Graphs and Combinatorics 2, 89–92 (1986)

Graphs and Combinatorics

Research Problems

In this column *Graphs and Combinatorics* publishes current research problems whose proposer believes them to be within reach of existing methods.

Manuscripts should preferably contain the background of the problem and all references known to the author. The length of the manuscript should not exceed two type-written pages. Manuscripts should be sent to:

P. Frankl or J. Akiyama
Graphs and Combinatorics
Department of Mathematics
Tokai University
Hiratsuka, 259–12
Japan

The Chromatic Index of Simple Hypergraphs

A hypergraph $H = (V, \mathscr{E})$ is a set of vertices $V = V(H)$ and a set $\mathscr{E} = \mathscr{E}(H)$ of non-empty subsets (called edges) of V. H is called *simple* (or linear, or (0, 1)-intersecting) if $|E \cap F| \leq 1$ holds for all pairs of edges E, $F \in \mathscr{E}$. Let us denote $|V(H)|$ by n. A *matching* in H is a collection of pairwise disjoint edges. By $q(H)$ we denote the *chromatic index* of H, it is the minimum number q such that one can decompose $\mathscr{E}$ into q matchings. Clearly, $q \geq D$ where $D = D(H) = \max_{x \in V} \deg_H(x) = \max|\{E \in \mathscr{E}: x \in E\}|$ is the *maximum degree.* The *neighborhood* of $x \in V$ is $N(x) =: \bigcup \{E - \{x\}: x \in E \in \mathscr{E}\}$. Define $N = \max_{x \in V} |N(x)|$. For graphs $D = N$. The well-known Vizing Theorem says.

Theorem 1. (Vizing [8]) *For every graph* $G = (V(G), \mathscr{E}(G))$ *we have* $q \leq D + 1$.

The following would be an interesting generalization:

Conjecture 2. For every simple hypergraph H, $q(H) \leq N(H) + 1$.

This would imply the Erdös-Faber-Lovász [5] conjecture

$$q(H) \leq |V(H)| \text{ holds for simple hypergraphs.} \tag{3}$$

Indeed, trivially $|N(x)| \leq |V(H)| - 1$ holds. The original formulation of (3) asserts that if the edge-set of a graph is the edge-disjoint union of t complete graphs on t vertices then its chromatic number is equal to t. The equivalence with (3) was pointed out by Hindman [6].

Let $\nu(H)$ denote the *matching number* of H, i.e., the maximum cardinality of pairwise disjoint edges in H. Note that (3) would imply $\nu(H) \geq |\mathscr{E}(H)|/n$. This weaker (but by far nontrivial) statement was proved by Seymour:

Theorem 4. (Seymour [7]) *For a simple hypergraph H one has* $\nu(H) \geq |\mathscr{E}(H)|/n$.

Seymour characterized the extremal families as well, i.e., those hypergraphs for which $\nu = |\mathscr{E}|/n$ holds. This is a generalization of the deBruijn-Erdös [2] theorem on combinatorial geometries. Among other partial results, using a computer search Hindman [6] proved (3) for $n \leq 10$.

Conjecture 2 appears to be very difficult. A first step could be to establish the following.

Conjecture 5. $\nu(H) \geq |\mathscr{E}(H)|/(N(H) + 1)$ holds for every simple hypergraph H.

Let H be a simple hypergraph and $\omega: \mathscr{E}(H) \to \mathbb{R}^+$ a non-negative real valued function on the edges, $\|\omega\| = \sum_{E \in \mathscr{E}(H)} \omega(E)$. If Conjecture 2 is true then the following is an easy consequence:

Conjecture 6. There exists a matching $\mathscr{M} \subset \mathscr{E}(H)$ such that

$$\sum_{M \in \mathscr{M}} \omega(M) \geq \frac{\|\omega\|}{N(H) + 1}. \tag{7}$$

Let us note that a weaker version of (7) namely, $\max \omega(\mathscr{M}) \geq \|\omega\|/n$, was conjectured by Seymour [7].

Now we settle the above conjectures for intersecting hypergraphs. Another special case, when H is a cyclic Steiner-system, was proved by J. and M. Colbourn [3]. We have to remark that Conjecture 2 was posed independently by C. Berge [1] and H. Meyniel (unpublished).

Theorem 8. *Suppose that H is a simple, intersecting hypergraph, i.e., $|E \cap F| = 1$ holds for all pairs of distinct edges E, $F \in \mathscr{E}(H)$. Then $|\mathscr{E}(H)| \leq N(H) + 1$.*

Before the proof we recall a definition and a lemma. A collection of sets $E_1, E_2, \ldots, E_s$ is called a *star* with kernel K if $E_i \cap E_j = K$ holds for every $1 \leq i < j \leq s$. The following lemma is a very special case of a theorem of Deza [4] on equidistant codes:

Lemma 9. *Let P be a simple, intersecting hypergraph and suppose that $|E| = k$ holds for all $E \in \mathscr{E}(P)$, $|\mathscr{E}(P)| \geq k^2 - k + 1$. Then either P is a star or it is isomorphic to a finite projective plane of order $k - 1$.* □

Proof of Theorem 8. Let E_0 be an edge of H of minimal size. Denote $|E_0|$ by k. The case $k = 1$ is trivial, so we will suppose that $k \geq 2$. Define $\mathscr{A} = \{E \in \mathscr{E}(H): |E| = k\}$, $\mathscr{B} = \mathscr{E} - \mathscr{A}$. By definition for every $x \in V$ we have

$$\sum_{E \ni x} (|E| - 1) = N(x)$$

i.e.,

$$(k - 1)\deg_{\mathscr{A}}(x) + k \deg_{\mathscr{B}}(x) \leq N.$$

Since H is 1-intersecting, we obtain

$$\sum_{x \in E_0} \deg_{\mathscr{A}}(x) = |\mathscr{A}| + k - 1$$

and

$$\sum_{x \in E_0} \deg_{\mathscr{B}}(x) = |\mathscr{B}|.$$

Hence the above three inequalities yield that

$$(k-1)\big(|\mathscr{A}| + k - 1\big) + k|\mathscr{B}| \le kN.$$

Rearranging we get

$$|\mathscr{E}(H)| \le N + \frac{1}{k}\big(|\mathscr{A}| - (k-1)^2\big). \tag{10}$$

Since $|\mathscr{E}(H)|$ is an integer the proof is complete unless $\frac{1}{k}\big(|\mathscr{A}| - (k-1)^2\big) \ge 2$, or equivalently, $|\mathscr{A}| \ge k^2 + 1$ holds. However this inequality implies, by Lemma 9, that $\mathscr{A}$ is a star. Denote by x_0 the center of the star, and let $A_1, A_2 \in \mathscr{A}$, $\deg_H(x_0) = d$. Every pair $x_1 \in A_1 - \{x_0\}$, $x_2 \in A_2 - \{x_0\}$ is contained in at most one member of H, hence

$$|\mathscr{E}(H)| \le d + (k-1)^2. \tag{11}$$

On the other hand

$$N \ge |N(x)| \ge d(k-1). \tag{12}$$

As $d \ge k^2 + 1$, (11) and (12) give $|\mathscr{E}(H)| \le N + 1$ finishing the proof.

The case of equality in Theorem 8. We need two more definitions. A *near-pencil* (of order n) is a hypergraph N $V(N) = \{1, 2, \ldots, n\}$ and $\mathscr{E}(N) = \{\{1, i\}\ 2 \le i \le n$ and $\{2, 3, \ldots, n\}\}$. Call the hypergraph a *star with a loop* it if has a one element edge $\{x\}$, and all the other edges have 2-elements and contain x. Now suppose that H is a simple, interesting hypergraph with $|\mathscr{E}(H)| = N(H) + 1$. Then by (10) we have $|\mathscr{A}| \ge k^2 - k + 1$. Hence Lemma 9 implies that one of the following three cases holds

(a) H is a star with a loop,
(b) H is a near-pencil,
(c) H is a finite projective plane.

References

1. Berge, C.: On the chromatic index of a linear hypergraph and the Chvátal conjecture. Annals of the New York Academy of Sciences, the proc. of 1985 meeting. (to appear)
2. deBruijn, N.G., Erdös, P.: A combinational problem. Indagations Math. **8**, 461–467 (1946)
3. Colbourn, J., Colbourn, M.: The chromatic index of cyclic Steiner 2-design. Int. J. Math. and Math. Sci. **5**, 823–825 (1982)
4. Deza, M.: Une propriété extrémale des plans projectives finis dans une classe de codes équidistants. Discrete Math. **6**, 343–352 (1973)
5. Erdös, P.: Problems and results in graph theory and combinational analysis. In: Proc. Fifth British Comb. Conf., pp 169–192. Congr. Numerantium 15. Winnipeg: Utilitas Mathematica 1976
6. Hindman, N.: On a conjecture of Erdös, Faber, and Lovász about n-colorings. Canad. J. Math. **33**, 563–570 (1981)
7. Seymour, P.: Packing nearly disjoint sets. Combinatorica **2**, 91–97 (1982)
8. Vizing, V.G.: On an estimate of the chromatic class of a p-graph (in Russian). Diskret. Analiz **3**, 25–30 (1964)

Z. Füredi
Mathematical Institute
of the Hungarian Academy of Science
P.O.B. 127
1364 Budapest
Hungary

Received: December 30, 1985