven that there exists a constant c ($c < 50$) such that $\dim P < ck(\log k)^2$. This improves an earlier result of Rödl and Trotter ($\dim P \leqslant 2k^2 + 2$). Our proof is nonconstructive, depending in part on Lovász' local lemma.

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1. Introduction and Statement of the Theorem

Let $(P, <)$ be a poset (partially ordered set) not necessarily finite. A permutation L of P is called a *(linear) extension* of P if $x <_P y$ implies $x <_L y$. (Note a permutation $x_1 x_2, \ldots, x_n$ of $[n]$ is regarded as the total order $x_1 < x_2 < \cdots < x_n$.) The *dimension* of P is the minimum number of extensions $L_1, \ldots, L_d$ such that $P = L_1 \cap \cdots \cap L_d$ (i.e., $x <_p y$ iff $x <_{L_i} y$ for all $1 \leqslant i \leqslant d$). Denote by $C(x)$ the set of elements of P comparable with x, i.e., $C(x) = \{y \in P : y < x \text{ or } x < y\}$, and let

$$k = k(P) = \max\{|C(x)| : x \in P\}.$$

Rödl and Trotter [4] proved that

$$\dim P \leqslant 2k^2 + 2. \tag{1.1}$$

Let $f(k) = \max\{\dim P : k(P) \leqslant k\}$. Considering the poset on $2k + 2$ elements whose Hasse diagram is $K_{k+1,k+1}$ with a complete matching removed, one can see that

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$$f(k) \geqslant k+1. \tag{1.2}$$

Our main result is

THEOREM 1.3. *There exists a c* ($c < 50$) *such that* $f(k) < ck(\log k)^2$.

After some preparations the proof is given in Section 5. Section 2 deals with a simple reformulation of the dimension; Section 3 deals with scrambling sets of permutations; and Section 4 contains a pair of results on hypergraph coloring, one elementary and the other a straightforward consequence of Lovász' local lemma.

The basic ingredients of the proof are Lemmas 2.2, 3.3, 4.2 and 4.3. Probably the most efficient way to read the paper is to scan the statements of these four results and then proceed directly to Section 5 to see how they fit together.

Before proceeding, let us mention that what we are really proving in Sections 3–5 is the following result, which may be of independent interest.

THEOREM 1.4. *There is a constant c such that for any n, k and any family* $\mathscr{H}$ *of subsets of size at most k of* $[n]$ *with each member of* $[n]$ *in at most k sets of* $\mathscr{H}$, *there is a set* $\pi_1, \ldots, \pi_d$ *of at most* $ck(\log k)^2$ *permutations of* $[n]$ *with the property that*

for each $X \in \mathscr{H}$ *and* $x \in [n] \setminus X$

$x <_{\pi_i} X$ *for some* i, $1 \leqslant i \leqslant d$.

2. A Reformulation

For our purposes it is convenient to work with the alternate description of dimension given in Lemma 2.2. Both the results below are presumably well known.

Let $(P, <)$ be a poset and π a permutation of the elements of P. If x, y are elements of P such that $x <_P y$ but $y <_\pi x$, we may in some sense bring π closer to being an extension of P by removing x from its position in π and inserting it immediately before y. We call such an operation a *left-shift.* Although a left shift may create new out-of-order pairs, it is easy to see that the process cannot cycle, that is, it is not possible to return to π via a sequence of left-shifts. We thus have

LEMMA 2.1. *Any permutation* π *of the elements of (a finite) P may be turned into a linear extension by a finite sequence of left-shifts.* □

For $x \in P$ let $U(x) = \{y \in P : y \geqslant x\}$. Let $u = \max\{|U(x)| : x \in P\}$.

LEMMA 2.2. *The dimension of P is the least* α *for which there exist permutations* $\pi_1, \ldots, \pi_\alpha$ *of the elements of P such that*

$$y \nless_P x \Rightarrow \text{for some } i,\ x <_{\pi_i} U(y).$$

Proof. Clearly α is a lower bound on the dimension. That it is also an upper bound follows from Lemma 2.1 once one observes that if $x <_\pi U(y)$ then also $x <_{\pi'} U(y)$ for any π' obtained from π by left-shifts. □

A set of permutations as in Lemma 2.2 will be said to be *destroying* (with respect to P). A simple consequence of Lemma 2.2 is

PROPOSITION 2.3. *With u as above,* $\dim P < 2(u+1)(\log |P|) + 1$.

Proof. Let $\pi_1, \ldots, \pi_d$, $d = \lceil 2(u+1)\log |P| \rceil$ be random permutations of P. Then for any $x, y \in P$ and $1 \leqslant i \leqslant d$,

$$\text{Prob}(x <_{\pi_i} U(y)) = \frac{1}{|U(y)|+1} \geqslant \frac{1}{u+1},$$

hence

$$\text{Prob}(\forall i\, x \not<_{\pi_i} U(y)) \leqslant \left(1 - \frac{1}{u+1}\right)^d < \frac{1}{|P|^2}.$$

This proves the proposition, since

$$\begin{aligned}&\text{Prob}(\pi_1, \ldots, \pi_d \text{ not destroying})\\ &\quad = \text{Prob}(\exists x, y : y \not<_P x,\ \forall i\, x \not<_{\pi_i} U(y))\\ &\quad < \binom{|P|}{2} / |P|^2\\ &\quad < 1.\end{aligned}$$

□

3. Scrambling Sets of Permutations

For n a positive integer we set $[n] = \{1, \ldots, n\}$. In this section we are interested in small sets of permutations of $[n]$ which are 'mixing' in the following sense.

DEFINITION 3.1. For $2 \leqslant t \leqslant n$ a set $S = \{\pi_1, \ldots, \pi_d\}$ of permutations of $[n]$ is *t-scrambling* if for each t-subset X of $[n]$ and $x \in X$ there is some $\pi_i \in S$ under which x is the smallest member of X (i.e., $x <_{\pi_i} (X - \{x\})$). The least cardinality of a t-scrambling set of permutations of $[n]$ is denoted $d(n, t)$.

The notion of a t-scramble is due to Dushnik [1] who found a simple formula for $d(n, t)$ when $2[\sqrt{n}] - 1 \leqslant t \leqslant n$. We note in passing that for all n, $d(n, 2) = 2$, and that $d(n, t)$ is nondecreasing in n and t.

For fixed t and large n, A. Hajnal and J. Spencer (see [5]) have shown

$$\log_2 \log_2 n \leqslant d(n, t) \leqslant t 2^t \log_2 \log_2 n. \tag{3.2}$$

In what follows we will need an upper bound on $d(n, t)$ for $t \sim \log n$. In this range (3.2) is not strong enough, but we can use the following simple bound.

LEMMA 3.3. *For all* n, t, $d(n, t) \leqslant t^2 (1 + \log(n/t))$.

The proof is similar to that of Lemma 2.3 (consider $d = [t^2(1 + \log(n/t))]$ random permutations of $[n]$) and we omit it. □

4. Coloring Hypergraphs

The only not quite elementary ingredient in the proof of Theorem 3.3. is the following powerful result of Lovász [2].

LEMMA 4.1 (Lovász' local lemma). *Let G be a graph on $[m]$ with maximum degree d and $A_1, \ldots, A_m$ events defined on some probability space such that for each i,*

$$\mathrm{Prob}(A_i) \leqslant \frac{1}{4d}.$$

Suppose further that each A_i is jointly independent of the events A_j for which $\{i, j\} \notin E(G)$. Then $\mathrm{Prob}(\bar{A}_1 \cdots \bar{A}_m) > 0$.

Recall that a *hypergraph* on a set X is a collection $\mathscr{H}$ of subsets of X. For $x \in X$ the *degree* of x in $\mathscr{H}$, denoted $\deg_{\mathscr{H}}(x)$, is the number of members of $\mathscr{H}$ that contain x. We write $\deg \mathscr{H}$ for $\max\{\deg_{\mathscr{H}}(x) : x \in X\}$. A *coloring* of X by s colors is just a partition $X = X_1 \cup \cdots \cup X_s$ (where $X_i = \phi$ is allowed). We denote by $[X]^{\leqslant a}$ the collection of subsets of size at most a of X.

The proof of Theorem 1.3 will begin by partitioning the set P into a relatively small number of sets $X_1 \cup \cdots \cup X_s$ such that each $|U(x) \cap X_i| \leqslant v$, where v is also small. Lemma 4.1 enables us to do this with s and v on the order of $k/\log k$ and $\log k$, respectively.

LEMMA 4.2. *Let $\mathscr{H}$ be a hypergraph, $\mathscr{H} \subseteq [X]^{\leqslant b}$ with $\deg \mathscr{H} \leqslant b$, where $b \geqslant 500$. Set $s = \lceil b/\log b \rceil$, $v = \lceil 4.7 \log b \rceil$. Then there is a coloring of X, $X = X_1 \cup \cdots \cup X_s$ such that $|H \cap X_i| \leqslant v$ for all $H \in \mathscr{H}$ and $1 \leqslant i \leqslant s$.*

Proof. Let $X_1 \cup \cdots \cup X_s$ be a random partition of X (i.e., for each $x \in X$ and i $\mathrm{Prob}(x \in X_i) = 1/s$ and events corresponding to distinct x's are mutually independent). Denote by $A(H, i)$ the event $|H \cap X_i| > v$ and define the graph G on the index set $\mathscr{H} \times [s]$ by $\{(H, i), (H', i')\} \in E(G)$ if $H \cap H' \neq \phi$. Then G has maximum degree at most $(1 + b(b-1))s$.

On the other hand, since $|H| \leqslant b$,

$$\begin{aligned}
\mathrm{Prob}(|H \cap X_i| > v) &\leqslant \sum_{t > v} \binom{b}{t} \left(\frac{1}{s}\right)^t \left(1 - \frac{1}{s}\right)^{b-t} \\
&< \frac{1}{3} \binom{b}{v} \left(\frac{1}{s}\right)^v \left(1 - \frac{1}{s}\right)^{b-v} \\
&< \frac{1}{3} \left(\frac{be}{vs}\right)^v \frac{1}{\sqrt{2\pi v}} \left(1 - \frac{1}{s}\right)^{b-v} \\
&< \frac{1}{4} \left(\frac{e}{4.5}\right)^{4.5 \log b} b^{-(1-(v/b))} \\
&< \frac{1}{4} b^{-3},
\end{aligned}$$

so Lemma 4.2 follows from Lemma 4.1. □

Finally we require the following fact, whose straightforward inductive proof we omit.

LEMMA 4.3. *Let* a, b *be positive integers and* $\mathscr{H} \subseteq [Y]^{\leqslant a}$ *a hypergraph with* $\deg \mathscr{H} \leqslant b$. *Then there exists a coloring* $Y = Y_1 \cup \cdots \cup Y_n$ *with* $n = (a-1)b + 1$ *such that* $|H \cap Y_i| \leqslant 1$ *for all* $H \in \mathscr{H}$ *and* $1 \leqslant i \leqslant n$.

5. Proof of Theorem 1.3

For $k < 500$ the theorem follows from (1.1), so we assume $k \geqslant 500$. Set $\mathscr{H} = \{U(x) : x \in P\}$. Then Lemma 4.2 applied to $\mathscr{H}$ yields a partion

$$P = X_1 \cup \cdots \cup X_s \quad \left(s = \left\lceil \frac{k+1}{\log(k+1)} \right\rceil\right)$$

such that

$$|U(x) \cap X_i| \leqslant v \quad (v = \lceil 4.7 \log(k+1) \rceil)$$

holds for all x and i.

Set $\mathscr{H}_i = \{U(x) \cap X_i : x \in P\}$. Since $\mathscr{H}_i \subseteq [X_i]^{\leqslant v}$ and $\deg \mathscr{H}_i \leqslant k+1$, Lemma 4.3 implies the existence of a partition $X_i = X_{i1} \cup \cdots \cup X_{in}$, $n = (v-1)(k+1) + 1$, such that $|U(x) \cap X_{ij}| \leqslant 1$ for all x, i, j. With $d = d(n, v+1)$ as defined in 3.1, we have, according to 3.3,

$$d \leqslant (v+1)^2 \left(1 + \log \frac{(v-1)(k+1)+1}{v+1}\right). \tag{5.1}$$

Let $\{\pi_1, \ldots, \pi_d\}$ be a $(v+1)$-scrambling set of permutations of $[n]$. Also, for each i, j, let R_{ij} be a fixed linear ordering of X_{ij}, and R_{ij}^c its converse (i.e., $x <_{R_{ij}^c} y$ iff $y <_{R_{ij}} x$). Finally, for each $i \in [s]$ and $l \in [d]$ we define two permutations $\Pi_{i,l}$ and $\Pi'_{i,l}$ of X:

$$\Pi_{i,l} = (R_{i,\pi_l(1)}, R_{i,\pi_l(2)}, \ldots, R_{i,\pi_l(n)}, X - X_i),$$
$$\Pi'_{i,l} = (R^c_{i,\pi_l(1)}, R^c_{i,\pi_l(2)}, \ldots, R^c_{i,\pi_l(n)}, X - X_i),$$

where in each case the ordering of $X - X_i$ is arbitrary.

We assert that this family of permutations is destroying. To see this let $x, y \in P$ with $y \nleqslant x$, let $x \in X_{i,\alpha}$ and set $T = \{j : U(y) \cap X_{i,j} \neq \phi\}$. Since $|T| \leqslant v$ there is an l for which $\alpha <_{\pi_l} T$ ($\alpha <_{\pi_l} (T - \{\alpha\})$ in case $\alpha \in T$). Then $x <_{\Pi_{i,l}} U(y)$ or $x <_{\Pi'_{i,l}} U(y)$ (or both if $\alpha \notin T$), verifying the assertion. But in view of Lemma 2.2 and (5.1) this gives

$$\dim P \leqslant 2sd < 50\, k (\log k)^2$$

and the theorem is proved. □

6. Final Remarks

(1) As pointed out to us by T. Trotter, the following result of R. J. Kimble shows that it would have been sufficient to consider posets of height 1 (i.e., having no chains of more than two elements).

THEOREM 6.1 ([3]). *Given a poset P, let Q be the poset defined on the set* $P \times \{0, 1\}$ by

$$(p, 0) < (q, 1) \quad \textit{iff } p \leqslant q.$$

Then $\dim P \leqslant \dim Q \leqslant \dim P + 1$.

Of course, $k(Q) \leqslant k(P) + 1$.

(2) If in analogy with the definition of u in Section 2 we set $D(x) = \{y \in P : y \leqslant x\}$ and $d = \max\{|D(x)| : x \in P\}$, then our method yields

THEOREM 6.2. $\dim P \leqslant 40(u + d) \log u \log d$.

(3) *Better bounds:* If it is true that $f(k)/(k \log k) \to \infty$ then Lemma 2.3 shows that $|P|$ is superpolynomial in k, so fairly large examples will be required. The most likely place for improvement of our result is in the bound of Lemma 3.3. But the more intriguing problem is to say anything at all about the lower bound beyond the rather trivial $f(k) \geqslant k + 1$.

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