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Extremal Problems Concerning Kneser Graphs

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Let $\mathscr{A}$ and $\mathscr{B}$ be two intersecting families of k-subsets of an n-element set. It is proven that $|\mathscr{A} \cup \mathscr{B}| \leqslant \binom{n-1}{k-1} + \binom{n-2}{k-1}$ holds for $n > \frac{1}{2}(3+\sqrt{5})\,k$, and equality holds only if there exist two points a, b such that $\{a, b\} \cap F \neq \varnothing$ for all $F \in \mathscr{A} \cup \mathscr{B}$. For $n = 2k + o(\sqrt{k})$ an example showing that in this case $\max|\mathscr{A} \cup \mathscr{B}| = (1 - o(1))\binom{n}{k}$ is given. This disproves an old conjecture of Erdös [7]. In the second part we deal with several generalizations of Kneser's conjecture. © 1986 Academic Press, Inc.

1. INTRODUCTION AND EXAMPLE

Let X be an n-element set. For notational simplicity we suppose $X = \{1, 2, \ldots, n\}$. The family of k-element subsets of X is denoted by $\binom{X}{k}$. A family of sets $\mathscr{F}$ is called *intersecting* if $A \cap B \neq \varnothing$ holds for all $A, B \in \mathscr{F}$.

For $n \geqslant 2k$ the vertex-set of the *Kneser graph* $K(n, k)$ is $\binom{X}{k}$ and two vertices $A, B \in \binom{X}{k}$ are connected by an edge if $A \cap B = \varnothing$. Let $\mathscr{F}_i = \{A \in \binom{X}{k}: \min A = i\}$ for $i = 1, 2, \ldots, n-2k+1$ and $\mathscr{F}_o = \{A \in \binom{X}{k}: A \subset \{n-2k+2, \ldots, n\}\}$. Each $\mathscr{F}_i$ is intersecting so this partition of $\binom{X}{k}$ shows that the chromatic number of the Kneser graph satisfies $\chi(K(n, k)) \leqslant n - 2k + 2$. Kneser [22] conjectured and Lovász [23] proved that here equality holds. Bárány [1] gave a simple proof. Erdös [7] suggested the investigation of the cardinality of colour classes of Kneser graphs, i.e., the cardinality of intersecting families of k-sets, especially the case of two intersecting families.

Let $f_t(n, k)$ denote $\max\{|\bigcup_{1 \leqslant i \leqslant t} \mathscr{F}_i| : \mathscr{F}_i \subset \binom{X}{k},\ \mathscr{F}_i$ is intersecting$\}$.

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Lovász's theorem says that $f_t(n, k) < \binom{n}{k}$ for $t \leqslant n - 2k + 1$. Erdös conjectured that

$$f_t(n, k) = \sum_{1 \leqslant i \leqslant t} \binom{n-i}{k-1} \tag{1}$$

for all $n \geqslant 2k + t - 1$. Equation (1) holds for $t = 1$, for all $n \geqslant 2k$, as was proved by Erdös, Ko, and Rado [8]. For $t \geqslant 2$ this conjecture turned out to be wrong for $n = 2k + t - 1$ as pointed out by Hilton [19] for $k = 3$ and the second author [14] for all k. However, Erdös [6] proved that (1) holds for n large enough. Example 1 shows that (1) can hold only for $n > 2k + t + \sqrt{k}$.

Knowing Hilton's example Erdös [7] made a weaker conjecture $f_2(n, k) < \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}$. Again Example 1 shows that for $n = 2k + o(\sqrt{k})$ we have $f_2(n, k) > \sum_{1 \leqslant i \leqslant t} \binom{n-i}{k-1}$ for any fixed t, if k is large enough.

EXAMPLE 1. Let $n = 2k + 2v$, $v \leqslant k$ and $X = X_1 \cup X_2$, $|X_1| = |X_2| = k + v$. Define $\mathscr{F}_i = \{F \subset X: |F| = k, |F \cap X_i| > (k+v)/2\}$ $(i = 1, 2)$.

Obviously, $\mathscr{F}_i$ is intersecting. Then

$$|\mathscr{F}_1 \cup \mathscr{F}_2| = \binom{n}{k} - \sum_{(k-v)/2 \leqslant i \leqslant (k+v)/2} \binom{n/2}{i}\binom{n/2}{k-i}$$
$$\geqslant \binom{n}{k} - (v+1)\binom{n/2}{k/2}\binom{n/2}{k/2} = \binom{n}{k}(1 - (v+1) \times$$
$$\binom{k}{k/2}\binom{n-k}{(n-k)/2}\Big/\binom{n}{n/2}\Big) \sim \binom{n}{k}(1 - (v+1)\sqrt{n/(n-k)\,k(\pi/2)});$$

Here we used the Stirling formula (see, e.g., [25]) which yields $\binom{x}{x/2} \sim 2^x/\sqrt{x(\pi/2)}$. Here we have

$$|\mathscr{F}_1 \cup \mathscr{F}_2| > \binom{n}{k}(1 - 3v/\sqrt{k}). \tag{2}$$

Now using the equality (see [25]) $\binom{x}{(x-t)/2} \sim \binom{x}{x/2}\exp(-t^2/x)$ we obtain that

$$|\mathscr{F}_1 \cup \mathscr{F}_2| > \sum_{1 \leqslant i \leqslant t} \binom{n-i}{k-1} \tag{3}$$

if $t<\log(k/v^2)$. (Some hints can be found in the end of Section 4.) For $t=2$, more careful calculation shows that for every fixed $c>0$ and for $n=2k+c\sqrt{k}$ we have

$$|\mathscr{F}_1 \cup \mathscr{F}_2| > (1+h(c))\left(\binom{n-1}{k-1}+\binom{n-2}{k-1}\right), \tag{4}$$

where $h(c)>0$ if $k>k_0(c)$. Thus (1) cannot hold for $n-t-2k=0(\sqrt{k})$.

2. Results for Two Intersecting Families

We prove that the Erdös conjecture is essentially true for $n=2k+\Omega(\sqrt{k})$. (Here $a_i=\Omega(b_i)$ means that $b_i/a_i \to 0$ whenever $i\to\infty$.)

Theorem 1. *If $n=2k+c\sqrt{k}$, where $c>0$, and $\mathscr{F}_1, \mathscr{F}_2$ are intersecting families of k-subsets of an n-element set then $|\mathscr{F}_1 \cup \mathscr{F}_2| \leqslant (1+c^{-4})(\binom{n-1}{k-1}+\binom{n-2}{k-1})$.*

The proof of this theorem and the proof of all the results in Sections 2 and 3 are postponed to the last sections of the paper.

Theorem 2. *If $n>\frac{1}{2}(3+\sqrt{5})\,k \sim 2.62k$ then*

$$|\mathscr{F}_1 \cup \mathscr{F}_2| \leqslant \binom{n-1}{k-1}+\binom{n-2}{k-1}. \tag{5}$$

Equality holds iff there exists two elements so that all members of $\mathscr{F}_1$ and $\mathscr{F}_2$ contain at least one of them.

This theorem is an improvement of an earlier result of the second author who proved the statement for $n>6k$ [14].

Similar theorems can be proved for two hypergraphs possessing shifting–stable properties. (cf. Sect. 4) We give 3 examples.

The family $\mathscr{F}\subset\binom{X}{k}$ is *r-intersecting* if $|F\cap F'|\geqslant r$ holds for all $F, F'\in\mathscr{F}$. Erdös *et al.* [8] proved that for $n\geqslant n_0(k,r)$ $|\mathscr{F}|\leqslant\binom{n-r}{k-r}$ holds. Here, if $|\mathscr{F}|=\binom{n-r}{k-r}$ then there exists an r-subset R of X such that $\mathscr{F}=\{F\in\binom{X}{k}: R\subset F\}$. The first author [11] determined the value of $n_0(k,r)=(r+1)(k-r+1)$ for $r\geqslant 15$ and recently Wilson [27] proved that this holds for all r.

Theorem 3. *Let $\mathscr{F}_1\subset\binom{X}{k}$, $\mathscr{F}_2\subset\binom{X}{k}$ be r_1-intersecting and r_2-intersecting families, respectively. If $n\geqslant n_0(k,r_1)+n_0(k,r_2)$ then $|\mathscr{F}_1\cup\mathscr{F}_2|\leqslant \binom{n-r_1}{k-r_1}+\binom{n-r_2}{k-r_2}-\binom{n-r_1-r_2}{k-r_1-r_2}$.*

Here for $n > n_0(k, r_1) + n_0(k, r_2)$ equality holds only if there exist two subsets $R_1, R_2 \subset X$, $|R_i| = r_i$, $R_1 \cap R_2 = \emptyset$ such that $\mathscr{F}_i \subset \{F \in \binom{X}{k}: R_i \subset F\}$.

We say $\mathscr{F} \subset \binom{X}{k}$ is *l-wise intersecting* if $F_1 \cap \cdots \cap F_l \neq \emptyset$ holds for all $F_1, ..., F_l \in \mathscr{F}$. The first author [10] proved that $|\mathscr{F}| \leqslant \binom{n-1}{k-1}$ holds for $n \geqslant [kl/(l-1)] = n_1(k, l)$. Moreover, for $n > n_1$ equality implies $\bigcap \mathscr{F} \neq \emptyset$.

THEOREM 4. *Let $\mathscr{F}_1, \mathscr{F}_2 \subset \binom{X}{k}$ be l_1-wise (l_2-wise) intersecting families, respectively. ($l_1, l_2 \geqslant 2$). If $n \geqslant n_1(k, l_1) + n_1(k, l_2)$ then $|\mathscr{F}_1 \cup \mathscr{F}_2| \leqslant \binom{n-1}{k-1} + \binom{n-2}{k-1}$ holds. Here for $n > n_1(k, l_1) + n_1(k, l_2)$ equality holds only if there exist two elements x_1, x_2 so that $\mathscr{F}_1 \cup \mathscr{F}_2 = \{F \in \binom{X}{k}: F \cap \{x_1, x_2\} \neq \emptyset\}$.*

We say that the matching-number of $\mathscr{F} \subset \binom{X}{k}$ is t if $\mathscr{F}$ does not contain $t+1$ pairwise disjoint (but contains t such) members. The above-mentioned Erdös theorem [6] says that $|\mathscr{F}| \leqslant \sum_{1 \leqslant i \leqslant t} \binom{n-i}{k-1}$ whenever $n \geqslant n_2(k, t)$. Moreover equality holds iff there exists a t-subset T such that $\mathscr{F} = \{F \in \binom{X}{k}: F \cap T \neq \emptyset\}$. Bollobás, Daykin, and Erdös [2] proved $n_2(k, t) < 2k^3 t$.

THEOREM 5. *Let $\mathscr{F}_1, \mathscr{F}_2 \subset \binom{X}{k}$ be such that $\mathscr{F}_i$ does not contain more than t_i pairwise disjoint members ($i = 1, 2$). Then for $n \geqslant n_2(k, t_1) + n_2(k, t_2)$ we have $|\mathscr{F}_1 \cup \mathscr{F}_2| \leqslant \sum_{1 \leqslant i \leqslant t_1 + t_2} \binom{n-i}{k-1}$. Here equality holds only if there exists a $(t_1 + t_2)$-subset T such that $\mathscr{F}_1 \cup \mathscr{F}_2 = \{F \in \binom{X}{k}: F \cap T \neq \emptyset\}$.*

3. Generalizations of Kneser's Conjecture

Theorem 5 is a small step forward verifying the following conjecture of Erdös [30] (also see Gyárfás [17]):

CONJECTURE 1. Let $\mathscr{F}_1 \cup \cdots \cup \mathscr{F}_\chi = \binom{X}{k}$ such that $\mathscr{F}_i$ does not contain more than t pairwise disjoint k-subsets ($1 \leqslant i \leqslant \chi$). Then $n \leqslant (\chi - 1)\, t + (tk + k - 1)$.

This conjecture, if it is true, generalizes Lovász' theorem which is the special case $t = 1$. Let $\chi(K_t(n, k))$ be the minimum value of χ for which such a partition exists. (Here $K_t(n, k)$ means a $(t+1)$-uniform hypergraph $\mathscr{H}$ with vertex-set $\binom{X}{k}$, and a collection $\{F_1, ..., F_{t+1}\} \in \mathscr{H}$ iff $|F_1| = \cdots = |F_{t+1}| = k$ and $F_i \cap F_j = \emptyset$ for all $1 \leqslant i < j \leqslant t+1$.)

Gyárfás [17] observed $\chi \leqslant 1 + (n - tk - k + 1)/t$: Let $\mathscr{F}_i = \{F \in \binom{X}{k}: F \cap [it - t + 1, it] \neq \emptyset\}$ for $1 \leqslant i \leqslant \chi - 1$ and $\mathscr{F}_0 = \mathscr{F} - \bigcup \mathscr{F}_i$. Then $|\bigcup \mathscr{F}_0| \leqslant tk + k - 1$, hence each $\mathscr{F}_i$ contains at most t pairwise disjoint members.

THEOREM 6. (Gyárfás [17]). *Conjecture* 1 *holds for the following cases*:

(1) $k=2$,

(2) $k=3, n\leqslant 5t-2$.

Case (1) follows from a theorem of Cockayne and Lorimer [5].

THEOREM 7. $\chi(K_t(n,k))\geqslant(n/t)-c_k$, *where* c_k *depends only on* k. $(c_k<k^4)$.

Theorems 6 and 7 indicate that Erdös' conjecture is very probably true. Examples analogous to Example 1, show that to give a purely combinatorial proof is unlikely.

Let $n\geqslant k\geqslant r$ be positive integers. Let us denote by $T(n,k,r)$ the minimum size of a family $\mathscr{F}\subset\binom{X}{r}$ such that every k-subset of X has an r-subset that belongs to $\mathscr{F}$.

Define a graph $K(n,k,r)$ with vertex-set $\binom{X}{k}$. Two vertices $A, B\in\binom{X}{k}$ are connected by an edge iff $|A\cap B|<r$. (For $r=1$ we get the usual Kneser graph.)

CONJECTURE 2. [13]. For $r\geqslant 2$ $\chi(K(n,k,r))=T(n,k,r)$ holds for $n>n_3(k,r)$.

This conjecture was proved in [13] for $r=2$. Here we prove Conjecture 2 in a weaker form, as it was mentioned in [13]:

THEOREM 8. *Let* k *and* r *be fixed. Then*

$$\chi(K(n,k,r))=(1+o(1))\,T(n,k,r).$$

An interesting extension of Kneser's conjecture was raised by Stahl [25]. Define for each graph $\mathscr{G}$ and for each natural number l the *l-chromatic number* $\chi_l(\mathscr{G})$ as the minimal number of colours needed to give each vertex of $\mathscr{G}$ l colours such that no colour occurs at two adjacent vertices. Otherwise stated, $\chi_l(\mathscr{G})$ is the minimal number of independent subsets of the vertex-set of $\mathscr{G}$ such that each vertex occurs in at least l of them.

CONJECTURE 3. [25]. $\chi_l(K(n,k))=\lceil l/k\rceil(n-2k)+2l$.

Stahl [26] proved his conjecture using Lovász's theorem for $1\leqslant l\leqslant k$ and also that the right-hand side is always an upper bound for $\chi_l(K(n,k))$. The conjecture was proved for $k=3$, $l=4$ by Garey and Johnson [15]. Further results were proved by Chvátal, Garey, and Johnson [4] and Geller and Stahl [16] (see, e.g., Brouwer and Schrijver [3]).

4. Lemmas for Theorems 1 and 2

Following Erdös, Ko, and Rado [8] we define a shifting operation S_{ij} for all $1 \leqslant i < j \leqslant n$. However, here we apply it to two intersecting families simultaneously. For $\mathscr{F} \in \mathscr{F}_1 \cup \mathscr{F}_2$ let

$$S_{ij}(F) = \begin{cases} F - \{j\} \cup \{i\} & \text{if } j \in F \in \mathscr{F}_1, i \notin F, F - \{j\} \cup \{i\} \notin \mathscr{F}_1, \\ F - \{i\} \cup \{j\} & \text{if } i \in F \in \mathscr{F}_2, j \notin F, F - \{i\} \cup \{j\} \notin \mathscr{F}_2, \\ F & \text{otherwise.} \end{cases}$$

Let $S_{ij}(\mathscr{F}_\alpha) = \{S_{ij}(F): F \in \mathscr{F}_\alpha\}$, $\alpha = 1, 2$.

Lemma 1. *$|S_{ij}(\mathscr{F}_\alpha)| = |\mathscr{F}_\alpha|$, $S_{ij}(\mathscr{F}_\alpha)$ is intersecting if $\mathscr{F}_\alpha$ is intersecting, $S_{ij}(\mathscr{F}_1) \cap S_{ij}(\mathscr{F}_2) = \varnothing$ if $\mathscr{F}_1 \cap \mathscr{F}_2 = \varnothing$.*

Proof. The first two statements were proved in several places, e.g., in [8] and the third one, which is also easy, in [14]. ∎

We call the intersecting family left (right) *stable* if $F \in \mathscr{F}$, $j \in F$, $i \notin F$ implies $F - \{j\} \cup \{i\} \in \mathscr{F}$ for all $1 \leqslant i < j \leqslant n$ (for all $1 \leqslant j < i \leqslant n$).

Lemma 2. *Let $\mathscr{F} \subset \binom{X}{k}$ be a left stable intersecting family, $|X| = n > 2k$. Denote by $\mathscr{F}_0 = \{F \in \mathscr{F}: 1 \notin F)$. Then $\mathscr{F}_0$ is 2-intersecting, i.e., $|F \cap F'| \geqslant 2$ holds for every $F, F' \in \mathscr{F}_0$.*

Proof. Suppose for contradiction that there exist F and F' such that $1 \notin F \cup F'$ and $F \cap F' = \{x\}$. Then $F'' = F' - \{x\} \cup \{1\} \in \mathscr{F}$, $F'' \cap F = \varnothing$ contradicting the intersecting property of $\mathscr{F}$. ∎

Lemma 3. *Let $\mathscr{F} \subset \binom{X}{k}$ be a left stable, intersecting family, $|X| = n > 2k$, $h \geqslant 0$. Denote by $\mathscr{F}^*$ those members F of $\mathscr{F}$ for which $|F \cap \{1,..., 2i\}| \geqslant i$ holds for some $i > h$. Then for all $F, F' \in (\mathscr{F} - \mathscr{F}^*)$ we have $F \cap F' \cap [1, 2h] \neq \varnothing$.*

Proof. Actually, we prove the following stronger statement:

Proposition 1. *Let $\mathscr{F} \subset \binom{X}{k}$ be a left stable, intersecting family on $X = [1, n]$, $A, B \in \mathscr{F}$. Then there exists an i such that $|A \cap [1, i]| + |B \cap [1, i]| > i$.*

Proof. Suppose the contrary and let A and B be a counterexample minimizing $\sum_{a \in A} a + \sum_{b \in B} b$. Set $i = \min A \cap B$. Since $|A \cap [1, i]| + |B \cap [1, i]| \leqslant i$, there exists a $j \in [1, i-1]$ such that $j \notin A \cup B$. Let $A' = A - \{i\} \cup \{j\}$. Now the pair $\{A', B\}$ is a counterexample again but $\sum_{a \in A'} a + \sum_{b \in B} b$ is smaller. This contradiction shows that such a pair does not exist. ∎

Finally, let $F, F' \in \mathscr{F} - \mathscr{F}^*$. By Proposition 1 we have an i such that $|F \cap [1, i]| + |F' \cap [1, i]| > i$. Then $F \cap F' \cap [1, i] \neq \emptyset$, hence we are done with Lemma 3 if $i \leqslant 2h$. But $i > 2h$ implies, e.g., for F, that $|F \cap [1, i]| > i/2$, i.e., $F \in \mathscr{F}^*$, a contradiction. ▮

LEMMA 4. *Let* $|X| = 2k + 2v = n$, $v < (k/4)$. *Let* $\mathscr{A} = \{F \subset X: |F| = k$ *and* $|F \cap [1, 2i]| \geqslant i$ *for some* $i \geqslant (k+v)/2\}$. *Then*

$$|\mathscr{A}| < \binom{n}{k} e^{-v^2/4k} \frac{2k}{v^2}.$$

Proof. Let $i(F) = \min\{i: |F \cap [1, 2i]| \geqslant i)$. Then $|F \cap [1, 2i(F)]| = i(F)$. We use the well-known fact that the number of 0-1 sequences consisting of i 0's and i 1's in which every initial segment contains more 0's then 1's is equal to $\binom{2i}{i}/(2i-1) = A_i$ (see, e.g., [24]). Hence $|\{F \in \binom{X}{k}: i(F) = i\}| = A_i \binom{n-2i}{k-i}$. It is easy to see that

$$\binom{n}{k} \geqslant A_1 \binom{n-2}{k-1} \geqslant A_2 \binom{n-4}{k-2} \geqslant \cdots \geqslant A_i \binom{n-2i}{k-i} \geqslant A_{i+1} \binom{n-2i-2}{k-i-1} \geqslant \cdots.$$

Hence we have

$$A_{k/4} \binom{n-k/2-2}{(3/4)k-1} \leqslant \frac{2}{k} \binom{n}{k}.$$

Moreover the ratio of the $(i+1)$th and the ith member equals to $q_i = 2(2i-1)(k-i)(n-k-i)/(i+1)(n-2i)(n-2i+1)$. Now $q_i < 1 - (n-2k)^2/(n-2i)^2 \leqslant 1 - v^2/k^2$ if $i \geqslant k/4$. This yields

$$\frac{1}{2j-1} \binom{2j}{j} \binom{n-2j}{k-j} = \binom{n}{k} \prod_{0 \leqslant i < j} q_i < \frac{2}{k} \binom{n}{k} \prod_{k/4 < i < j} q_i < \frac{2}{k} \binom{n}{k} \left(1 - \frac{v^2}{k^2}\right)^{j-(k/4)}.$$

Summing up these inequalities for $(k+v)/2 \leqslant i \leqslant k$ we get

$$|\mathscr{A}| \leqslant \frac{2}{k} \binom{n}{k} \sum_{j \geqslant (k+v)/2} \left(1 - \frac{v^2}{k^2}\right)^{j-(k/4)} < \frac{2}{k} \binom{n}{k} \frac{k^2}{v^2} \left(1 - \frac{v^2}{k^2}\right)^{k/4} < \binom{n}{k} \frac{2k}{v^2} e^{-v^2/4k}. \quad ▮$$

Remark. A very similar calculation shows that for $v = O(\sqrt{k})$ we have $|\mathscr{A}| > \frac{1}{10}\binom{n}{k}(k/v^2)\, e^{-v^2/k}$. This can be used to verify (4).

5. The Proof of Theorem 1

By Lemma 1 we can suppose that $\mathscr{F}_1$ is a left-stable and $\mathscr{F}_2$ is a right-stable intersecting family. Let $X = X_1 \cup X_2$, where $X_1 = \{1, 2,..., k+v\}$, $X_2 = X - X_1$, $\| X_1| - |X_2\| \leqslant 1$ $(v = c\sqrt{k}/2)$. Let $\mathscr{A}$ be the family given by Lemma 4. Set $\mathscr{F}'_1 = \mathscr{F}_1 - \mathscr{A}$. By Lemma 3

$$F \cap F' \cap X_1 \neq \varnothing \qquad \text{holds for all } F, F' \in \mathscr{F}'_1. \tag{6}$$

Define $\mathscr{A}_i = \{F \cap X_1 : F \in \mathscr{F}'_1,\ |F \cap X_1| = i\}$. By the Erdös–Ko–Rado theorem and (6) we have

$$|\mathscr{A}_i| \leqslant \binom{|X_1| - 1}{i-1}. \tag{7}$$

To each $A \in \mathscr{A}_i$ there are at most $\binom{|X_2|}{k-i}$ $F \in \mathscr{F}'_1$ satisfying $A \subset F$, $F \cap X_1 = A$. Hence we get

$$|\mathscr{F}'_1| \leqslant \sum_i |\mathscr{A}_i| \binom{|X_2|}{k-i}. \tag{8}$$

Similarly, set $\mathscr{F}'_2 = \mathscr{F}_2 - \mathscr{B}$, where $\mathscr{B} = \{F \in \binom{X}{k}:\ |B \cap [n-2i+1, n]| \geqslant i$ holds for some $i \geqslant (k+v)/2\}$. By symmetry $|\mathscr{B}| = |\mathscr{A}|$. Let $\mathscr{B}_i = \{F \cap X_2 : F \in \mathscr{F}'_2,\ |F \cap X_2| = i\}$. We have

$$|\mathscr{B}_i| \leqslant \binom{|X_2| - 1}{i-1} \tag{9}$$

and

$$|\mathscr{F}'_2| \leqslant \sum_i |\mathscr{B}_i| \binom{|X_1|}{k-i}. \tag{10}$$

In the estimations (8) and (10) we count twice the sets $G = A \cup B$, $A \in \mathscr{A}_i$, $B \in \mathscr{B}_{k-i}$. So we have

$$|\mathscr{F}'_1 \cup \mathscr{F}'_2| \leqslant \sum_i |\mathscr{A}_i| \binom{|X_2|}{k-i} + |B_{k-i}| \binom{|X_1|}{i} - |\mathscr{A}_i|\, |\mathscr{B}_{k-i}|.$$

The coefficient of $|\mathscr{A}_i|$ is nonnegative by (9) so we can replace $|\mathscr{A}_i|$ by $\binom{|X_1|-1}{i-1}$ without spoiling the inequality:

$$\begin{aligned}|\mathscr{F}'_1 \cup \mathscr{F}'_2| &\leqslant \sum_i \binom{|X_1|-1}{i-1}\left(\binom{|X_2|}{k-i} - |\mathscr{B}_{k-i}|\right) + |\mathscr{B}_{k-i}|\binom{|X_1|}{i} \\ &= \sum_i \binom{|X_1|-1}{i-1}\binom{|X_2|}{k-i} + \sum_i |\mathscr{B}_{k-i}|\binom{|X_1|-1}{i} \\ &\leqslant \sum_i \binom{|X_1|-1}{i-1}\binom{|X_2|}{k-i} + \sum_i \binom{|X_2|-1}{k-i-1}\binom{|X_1|-1}{i} \\ &= \binom{n-1}{k-1} + \binom{n-2}{k-1}.\end{aligned}$$

Finally, $|\mathscr{F}_1 \cup \mathscr{F}_2| \leqslant |\mathscr{F}'_1 \cup \mathscr{F}'_2| + |\mathscr{A}| + |\mathscr{B}|$. However, $|\mathscr{A}| < \binom{n-1}{k-1}\, 8c^{-2}$ $e^{-c^2/16}$ by Lemma 4. This finishes the proof of Theorem 1. ∎

6. The Proof of Theorem 2

Again by Lemma 1 we can suppose that $\mathscr{F}_1$ is a left-stable and $\mathscr{F}_2$ is a right-stable intersecting family. Let $\mathscr{K} = \{F \in \binom{X}{k}\colon\ 1 \in F, n \in F\}$, $\mathscr{A} = \mathscr{F}_1 - \mathscr{K}$, $\mathscr{B} = \mathscr{F}_2 - \mathscr{K}$. Moreover $\mathscr{A}_1 = \{A \in \mathscr{A}\colon 1 \in A$ (and $n \notin A)\}$, $\mathscr{A}_0 = \{A \in \mathscr{A}\colon 1 \notin A$ and $n \notin A\}$ and $\mathscr{A}_n = \{A \in \mathscr{A}\colon n \in A$ (and $1 \notin A)\}$.

For a family $\mathscr{H}$ define $\Delta_l \mathscr{H}$ as its *shadow* of order l, i.e., $\Delta_l \mathscr{H} = \{L\colon |L| = l$ and $\exists H \in \mathscr{H}, H \supset L\}$. Katona [21] proved the following

If $\mathscr{H}$ is a family of r-sets and $|F \cap F'| \geqslant t$ holds for all $F, F' \in \mathscr{H}$ then $|\Delta_l \mathscr{H}| \geqslant |\mathscr{H}| \binom{2r-t}{l} / \binom{2r-t}{r}$ holds for $r - t \leqslant l \leqslant r$. (1)

The set-system $\mathscr{A}_0$ is 2-intersecting, by Lemma 2, hence $\mathscr{A}_0^c = \{[2, n-1] - A\colon A \in \mathscr{A}_0\}$ is an $(n-k-2)$-uniform $(n-2k)$-intersecting system. Using (11) we have

$$|\Delta_{k-1} \mathscr{A}_0^c| \geqslant |\mathscr{A}_0^c| \binom{n-4}{k-1} \Big/ \binom{n-4}{n-k-2} = |\mathscr{A}_0| \frac{n-k-1}{k-1}. \tag{12}$$

If $A \in \mathscr{A}_n$ and $1 < x \notin A$ then $(A - \{n\} \cup \{x\}) \in \mathscr{A}_0$.

This yields

$$|\mathscr{A}_0| \geqslant \frac{1}{k} |\{(x, A)\colon 1 < x \in A \in \mathscr{A}_n\}| = \frac{n-k-1}{k} |\mathscr{A}_n|. \tag{13}$$

Finally, we have $|\mathscr{A}_1| \leqslant \binom{n-2}{k-1} - |\Delta_{k-1}\mathscr{A}_0^c|$ hence using (12) and (13)

$$|\mathscr{A}| = |\mathscr{A}_1| + |\mathscr{A}_0| + |\mathscr{A}_n| \leqslant \binom{n-2}{k-1} - |\mathscr{A}_0| \frac{n-k-1}{k-1} + |\mathscr{A}_0| + |\mathscr{A}_0| \frac{k}{n-k-1}.$$

Here the coefficient of $|\mathscr{A}_0|$ is less than 0 if $n > \frac{1}{2}(3+\sqrt{5})\,k$. Similarly, we get $|\mathscr{B}| < \binom{n-2}{k-1}$, or $n \in (\cap \mathscr{B})$, yielding Theorem 2. ∎

Remark. We proved that if $\mathscr{A}$ is left-stable then $|\mathscr{A} - \mathscr{K}| \lesssim \binom{n-2}{k-1}$ holds for $n > n_0(k)$. This statement does not hold for $n < \frac{1}{2}(3+\sqrt{5})\,k$ as an easy calculation and the following example show: $\mathscr{A} = \{A \in \binom{X}{k}: |A \cap [1,3]| \geqslant 2\}$.

7. The Proof of Theorems 3, 4, and 5

We prove only Theorem 3. The other proofs are similar and are left to the reader. We use the following Lemma which was proved in several places (e.g., [8, 10]):

Lemma 5. *Let $\mathscr{F} \subset \binom{X}{k}$ be a family of sets having property P. Then $S_{ij}(\mathscr{F})$ has property P, too.*

Here property P can be, e.g.,

$$|F \cap F'| \geqslant r \qquad \text{for all } F, F' \in \mathscr{F} \tag{14}$$

$$F_1 \cap \cdots \cap F_l \neq \varnothing \qquad \text{for all } F_1, \ldots, F_l \in \mathscr{F} \tag{15}$$

$$|F_1 \cup \cdots \cup F_{t+1}| \leqslant (t+1)\,k - 1 \qquad \text{for all } F_1, \ldots, F_{t+1} \in \mathscr{F}. \tag{16}$$

The following lemma is analogous to Lemma 3, but is in a weaker form.

Lemma 6. *Let $\mathscr{F} \subset \binom{X}{k}$ be a left-stable family having one of the properties P defined in* (14)–(16), *and let $n_p(k)$ be the threshold function for this property* (i.e., *$n_p(k)$ is one of $n_0(k,r)$, $n_1(k,l)$ and $n_2(k,t)$*). *Then the family $\mathscr{F}_0 = \{F \cap [1, n_p(k)]: F \in \mathscr{F}\}$ has property P, as well.*

The proof is easy. We have to use only that $n_0(k,r) \geqslant 2k-r$, $n_1(k,l) \geqslant lk/(l-1)$, $n_2(k,t) > tk$. We present only the proof of (14). Suppose for contradiction $|F \cap F' \cap [1, 2k-r]| < r$ and $F, F' \in \mathscr{F}$ are such that $|F \cap F'|$ is minimal. Now we may choose an element i $(1 \leqslant i \leqslant 2k-r)$, $i \notin F \cup F'$, and $j > 2k-r$, $j \in F \cap F'$. Then $F' - \{j\} \cup \{i\} = F'' \in \mathscr{F}$. However $|F \cap F'' \cap [1, 2k-r]| < r$ and $|F \cap F''| < |F \cap F'|$, a contradiction.

Proof of Theorem 3. It goes similarly to the proof of Theorem 1. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be an r_1- and r_2-intersecting family, respectively. By Lemmas 5 and 1 we can suppose that $\mathscr{F}_1$ is left-stable and $\mathscr{F}_2$ is right-stable. Split X into two parts $X = X_1 \cup X_2$ such that $X_1 = [1, n_0(k, r_1)]$, $X_2 = X - X_1$. By hypothesis $|X_2| \geqslant n_0(k, r_2)$. Now define $\mathscr{A}_i = \{F \cap X_1 : F \in \mathscr{F}_1, |F \cap X_1| = i\}$, $\mathscr{B}_j = \{F \cap X_2 : F \in \mathscr{F}_2, |F \cap X_2| = j\}$. By Lemma 6, and the monotonicity of $n_0(k, r_1)$ we can use the Erdös–Ko–Rado theorem, saying

$$|\mathscr{A}_i| \leqslant \binom{|X_1| - r_1}{i - r_1}, \tag{17}$$

$$|\mathscr{B}_j| \leqslant \binom{|X_2| - r_2}{j - r_2}. \tag{18}$$

To each $A \in \mathscr{A}_i$ there are at most $\binom{|X_2|}{k-i}$ $F \in \mathscr{F}_1$ satisfying $F \cap X_1 = A$. Hence we get

$$|\mathscr{F}_1| \leqslant \sum_i \quad |\mathscr{A}_i| \binom{|X_2|}{k-i}, \tag{19}$$

$$|\mathscr{F}_2| \leqslant \sum_j \quad |\mathscr{B}_j| \binom{|X_1|}{k-j}. \tag{20}$$

In the estimations (19) and (20) we count twice the sets $G = A \cup B$, $A \in \mathscr{A}_i$, $B \in \mathscr{B}_{k-i}$. So we have

$$|\mathscr{F}_1 \cup \mathscr{F}_2| \leqslant \sum_i |\mathscr{A}_i| \binom{|X_2|}{k-i} + |\mathscr{B}_{k-i}| \binom{|X_1|}{i} - |\mathscr{A}_i|\,|\mathscr{B}_{k-i}|.$$

From now on the proof coincides with the proof of Theorem 1, i.e., because the coefficient of $|\mathscr{A}_i|$ is nonnegative by (18), we can replace $|\mathscr{A}_i|$ by $\binom{|X_1| - r_1}{i - r_1}$ without spoiling the inequality:

$$\begin{aligned} |\mathscr{F}_1 \cup \mathscr{F}_2| &\leqq \sum_i \binom{|X_1| - r_1}{i - r_1} \left(\binom{|X_2|}{k-i} - |\mathscr{B}_{k-i}| \right) + |\mathscr{B}_{k-i}| \binom{|X_1|}{i} \\ &= \sum_i \binom{|X_1| - r_1}{i - r_1} \binom{|X_2|}{k-i} + |\mathscr{B}_{k-i}| \left(\binom{|X_1|}{i} - \binom{|X_1| - r_1}{i - r_1} \right). \end{aligned}$$

Using (18) we have

$$\begin{aligned} |\mathscr{F}_1 \cup \mathscr{F}_2| &\leqslant \sum_i \binom{|X_1| - r_1}{i - r_1} \binom{|X_2|}{k-i} + \binom{|X_2| - r_2}{k - i - r_2} \left(\binom{|X_1|}{i} - \binom{|X_1| - r_1}{i - r_2} \right) \\ &= \binom{n - r_1}{k - r_1} + \binom{n - r_2}{k - r_2} - \binom{n - r_1 - r_2}{k - r_1 - r_2}. \quad \blacksquare \end{aligned}$$

8. The Proof of Theorem 7

We use the following lemma of Hajnal and Rothschild [18]. (It was proved for $t=1$ by Hilton and Milner [20] in a more exact form.) Here we state it in a slightly stronger form, which was proved by Bollobás, Daykin and Erdös [2].

Lemma 7 [2]. *Let $\mathscr{F} \subset \binom{X}{r}$ and suppose that $\mathscr{F}$ contains at most t pairwise disjoint members. Then either*

(a) *there exists an element $x \in X$ such that $\mathscr{F}(\neg x) =^{\text{def}} \{F \in \mathscr{F} : x \notin F\}$ contains at most $(t-1)$ pairwise disjoint members (i.e., $\mathscr{F}(\neg x) = \emptyset$ for $t=1$, in this case). Or*

(b) $|\mathscr{F}| < r^2 t^2 \binom{n-2}{r-2}$.

Call the element $x \in X$ *extremal* for $\mathscr{F}$ if the maximum number of disjoint edges in $\mathscr{F}(\neg x)$ is less than in $\mathscr{F}$, i.e., $t(\mathscr{F}) > t(\mathscr{F}(\neg x))$.

Now we are ready to prove Theorem 6. Suppose $\mathscr{F}_1 \cup \mathscr{F}_2 \cup \cdots \cup \mathscr{F}_\chi = \binom{X}{k}$ and $t(\mathscr{F}_i) \leqslant t$. Let $\mathscr{F}_i^0 = \mathscr{F}_i$, $X^0 = X$, $Y_i^0 = \emptyset$ for $1 \leqslant i \leqslant \chi$. If we define the systems $\{\mathscr{F}_i^\alpha\}$, $\{Y_i^\alpha\}$, X^α $(1 \leqslant i \leqslant \chi)$ and there exists an $\mathscr{F}_j^\alpha$ having an extremal point x then set $\mathscr{F}_i^{\alpha+1} = \mathscr{F}_i^\alpha(\neg x)$ for all $1 \leqslant i \leqslant \chi$,

$$Y_i^{\alpha+1} = \begin{cases} Y_i^\alpha \cup \{x\} & \text{for } i=j \\ Y_i^\alpha & \text{otherwise,} \end{cases}$$

and $X^{\alpha+1} = X^\alpha - \{x\}$. Then we have $\sum_i t(\mathscr{F}_i^\alpha) > \sum_i t(\mathscr{F}_i^{\alpha+1})$ in view of $t(\mathscr{F}_i^{\alpha+1}) \leqslant t(\mathscr{F}_i^\alpha)$ and $t(\mathscr{F}_j^{\alpha+1}) < t(\mathscr{F}_j^\alpha)$. Continue this procedure till there exists no extremal point. Suppose that our procedure stops after the sth step. We have $t(\mathscr{F}_i^s) + |Y_i^s| \leqslant t$. Let $\mathscr{F}_{i_1}^s, \mathscr{F}_{i_2}^s, \ldots, \mathscr{F}_{i_u}^s$ be those families which satisfy $t(\mathscr{F}_i^s) > 0$, let $t_j = t(\mathscr{F}_{i_j}^s)$ $(1 \leqslant j \leqslant u)$, $|X^s| = m$. By Lemma 7 each family $\mathscr{F}_{i_j}^s$ *contains less than* $k^2 t_j^2 \binom{m-2}{k-2}$ members. Hence we get

$$\sum_{1 \leqslant j \leqslant u} k^2 t_j^2 \binom{m-2}{k-2} \geqslant \sum_{1 \leqslant j \leqslant u} |\mathscr{F}_{i_j}^s| = \left| \binom{|X_s|}{k} \right| = \binom{m}{k}.$$

Comparing the two extreme sides yields

$$\left(\sum_{1 \leqslant j \leqslant u} t_j^2 \right) k^4 > m^2. \tag{21}$$

Moreover, we know that $|\bigcup Y_i^s| = \sum_i |Y_i^s| \leqslant \chi t - \sum_{1 \leqslant j \leqslant u} t_j$. This yields

$$n = |X| = m + \sum |Y_i^s| \leqslant m + \chi t - \sum_{1 \leqslant j \leqslant u} t_j.$$

Using (21) we get

$$n-k^2\sqrt{\sum t_j^2}+\sum t_j\leqslant \chi t. \tag{22}$$

Now, it is easy to see that $k^2(\sum_{1\leqslant j\leqslant u} t_j^2)^{1/2}-\sum t_j\leqslant \frac{1}{4}k^4t$ independently of u. (As $t_j\leqslant t$ we have $\sum t_j\geqslant (\sum t_j^2)/t$. Set $T=\sqrt{\sum t_j^2}$, we have $k^2(\sum t_j^2)^{1/2}-\sum t_j\leqslant k^2T-T^2/t\leqslant k^4t/4$.) Hence (22) gives

$$n-\frac{k^4t}{4}\leqslant \chi t$$

as desired. ∎

Remark. We can prove in Lemma 7(b) that $|\mathscr{F}|<2rt^2\binom{n-2}{r-2}$. Using similar calculations we can show that $\chi\geqslant (n/t)-k^3$.

9. The Proof of Theorem 8

Let $\mathscr{H}$ be an r-graph on v elements (i.e., $|\cup\mathscr{H}|=v$). As usual, denote by $\mathrm{ex}(n,\mathscr{H})=\max\{|\mathscr{F}|: \mathscr{F}\subset\binom{X}{r}, |X|=n, \mathscr{F}$ does not contain $\mathscr{H}$ as a subsystem$\}$. By this notation we have $T(n,k,r)=\binom{n}{r}-\mathrm{ex}(n,\mathscr{K}_r^k)$. ($\mathscr{K}_r^k$ denotes the hypergraph consisting of all r-subsets of a k-set.) It is well known (see, e.g., Erdös, Simonovits [9]).

LEMMA 8 [9]. *If $\mathscr{F}\subset\binom{X}{r}$, $|X|=n$, $|\mathscr{F}|>\mathrm{ex}(n,\mathscr{H})+\varepsilon\binom{n}{r}$ then $\mathscr{F}$ contains at least $\varepsilon'\binom{n}{v}$ copies of $\mathscr{H}$, where ε' depends only on v and ε.*

We will use the following generalization of a theorem of Hilton and Milner [20]. It was proved by the first author in a more exact form.

LEMMA 9 [12]. *If $\mathscr{F}\subset\binom{X}{k}$, $|X|=n$, $\mathscr{F}$ is r-intersecting (i.e., $|F\cap F'|\geqslant r$ holds for all $F, F'\in\mathscr{F}$) and $|\bigcap\mathscr{F}|<r$ then $|\mathscr{F}|<n^{k-r-1}$ holds for $n>n_0(k)$.*

Proof of Theorem 8. Let $\binom{X}{k}=\mathscr{F}_1\cup\cdots\cup\mathscr{F}_m$, where $\mathscr{F}_i$ is an r-intersecting family ($1\leqslant i\leqslant m$). Suppose that $|\bigcap\mathscr{F}_i|\geqslant r$ holds for $1\leqslant i\leqslant s$, but $|\bigcap\mathscr{F}_i|<r$ for $s<i\leqslant m$. Let $R_i\subset\bigcap\mathscr{F}_i$, $|R_i|=r$, $\mathscr{R}=\{R_i: 1\leqslant i\leqslant s\}$. By Lemma 9 we have $|\mathscr{F}_j|<n^{k-r-1}$ holds for $j>s$, hence

$$\sum_{j>s}|\mathscr{F}_j|\leqslant (m-s)\,n^{k-r-1}\leqslant m\,n^{k-r-1}. \tag{25}$$

Suppose for contradiction that $m < (1-\varepsilon)\, T(n, k, r)$. This implies

$$\left|\binom{X}{r} - \mathscr{R}\right| = \binom{n}{r} - s \geqslant \binom{n}{r} - m \geqslant \left(\binom{n}{r} - T(n, k, r)\right) + \varepsilon\, T(n, k, r)$$

$$= \mathrm{ex}(n, \mathscr{K}_r^k) + \varepsilon \binom{n}{r} \Big/ \binom{k}{r} > \mathrm{ex}(n, \mathscr{K}_r^k) + \varepsilon_0 \binom{n}{r}.$$

Lemma 8 yields that $|\{F\colon |F| = k,\ \binom{F}{r} \cap \mathscr{R} = \varnothing\}| > \varepsilon' \binom{n}{k}$ holds. Now (25) gives

$$m\, n^{k-r-1} \geqslant \sum_{j > s} |\mathscr{F}_j| \geqslant |\{F\colon |F| = k,\ \nexists R_i \subset F\}| > \varepsilon' \binom{n}{k}.$$

This implies $m > \varepsilon' n^{r+1}/k! > \binom{n}{r} \geqslant T(n, k, r)$ if n is sufficiently large. ∎

Note added in proof. During the last two years, the following progress was made. Hujter [31] proved Proposition 1 even for t-intersecting families. Conjecture 1 was proved by Alon, Frankl and Lovász [28]. They used a generalization of Borsuk's theorem given in [29]. They also generalized Theorem 7.

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