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t-Expansive and t-wise Intersecting Hypergraphs

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Dedicated to Tibor Gallai

Abstract. We give a hypergraph generalization of Gallai's theorem about factor-critical graphs. This result can be used to determine $\tau^*(r,t)$ for $r < 3t/2$, where $\tau^*(r,t)$ denotes the maximum value of the fractional covering numbers of t-wise intersecting hypergraphs of rank r.

1. Introduction. t-Expansive Graphs and Hypergraphs

Let $\mathscr{H}$ be a finite set-system with non-empty members (i.e. $\mathscr{H}$ is a hypergraph) and let us denote by $t(\mathscr{H})$ the minimum integer t for which

$$|\bigcup \mathscr{H}'| \leq |\mathscr{H}'| + t \tag{1.1}$$

holds for all $\mathscr{H}' \subset \mathscr{H}$. If $t(\mathscr{H}) = t$, then we call $\mathscr{H}$ *t-expansive*. If $\mathscr{H}$ is 2-uniform, i.e. $|H| = 2$ for all $H \in \mathscr{H}$, i.e. $\mathscr{H}$ is a graph, then its expansion number $t(\mathscr{H})$ equals to the matching number $\nu(\mathscr{H})$, where the matching number denotes the maximum cardinality of an edge-set of $\mathscr{H}$ containing pairwise disjoint edges.

Let us denote by $\mathscr{H} - x$ the set-system $\{H \in \mathscr{H}: x \notin H\}$. $\mathscr{H}$ is *t-stable* (or critically t-expansive) if $t(\mathscr{H} - x) = t(\mathscr{H})$ holds for all $x \in \bigcup \mathscr{H}$. Similarly, the graph $\mathscr{G}$ (or a hypergraph $\mathscr{H}$) is *ν-stable* if $\nu(\mathscr{G} - x) = \nu(\mathscr{G})$ holds for all point x. E.g., the complete graph $K_{2\nu+1}$ and the circuit $C_{2\nu+1}$ are ν-stable. The ν-stability and t-stability coincide for graphs.

More than 20 years ago Gallai [11] proved that if a graph $\mathscr{G}$ is ν-stable and *connected* then it is *factor-critical*, i.e. $\mathscr{G} - x$ has a one-factor for all $x \in \bigcup \mathscr{G}$. Hence $|\bigcup \mathscr{G}| \leq 3\nu$ holds for all (not necessary connected) ν-stable graphs. Here equality holds only in the case when $\mathscr{G}$ is the disjoint union of ν triangles. (Triangle means the complete graph K_3.) This result plays an important role in the Edmonds-Gallai structure theorem (see [6], [11], [13, Problems 7.26–7.32]).

In this paper a similar theorem is proved about t-stable hypergraphs which generalizes Gallai's result. Our theorem shows that the natural extension of the matching number of graphs to hypergraphs is the expansion number (and not the usual matching number).

We have to mention that the concept of t-expansion is not unknown in hypergraph theory. E.g., Brace and Daykin [3] proved that $t(\mathscr{H}) = t$, $|\bigcup \mathscr{H}| = n$ implies $|\mathscr{H}| \leq (n - t + 1)2^t$, where equality holds iff $\bigcup \mathscr{H} = X = A \cup B$, $|A| = n - t$, $|B| = t$ and $\mathscr{H} = \{H \subset X: |H \cap A| \leq 1\}$. Daykin [4] proved for $m \geq 2t$, Bang, Sharp and

Winkler [2] proved for $m \geq 1.3t$ and Daykin and Frankl [5] proved for $m \geq t + 25$ that if $\mathscr{H}$ is a t-expansive hypergraph on the m-element set X then $\min_{x \in X} \deg_{\mathscr{H}}(x) \leq 2^t$, where $\deg_{\mathscr{H}}(x) = |\{H \in \mathscr{H}: x \in H\}|$, the *degree* of x in the hypergraph $\mathscr{H}$.

2. Critical t-Expansive Hypergraphs

Theorem 2.1. *Let $\mathscr{H}$ be a finite set-system consisting of at least 2-element members, and let t be an integer. Suppose that $\mathscr{H}$ is t-stable (i.e., (1.1) holds for $\mathscr{H}$ and for all $\mathscr{H} - x$.) If $\mathscr{H}$ is connected, then $|\bigcup \mathscr{H}| \leq 2t + 1$. Here equality holds if the graph $\mathscr{G} = \{E: E \subset H \in \mathscr{H}, |E| = 2\}$ is a factor-critical graph on $2t + 1$ vertices.*

Corollary 2.2. *Suppose $\mathscr{H}$ is a t-stable hypergraph in which every edge has size ≥ 2. Then $|\bigcup \mathscr{H}| \leq 3t$. Here equality holds iff $\mathscr{H}$ is the disjoint union of t triangles.*

Let $\mathscr{H}$ be a ν-stable hypergraph of *rank r* (i.e., $\forall H \in \mathscr{H}$ we have $|H| \leq r$). Erdös and Lovász [7] and Lovász [14] proved that $|\bigcup \mathscr{H}| \leq \frac{r}{2}\binom{r\nu + r - 1}{r}$. Lovász conjectures that there exists a constant c_r (depending only on r) such that $|\bigcup \mathscr{H}| < c_r \nu$. (This c_r cannot be less than $4^r/10r$. The best upper bound is due to Tuza [17], $|\bigcup \mathscr{H}| \leq \binom{r\nu + r}{r}$.) The following special case, which arose in a conversation with P. Erdös, seems to be more hopeful.

Conjecture 2.3. Let $\mathscr{G}$ be a graph and $\mathscr{H}$ be the hypergraph consisting of the vertices of the triangles of $\mathscr{G}$. If $\mathscr{H}$ is ν-stable, then $|\bigcup \mathscr{H}| \leq 5\nu$, and here equality holds only if $\mathscr{G}$ consists of ν disjoint K_5.

3. Application. Fractional Covering Number of t-wise Intersecting Hypergraphs

The *covering number* $\tau(\mathscr{H})$ of a hypergraph $\mathscr{H}$ denotes the minimum cardinality of a cover T (i.e., $T \cap H \neq \varnothing$ for all $H \in \mathscr{H}$). The real function $t: (\bigcup \mathscr{H}) \to \mathbb{R}$ is called *fractional cover* of $\mathscr{H}$ if $t(x) \geq 0$ for all x and $\sum_{x \in H} t(x) \geq 1$ for all $H \in \mathscr{H}$. The sum $\|t\| = \sum \{t(x): x \in \bigcup \mathscr{H}\}$ is the *value* of the fractional cover t. The *fractional covering number* of $\mathscr{H}$, denoted by $\tau^*(\mathscr{H})$, equals to $\min\{\|t\|: t$ is a fractional cover of $\mathscr{H}\}$. Analogously, the function $w: \mathscr{H} \to \mathbb{R}$ is a *fractional matching* if $w \geq 0$ and $\sum_{H \ni y} w(H) \leq 1$ holds for all $y \in \bigcup \mathscr{H}$. The *fractional matching number*, $\nu^*(\mathscr{H})$, is the maximum value of $\|w\|$, where $\|w\| = \sum_{H \in \mathscr{H}} w(H)$. The Duality Theorem of linear programming implies (see, e.g., [13, Problem 13.48], [15]) that $\nu^* = \tau^*$ holds for all hypergraphs $\mathscr{H}$. Trivially, $\nu \leq \nu^* = \tau^* \leq \tau$ holds.

A hypergraph $\mathscr{H}$ is called *t-wise intersecting* if $H_1 \cap H_2 \cap \cdots \cap H_t \neq \varnothing$ holds for all $H_1, \ldots, H_t \in \mathscr{H}$. The 2-wise intersecting hypergraphs are called briefly intersecting.

Verifying a conjecture of Lovász [12] the author proved [10] that $\tau^*(\mathscr{H}) \leq r - 1 + (1/r)$ holds for every intersecting hypergraph $\mathscr{H}$ of rank r, and here equality holds iff $\mathscr{H}$ is the hypergraph consisting of the lines of a finite projective plane of order $r - 1$. This result was generalized together with Frankl [9] for t-wise intersecting hypergraphs: If $r = q^{t-1} + q^{t-2} + \cdots + 1$ and $\mathscr{H}$ is a t-wise intersecting hypergraph of rank r then $\tau^*(\mathscr{H}) \leq q + (1/r)$ where equality holds iff $\mathscr{H}$ is the hypergraph obtained as the set of hyperplanes of the t-dimensional finite projective space of order q. Denote by $\tau^*(r,t) =: \max\{\tau^*(\mathscr{H})\colon \mathscr{H}$ is t-wise intersecting and its rank is $r\}$. Theorem 2.1 helps us to determine the function $\tau^*(r,t)$ for small values of r, namely:

Theorem 3.1. *Let $2 \le t \le r$ be integers. If $r \le \frac{3}{2}t - 1$ then $\tau^*(r,t) = 1 + 2/(3t - r)$. Moreover, $\tau^*(r,t) = 1 + 2/r$ for $r = (3t-1)/2$.*

If the rank r of the t-wise intersecting hypergraph $\mathscr{H}$ is less than t then $\bigcap \mathscr{H} \neq \varnothing$, hence $\tau(\mathscr{H}) = \tau^*(\mathscr{H}) = 1$, i.e., in this case $\tau^*(r,t) = 1$.

Example 3.2. Let $2 \le t \le r \le (3t-1)/2$. Let D be an $(r+1)$-set and $D = D_0 \cup D_1 \cup \cdots \cup D_{r-t}$ be a partition, where $|D_0| = 3t - 2r + 1\ (\ge 2)$, and $|D_1| = \cdots = |D_{r-t}| = 3$. Define the hypergraph $\mathscr{H}(r,t)$ on the vertex-set D as follows: $\mathscr{H} = \mathscr{H}_0 \cup \mathscr{H}_1$ where $\mathscr{H}_0 = \{H \subset D\colon |H| = r, D_0 \not\subset H\}$ and $\mathscr{H}_1 = \{H \subset D\colon |H| = r - 1,\ \exists 1 \le i \le r - t$ such that $|D_i \cap H| = 1\}$. I.e., the complements of the members of $\mathscr{H}$ form a graph over D consisting of $(3t - 2r + 1)$ isolated vertices and $(r - t)$ disjoint triangles.

We will use the following notation: $\mathscr{H}(r,t) = ((3t - 2r + 1)K_1 + (r - t)K_3)^c$ where $\mathscr{A} + \mathscr{B}$ denotes the disjoint union of the hypergraphs $\mathscr{A}$ and $\mathscr{B}$ and $\mathscr{H}^c$ the hypergraph consisting of the complements of the edges of $\mathscr{H}$. Moreover, K_1 denotes the hypergraph consisting of a single 1-element set and K_3 the triangle.

It is easy to calculate that $\tau^*(\mathscr{H}(r,t)) = 1 + 2/(3t - r)$. The fractional cover $t\colon D \to \mathbb{R}$

$$t(x) = \begin{cases} 2\alpha & \text{for } x \in D_0 \\ \alpha & \text{for } x \in \bigcup\limits_{1 \le i \le r-t} D_i \end{cases}$$

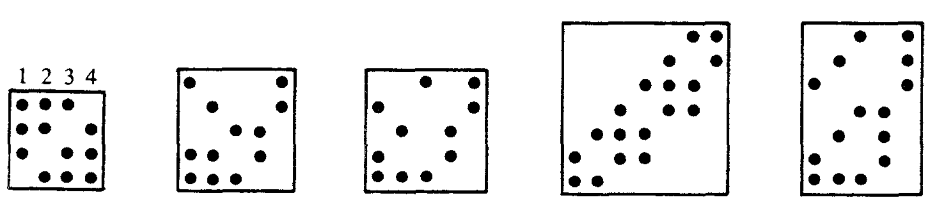
Shows that $\tau^*(\mathscr{H}) \le \|t\| = (3t - r + 2)\alpha$. (Here $\alpha = 1/(3t - r)$.) The fractional matching $w\colon \mathscr{H} \to \mathbb{R}$

$$w(H) = \begin{cases} 2\alpha & \text{for } H \in \mathscr{H}_0 \\ \alpha & \text{for } H \in \mathscr{H}_1 \end{cases}$$

shows that $(3t - r + 2)\alpha = \|w\| \le \nu^*(\mathscr{H}) = \tau^*(\mathscr{H})$. The fractional covering numbers of the following Examples can be calculated in the same way.

Example 3.3. Let t be an odd integer and $r = (3t-1)/2$. Let $\mathscr{G}(r,t) = (\frac{1}{2}(t+1)K_3)^c$. Then $\tau^*(\mathscr{G}) = (r+2)/r$.

Define the hypergraphs $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{C}'$ and $\mathscr{D}$ by their incidency matrices (see Fig. 1). I.e., let



$\mathscr{A}$ $\mathscr{B}$ $\mathscr{C}$ $\mathscr{D}$ $\mathscr{E}$

Fig. 1.

$$\mathscr{A} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\},$$

$$\mathscr{B} = \{\{1,2,3\},\{1,2,4\},\{3,4\},\{2,5\},\{1,5\}\},$$

$$\mathscr{C} = \{\{1,2,3\},\{1,4\},\{2,4\},\{1,5\},\{3,5\}\},$$

$$\mathscr{C}' = \mathscr{C} \cup \{\{3,4\},\{2,5\}\},$$

$$\mathscr{D} = \{\{1,2\},\{1,3,4\},\{2,3,4\},\{3,5,6\},\{4,5,6\},\{5,7\},\{6,7\}\}.$$

Example 3.4. Let $t + 2 \le r$ $(\le (3t-1)/2)$ and define $\mathscr{H}_a(r,t) = ((3t - 2r + 3)K_1 + \mathscr{A} + (r - t - 2)K_3)^c$. Then $\tau^*(\mathscr{H}_a) = 1 + 2/(3t - r + 2/3)$.

Example 3.5. Let $t + 2 \le r$ and define $\mathscr{H}_b(r,t) = ((3t - 2r + 2)K_1 + \mathscr{B} + (r - t - 2)K_3)^c$. Then $\tau^*(\mathscr{H}_b) = 1 + 2/(3t - r + 0.5)$.

Example 3.6. Let $t + 2 \le r$ and define $\mathscr{H}_c(r,t) = ((3t - 2r + 2)K_1 + \mathscr{C} + (r - t - 2)K_3)^c$. Then $\tau^*(\mathscr{H}_c) = 1 + 2/(3t - r + 2/3)$.

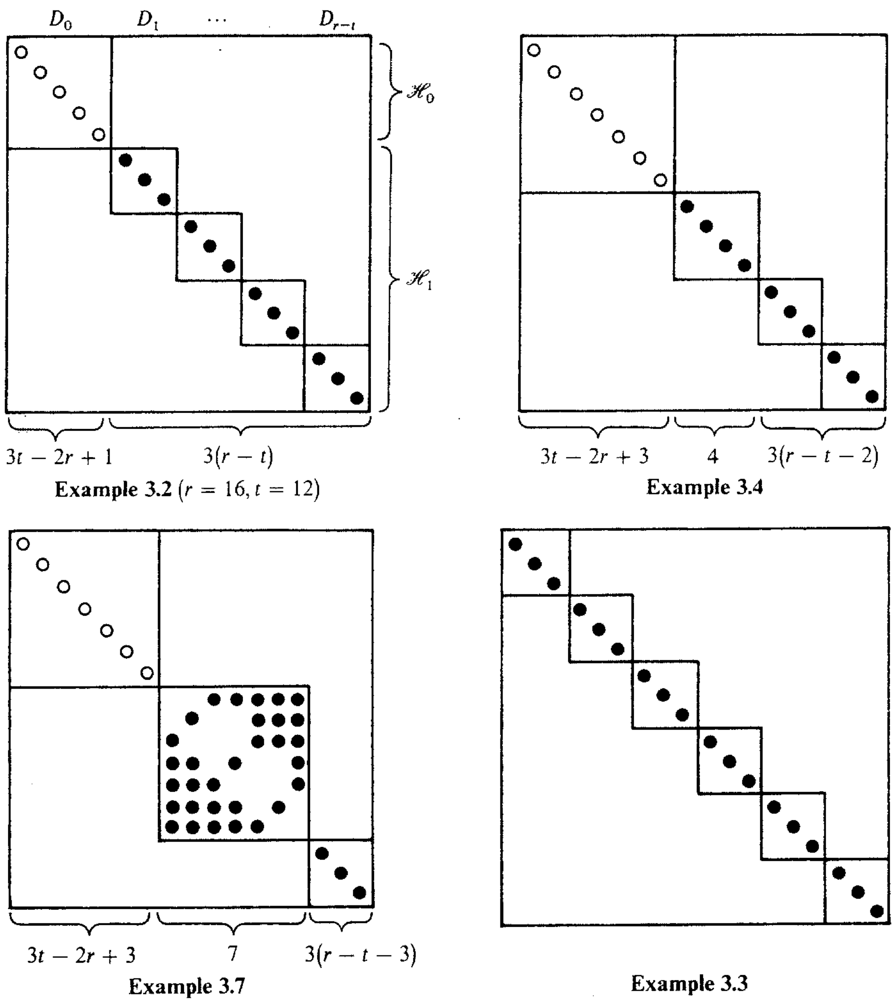


Fig. 2. $(r = 16)$

Example 3.7. Let $t + 3 \leq r$ and define $\mathscr{H}_d(r, t) = ((3t - 2r + 3)K_1 + \mathscr{D} + (r - t - 3)K_3)^c$. Then $\tau^*(\mathscr{H}_d) = 1 + 2/(3t - r + 0.75)$.

All of these Examples are t-wise intersecting. We are going to prove Theorem 3.1 in the following sharpened form.

Theorem 3.8. *Let $\mathscr{H}$ be a t-wise intersecting hypergraph of rank r. Suppose that $\tau^*(\mathscr{H}) > \tau^*(r - 1, t)$. Then $\mathscr{H}$ has a minor $\mathscr{H}_0$ isomorphic to one of the Examples* 3.2–3.7.

Here $\mathscr{H}_0$ is a minor of $\mathscr{H}$ if its incidency matrix is obtained from the incidency matrix of $\mathscr{H}$ by deleting rows and columns.

4. The Proof of Theorem 2.1.

Let $\mathscr{H}$ be a t-stable hypergraph with at least 2-element edges. If $H \in \mathscr{H}$ and $A \subset H$, $|A| \geq 2$ then the hypergraph $\mathscr{H} \cup \{A\}$ is also t-stable on the same underlying set. Hence we can suppose that $\mathscr{H}$ is almost a downset, i.e., if $|A| \geq 2$, $A \subset H \in \mathscr{H}$ then $A \in \mathscr{H}$. Let us define $\mathscr{G} = \{A \in \mathscr{H}: |A| = 2\}$. $\mathscr{H}$ is connected, so $\mathscr{G}$ is a connected graph as well. As usual, $\Gamma(x)$ denotes the neighbourhood of the point x in the graph $\mathscr{G}$, i.e., $\Gamma(x) = \{y: \{x, y\} \in \mathscr{G}\}$.

A subsystem $\mathscr{A} \subset \mathscr{H}$ is called *maximal* if $|\bigcup \mathscr{A}| = |\mathscr{A}| + t$, the members of $\mathscr{A}$ are pairwise disjoint and $|\bigcup \mathscr{A}|$ is maximal with respect to these constraints. Let us choose a maximal subsystem and denote it by $\mathscr{B}$. Let $\bigcup \mathscr{B} = X$.

Lemma 4.1. *For each $x \in X$ there exists a t-expansive set-system $\mathscr{C}_x$ such that $\bigcup \mathscr{C}_x$ covers $X - \{x\}$, $x \notin \bigcup \mathscr{C}_x$ and it consists of pairwise disjoint edges of $\mathscr{H}$.*

Proof. $\mathscr{H}$ is a stable t-expansive hypergraph, hence there exists a set-system $\mathscr{C} = \{C_1, \ldots, C_l\} \subset \mathscr{H} - x$, such that $|\bigcup \mathscr{C}| = l + t$. Let $C_i' = C_i - \bigcup \{C_j: j < i\}$. The existence of the system $\{C_i': |C_i'| \geq 2\}$ shows that the following family of subsystems is non-empty:

$\underline{\underline{C}}_x = \{\mathscr{D}: \mathscr{D} \subset \mathscr{H}, \mathscr{D}$ contains disjoint members, $\mathscr{D}$ is t-expansive and $x \notin \bigcup \mathscr{D}\}$.

Let $\mathscr{C}_x$ denote a subsystem belonging to $\underline{\underline{C}}_x$ for which $|\mathscr{B} \cap \mathscr{C}_x|$ (i.e., the number of the common members) is maximal. We are going to show that $X - \{x\} \subset \bigcup \mathscr{C}_x$. Suppose for contradiction that $y \in X - \{x\}$ but $y \notin \bigcup \mathscr{C}_x$. Let $B \in \mathscr{B}$ the edge for which $y \in B$. We distinguish two cases. If $x \in B$ then $\{x, y\} \in \mathscr{H}$ and the subsystem $\mathscr{C}_x \cup \{x, y\}$ would be $(t + 1)$-expansive. If $x \notin B$ then let $\{C_1, \ldots, C_l\} = \{C \in \mathscr{C}_x: C \cap B \neq \varnothing\}$. The set-system $\mathscr{C}_x - \{C_1, \ldots, C_l\} \cup \{C_i - B: |C_i - B| > 1\} \cup \{B\}$ belongs to $\underline{\underline{C}}_x$, too, and has more common members with $\mathscr{B}$ than $\mathscr{C}_x$. This contradiction proves, that such a y does not exist, i.e., $(\bigcup \mathscr{C}_x) \cap X = X - \{x\}$. □

Proposition 4.2. $|\bigcup \mathscr{B}| = |X| \leq 2t$. *Here equality holds if $\mathscr{B} \subset \mathscr{G}$.*

Proof. $\mathscr{B}$ consists of at least two-element disjoint sets, hence we get $2|\mathscr{B}| \leq |\bigcup \mathscr{B}| = |\mathscr{B}| + t$, i.e., $|\mathscr{B}| \leq t$, hence $|\bigcup \mathscr{B}| \leq 2t$. □

Proposition 4.3. *The sets $\bigcup \mathscr{C}_x (x \in X)$ and the set $\bigcup \mathscr{B}$ cover $\bigcup \mathscr{H}$.*

Proof. Suppose for contradiction that $y \in \bigcup \mathscr{H} - \bigcup \{\bigcup \mathscr{C}_x: x \in X\} - \bigcup \mathscr{B}$. Then there exists a 2-element subset $\{x, y\} \in \mathscr{H}$. Joining the set $\{x, y\}$ to $\mathscr{C}_x$ we get a $(t+1)$-expansive subsystem, which is a contradiction. □

Returning to the proof of Theorem 2.1 we distinguish two cases.

1) If each $(\bigcup \mathscr{C}_x) \subset X$ then Proposition 4.3 yields that $|\bigcup \mathscr{H}| = |\bigcup \mathscr{B}|$. Now $|\bigcup \mathscr{H}| \le 2t$, by Proposition 4.2, and we are ready.
2) From now on we can suppose that there exists a $\mathscr{C}_x$ such that $(\bigcup \mathscr{C}_x) - X \ne \varnothing$. Then $|\bigcup \mathscr{C}_x| \ge |X|$. But $\mathscr{B}$ is maximal, hence $|\bigcup \mathscr{C}_x| = |X|$ and $|(\bigcup \mathscr{C}_x) - X| = 1$. Let us denote by $\bigcup \mathscr{C}_x - X = \{z\}$. Obviously, $\Gamma(z) \subset X$ $(y \in \Gamma(z) - X$ implies that $\mathscr{B} \cup \{z, y\}$ is $(t+1)$-expansive, which is a contradiction.

Proposition 4.4. *Let $u \in \Gamma(z)$. Then $\mathscr{C}_u$ is maximal t-expansive and $(\bigcup \mathscr{C}_u) \subset (X \cup \{z\})$.* ($\mathscr{C}_u$ is defined by Lemma 4.1.)

Proof. By definition (more exactly by Lemma 4.1) $(\bigcup \mathscr{C}_u) \cap X = X - \{u\}$. If $z \notin \bigcup \mathscr{C}_u$ then the system $\mathscr{C}_u \cup \{z, u\}$ is $(t+1)$-expansive, which is a contradiction. Hence we get $\bigcup \mathscr{C}_u \supset (X - \{u\} \cup \{z\})$. This yields that $|\bigcup \mathscr{C}_u| \ge |X|$, i.e. $\mathscr{C}_u$ is maximal, as well. So we have $\bigcup \mathscr{C}_u = X - \{u\} \cup \{z\}$. □

Change the role of $\mathscr{B}$ and z with $\mathscr{C}$ and u. We get $\Gamma(u) \subset X \cup \{z\}$. This yields that $\Gamma(\Gamma(z)) \subset X \cup \{z\}$. Continuing procedure we get that the component of $\mathscr{G}$ which contains z is contained in $X \cup \{z\}$. Hence $\bigcup \mathscr{H} = \bigcup \mathscr{G} \subset X \cup \{z\}$. Finally, $|X \cup \{z\}| \le 2t + 1$, by Proposition 4.2. The case of equality is clear. □

Remark 4.5. In the case $|\bigcup \mathscr{H}| = 2t + 1$ the hypergraph $\mathscr{H}$ is not necessarily 2-uniform. E.g., $\mathscr{H} = \{\{2,3\}, \{3,4\}, \{4,5\}, \{5,1\}$ and $\{1,2,4\}\}$.

Remark 4.6. The crucial point of Gallai's proof is the following statement: If the graph $\mathscr{G}$ is ν-stable then $\nu(\mathscr{G} - x - y) < \nu(\mathscr{G})$, $\nu(\mathscr{G} - y - z) < \nu(\mathscr{G})$ imply $\nu(\mathscr{G} - x - z) < \nu(\mathscr{G})$. This means that the relation $x \sim y$: $\nu(\mathscr{G} - x - y) < \nu(\mathscr{G})$ is an equivalence relation on $\bigcup \mathscr{G}$. A similar statement for hypergraphs does not hold. E.g., the hypergraph $\mathscr{H}$ given on the pointset $\{1,2,3,4,5,6,7,8\}$, $\mathscr{H} = \{\{4,6,8\}, \{3,5,7\}, \{2,6,7\}, \{1,5,8\}, \{2,3,8\}, \{1,4,7\}, \{2,4,5\}, \{1,3,6\}\}$ is critical 4-expansive and $t(\mathscr{H} - \{1,3\}) < 4$, $t(\mathscr{H} - \{3,2\}) < 4$ but $t(\mathscr{H} - \{1,2\}) = 4$ (Fig. 3).

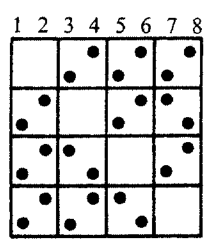


Fig. 3.

5. Lemmas for Theorem 3.8.

The case $r = t$ is trivial. (If $\mathscr{H}$ is a t-wise intersecting hypergraph of rank t and $\bigcap \mathscr{H} = \varnothing$ then it is the complete t-graph over $t+1$ vertices.) From now on we suppose that $t < r < \frac{3}{2}t$.

Let $\mathscr{H}$ be a hypergraph and denote by $\mu(\mathscr{H})$ the optimal value of the following linear program

$$\mu(\mathscr{H}) =: \max_m \left\{ \sum_{x \in V(\mathscr{H})} m(x) : m(x) \geq 0, \sum_{x \in E} m(x) \leq 1 \text{ for all } E \in \mathscr{H} \right\}.$$

(Properly, $\mu(\mathscr{H}) = v^*(\mathscr{H}^T)$ where $\mathscr{H}^T$ denotes the dual of $\mathscr{H}$.)

Proposition 5.1. *Let* $\mathscr{H} = (\mathscr{H}_1 + \cdots + \mathscr{H}_s)^c$. *Then* $\tau^*(\mathscr{H}) = \sum/(\sum - 1)$ *where* $\sum = \sum_{1 \leq i \leq s} \mu(\mathscr{H}_i)$.

Proof. Trivial. It follows from the definitions of μ and τ^*. □

Definition 5.2. For any hypergraph $\mathscr{H}$ denote by $f(\mathscr{H})$ the following

$$f(\mathscr{H}) =: \mu(\mathscr{H}) - |\bigcup \mathscr{H}| + \tfrac{3}{2} t(\mathscr{H}),$$

where $t(\mathscr{H})$ denotes the expansion number.

The aim of this section is to prove the following two lemmas.

Lemma 5.3. $f(\mathscr{H}) \geq 0$ *for every* $\mathscr{H}$.

Lemma 5.4. *Suppose* $\mathscr{H}$ *is connected and* $f(\mathscr{H}) < 1/2$. *Then* $|\bigcup \mathscr{H}| \leq 7$, $t(\mathscr{H}) \leq 3$ *and one of the following six cases holds*:

1) $\mathscr{H} \approx K_1$ *and then* $t = 0, f = 0$,
2) $\mathscr{H} \approx K_3$ *and then* $t = 1, f = 0$,
3) $\mathscr{H} \approx \mathscr{A}$ *and then* $t = 2, f = 1/3$,
4) $\mathscr{H} \approx \mathscr{B}$ *and then* $t = 2, f = 1/4$,
5) $\mathscr{C} \subseteq \mathscr{H} \subseteq \mathscr{C}'$ *and then* $t = 2, f = 1/3$,
6) $\mathscr{H} \approx \mathscr{D}$ *and then* $t = 3, f = 3/8$.

Proof of Lemma 5.3. We need a series of propositions. We will use induction on $|\bigcup \mathscr{H}| =: v$. The case $v = 1$ is trivial.

Proposition 5.5. *If* $\mathscr{H}$ *is not t-critical then* $f(\mathscr{H}) \geq 1/2$.

Proof. We have a vertex $x \in V(\mathscr{H})$ such that $t(\mathscr{H} - x) < t$. Use the inductional hypothesis for $\mathscr{H} - x$. We have

$$f(\mathscr{H} - x) = \mu(\mathscr{H} - x) - (v - 1) + \tfrac{3}{2}(t - 1) \geq 0.$$

As $\mu(\mathscr{H}) \geq \mu(\mathscr{H} - x)$ we obtain $f(\mathscr{H}) \geq 1/2$. □

Proposition 5.6. *If there exists an* $H \in \mathscr{H}$ *with* $|H| \geq 4$, *then* $f(\mathscr{H}) \geq 1/2$.

Proof. It is similar to Proposition 5.5. Consider $\mathscr{H} - H$ and apply the inductional hypothesis. We have

$$f(\mathscr{H} - H) = \mu(\mathscr{H} - H) - (v - |H|) + \tfrac{3}{2}(t - |H| + 1) \geq 0.$$

Hence

$$f(\mathscr{H}) \geq \mu(\mathscr{H} - H) - v + \tfrac{3}{2} t \geq \tfrac{1}{2}(|H| - 3). \tag{5.7}$$

□

Corollary 5.8. *If there exists an edge* $H \in \mathscr{H}$ *with* $|H| \geq 3$, *then* $f(\mathscr{H}) \geq 0$.

Proof. It follows from (5.7). □

Proposition 5.9. *Let* $\mathscr{H} = \sum_{1 \le i \le s} \mathscr{H}_i$. *Then* $f(\mathscr{H}) = \sum_i f(\mathscr{H}_i)$. □

Finally, Proposition 5.5, 5.8 and 5.9 imply $f \geq 0$ if we prove it for connected t-critical hypergraphs with at most 2-elements edges. In this case $\mu \geq v/2$ and $v \leq 2t + 1$ by Theorem 2.1. I.e.,

$$f(\mathscr{H}) \geq \frac{v}{2} - v + \frac{3}{2}t \geq -\frac{2t+1}{2} + \frac{3}{2}t = (t-1)/2 \geq 0$$

if $t \geq 1$. The case $t = 0$ is trivial. The proof of Lemma 5.3 is complete. □

Proof of Lemma 5.4. We need three more propositions. We suppose that $\mathscr{H}$ is connected and $f(\mathscr{H}) < 1/2$. Then by Proposition 5.5 and 5.6 we have that $\mathscr{H}$ is t-critical and $|H| \leq 3$ for all $H \in \mathscr{H}$.

Proposition 5.10. *Either* $t \leq 6$ *and* $v = 2t + 1$ *or* $t \leq 2$ *and* $v = 2t$ *holds.*

Proof. We have $\mu \geq v/3$ hence by Theorem 2.1 $f(\mathscr{H}) \geq -\frac{2}{3}v + \frac{3}{2}t \geq -\frac{2}{3}(2t+1) + \frac{3}{2}t = (t-4)/6 \geq 1/2$ for $t \geq 7$. If $v \leq 2t$ then the above inequality gives that $f \geq t/6 \geq 1/2$ for $t \geq 3$. □

From now on we have to consider only finitely many cases. Hence we only sketch the proof. If the reader believes that the author has examined all (finitely many, there are $\leq 10^{51}$ 13×13 0–1 matrices) cases of the hypergraphs with at most 13 vertices, then he or she can continue reading Chapter 6. It is easy to check the cases $t \leq 2$ $(v \leq 5)$.

Proposition 5.11. *Let* $H \in \mathscr{H}$, $|H| = 3$. *Then* $\mathscr{H} - H$ *is* $(t-2)$*-critical.*

Proof. It follows from Proposition 5.5 and (5.7). □

Let $\mathscr{H}(3) =: \{H \in \mathscr{H} : |H| = 3\}$.

Proposition 5.12. *Set* $\tau_3 = \tau(\mathscr{H}(3))$. *Then* $\tau_3 \geq t - 1$.

Proof. If we have a $T \subset \bigcup \mathscr{H}$, $|T| \leq t - 2$ such that $|T \cap H| \geq 1$ for all $H \in \mathscr{H}$, $|H| = 3$, then define the function $m: \bigcup \mathscr{H} \to \mathbb{R}$ as follows

$$m(x) = \begin{cases} 1/2 & \text{if } x \in \bigcup \mathscr{H} - T \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu(\mathscr{H}) \geq \sum_x m(x) \geq ((2t+1) - (t-2))/2 = (t+3)/2$. This implies $f(\mathscr{H}) \geq 1/2$. (We used that $v = 2t + 1$.) □

The case $t = 3$. First we prove that there exist $H_1, H_2 \in \mathscr{H}(3)$ such that $|H_1 \cap H_2| = 1$. ($\mathscr{H}(3)$ is intersecting, and $\tau_3 > 1$. Hence, if $\mathscr{H}(3)$ is 2-intersecting then $\mathscr{H}(3) \approx \mathscr{A}$ which leads to a contradiction). Next we prove that $\mathscr{H}(3)$ does not contain a triangle, (i.e., $H^1, H^2, H^3 \in \mathscr{H}(3)$, $|H^i \cap H^j| = 1$ implies $H^1 \cap H^2 \cap H^3 \neq \varnothing$.) Suppose $H_1 = \{1,2,3\}$, $H_2 = \{3,4,5\}$. As $\tau_3 > 1$ we have an $H_3 \in \mathscr{H}(3)$ such that $3 \notin H_3$. Then $H_3 \subset H_1 \cup H_2$, e.g., $H_3 = \{1,2,4\}$. There exists a $H_4 \not\subset H_1 \cup H_2$ (otherwise $m(\mathscr{H}) \geq 5 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = 3$), e.g., $6 \in H_4$. Then $3 \in H_4$ and $H_3 \cap H_4 \neq \varnothing$. If $H_4 = \{1,3,6\}$ then H_2, H_3 and H_4 form a triangle, hence $H_4 = \{3,4,6\}$. $\mathscr{H} - H_1 \approx K_3 + K_1$, whence we have a $H_5 \supset \{5,6\}$. There exists an edge H_6 such that $7 \in H_6$. Then $H_6 \subset \{1,2,7\}$ and $|H_6| \leq 2$. This implies that $H_5 = \{5,6\}$ and $H_6 = \{1,7\}$ or $\{2,7\}$. We obtained the family $\mathscr{D}$. □

The case $t = 4$. Then $v = 9$. Let $H_1 \in \mathscr{H}(3)$. By Proposition 5.11 we have that either

4/1. $\mathscr{H} - H_1 \approx \mathscr{A} + 2K_1$, or

4/2. $\approx \mathscr{B} + K_1$, or

4/3. $\approx \mathscr{F} + K_1$ where $\mathscr{C} \subseteqq \mathscr{F} \subseteqq \mathscr{C}'$, or

4/4. $\approx 2K_3$.

The case 4/1. Let $H_1 = \{1,2,3\}$, and $\mathscr{H} - H_1 \approx \{4\} + \{5\} + \{H_2, H_3, H_4, H_5\}$ where $H_i = \{6,7,8,9\} - \{i+4\}$. Consider $\mathscr{H} - H_i$ $(2 \leq i \leq 5)$. Again we have 4 cases. But 4/4 is impossible ($\mathscr{H} - H_i$ contains a 3-element set), 4/1 is impossible (then 4 or 5 would be an isolated point in $\mathscr{H}$) hence $\mathscr{H} - H_i$ has two components one with 5 elements and the other is an isolated vertex. This isolated vertex cannot be 4 or 5 and in $\{1,2,3\}$, so it is $i + 4$. But then $\mathscr{H}$ has two components $\{1,2,3,4,5\}$ and $\{6,7,8,9\}$, a contradiction. Hence case 4/1 is impossible.

Proposition 5.13. *$\mathscr{H}(3)$ contains two disjoint members.*

Otherwise $\mathscr{H}(3)$ is an intersecting family with $\tau(\mathscr{H}(3)) = 3$ by Proposition 5.12. Hence we can use a theorem of Meyer [16] or Hansen and Toft [12] which says that in this case $|\bigcup\mathscr{H}(3)| \leq 7$. Let $\{1,2\} \subset V(\mathscr{H}) - (\bigcup\mathscr{H}(3))$, and define $m(1) = m(2) = 2/3$, $m(x) = 1/3$ for $x \geq 3$. Then $\sum m(x) \geq 3.66$ a contradiction, except if this m does not fulfil its constraints, i.e., there is an edge $H \in \mathscr{H}$ with $\{1,2\} \subset H$. Then $\mathscr{H}(3)$ is a ≤ 3-expansive family, and so it does not contain 3 members with one common element (H_1, H_2, $H_3 \in \mathscr{H}(3)$, $|H_i \cap H_j| = 1$, $H_1 \cap H_2 \cap H_3 \neq \varnothing$). Then the above mentioned theorem of Hansen and Toft says that $|\bigcup\mathscr{H}(3)| \leq 6$. In this case the following function m: $V(\mathscr{H}) \to \mathbb{R}$ shows that $\mu(\mathscr{H}) \geq 3.5$ contradiction. $m(x) = 1/3$ for $x \in \bigcup\mathscr{H}(3)$, otherwise $m(x) = 1/2$. □

Let $H_1, H_2 \in \mathscr{H}(3)$, $H_1 = \{1,2,3\}$, $H_2 = \{7,8,9\}$. Then $\mathscr{H} - H_i$ $(i = 1,2)$ has two components, one of them an isolated vertex x_i. If $x_1 = x_2$ then it is isolated in $\mathscr{H}$, a contradiction. Let $x_2 = 4$, $x_1 = 6$. Denote $\mathscr{G}(\mathscr{H}) =: \{\{u,v\}: \exists H \in \mathscr{H}$ such that $\{u,v\} \subset H\}$. Clearly, $\{4,6\} \notin \mathscr{G}(\mathscr{H})$.

If $\{4,6\} \cap (\bigcup\mathscr{H}(3)) = \varnothing$ then $\mu(\mathscr{H}) \geq 3 + (2/3)$. Suppose there exists $H_3 \in \mathscr{H}(3)$, $4 \in H_3$. Then $H_3 \subset \{1,2,3,4,5\}$ hence $\mathscr{H}|\{1,2,3,4,5\} \approx \mathscr{B}$. E.g., $H_3 = \{1,2,4\}$ and $\mathscr{H}|\{1,2,3,4,5\} \approx \{\{H_1, H_3\}, \{3,4\}, \{1,5\}, \{2,5\}\} \cup \{\text{some one-element members}\}$. Consider $\mathscr{H} - H_3$. Then 3 is an isolated vertex, hence $\{3,4\} \in \mathscr{H}$. If 6 is not covered by any 3-element members of $\mathscr{H}$ then the following m shows $\mu(\mathscr{H}) \geq 3.5$: $m(1) = m(2) = 1/4$, $m(3) = m(4) = 1/2$, $m(6) = 2/3$, otherwise $m(x) = 1/3$.

If $6 \in (\bigcup\mathscr{H}(3))$ then $(\mathscr{H} - H_1) \approx K_1 + \mathscr{B}$, e.g., $\{6,8,9\} \in \mathscr{H}$. Then the following m shows that $\mu(\mathscr{H}) \geq 3.5$: $\mu(i) = 1/2$ for $3 \leq i \leq 7$, $\mu(j) = 1/4$ otherwise. So we have proved that in every case $\mu(\mathscr{H}) \geq 3.5$, i.e., $f(\mathscr{H}) \geq 1/2$ in the case $t = 4$.

Proposition 5.14. *Suppose $\mathscr{H}$ is t-critical, connected, $|V(\mathscr{H})| = 2t + 1$, $x \in V(\mathscr{H})$. Then the components of $\mathscr{H} - x$ have an even number of vertices.*

Proof. This trivially follows from the second half of Theorem 2.1. □

Proposition 5.15. *Let $t = 5$ or 6. If $\mathscr{H}$ has a vertex x which is not covered by a 3-element member then $f(\mathscr{H}) \geq 1/2$.*

Proof. Let $m(i) = 1/3$ for $i \neq x$ and $m(x) = 2/3$. Then $\mu(\mathscr{H}) \geq \|m\| = (2t+2)/3$ yields $f(\mathscr{H}) \geq (t-2)/6 \geq 1/2$. □

The case $t = 5$. Let $H \in \mathscr{H}(3)$, consider $\mathscr{H} - H$ and apply Proposition 5.11. We have 3 possibilities:

5/1. $\mathscr{H} - H \approx \mathscr{D} + K_1$

5/2. $\approx \mathscr{A} + K_3 + K_1$

5/3. $\approx \mathscr{B}$ or $\mathscr{F}$ (where $\mathscr{C} \subset \mathscr{F} \subset \mathscr{C}'$) $+ K_3$.

First we prove that the case 5/2 is impossible. The proof is similar to the case 4/1. □

Now we prove that the case 5/1 is impossible. Let $H_1 = \{1,2,3\}$, $V(\mathscr{H}) = \{1,\ldots,11\}$, 11 is the isolated point in $\mathscr{H} - H_1$. By Proposition 5.15 we have a 3-element set covering 11, e.g., $H_2 = \{1,2,11\}$. Consider $\mathscr{H} - H_2$. It is isomorphic to $\mathscr{D} + K_1$ as well, hence its isolated vertex is 3. Now we can suppose that $\{4,6,7\}$, $\{5,6,7\}$, $\{6,8,9\}$, $\{7,8,9\} \in \mathscr{H}$ and $\{4,5\}$, $\{8,10\}$ and $\{9,10\} \in \mathscr{G}(\mathscr{H})$. We have a 3-element set $H_3 \in \mathscr{H}$ through 11. If $3 \in H_3$, e.g., $H_3 = \{1,3,11\}$ then considering $\mathscr{H} - H_3 \approx K_1 + \mathscr{D}$, we have that the point 1 is a cutpoint with two odd components, which contradicts to Proposition 5.14. ($\mathscr{H} - \{1\} = \{2,3,11\} \cup \{4,\ldots,10\}$.) Hence $3 \notin H_3$, i.e., $H_3 = \{1, 2, 11\}$. Consider the following function:

i	1	2	3	4	5	6	7	8	9	10	11
$m_1(i)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{2}$

Then $\|m\| > 4$ and $\sum_{x \in E} m(x) \leq 1$ for all $E \in \mathscr{H}$ except if $E \cap \{1,2\} \neq \varnothing$ and $E \supset \{4,5\}$. So we are ready if such an edge E does not exist. Suppose $H_4 = \{1,4,5\} \in \mathscr{H}$. Then $\mathscr{H} - H_4$ has two components ($\{2,3,11\}$ and $\{6,\ldots,10\}$) Consider the following m_2: $m_2(1) = 0$, $m_2(2) = \frac{1}{2}$ and $m_2(i) = m_1(i)$ for $i \geq 3$. Then $\sum_{i \in E} m_2(i) \leq 1$ except for the edge $H_5 = \{2,4,5\}$ (if it exists.) Suppose that $H_5 \in \mathscr{H}$. Then the following function m_3 shows that $\mu(\mathscr{H}) > 4$:

i	1	2	3	4	5	6	7	8	9	10	11
$m_3(i)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{1}{2}$

Now we consider the case 5/3. Let $H_1 = \{1,2,3\} \in \mathscr{H}$, $\mathscr{H} - H_1 \approx K_3 + \mathscr{F}$. Let $H_2 \in \mathscr{F} \cap \mathscr{H}(3)$. Consider $\mathscr{H} - H_2$. It contains a 7- and a 1-element component. But the case 5/1 is impossible, as we have seen above. □

The case $t = 6$. Let $H_1 \in \mathscr{H}(3)$ and apply Proposition 5.11 to the family $\mathscr{H} - H_1$. It is critical 4-expansive on 10 vertices. Consider the components of $\mathscr{H} - H_1 = \mathscr{H}_1 + \cdots + \mathscr{H}_s$ where $\mathscr{H}_i$ is a critical t_i-expansive hypergraph. Apart from the 0's we have 5 possibilities to partition 4 into non-negative integers:

6/1. $4 = 1 + 1 + 1 + 1\,(+0)$

6/2. $\quad = 2 + 1 + 1\,(+0)$

6/3. $\quad = 3 + 1\,(+0)$

6/4. $\quad = 2 + 2\,(+0)$

6/5. $\quad = 4\,(+0)$.

By Proposition 5.6 we have that $f(\mathscr{H}) \geq \sum_{i=1}^{s} f(\mathscr{H}_i)$. Hence using the previous results we have in the case

6/5. $f(\mathscr{H}) \geq 1/2\,(+0)$

6/4. $f(\mathscr{H}) \geq 1/4 + 1/4\,(+0)$

6/1. $|V(\mathscr{H} - H_1)| \geq 4 \cdot 3 = 12$

which are contradictions. In the case 6/2 we have $\mathscr{H} - H_1 \approx \mathscr{A} + 2K_3$. Let $H_2 \in \mathscr{A}$ and consider $\mathscr{H} - H_2$. Then (because only the cases 6/2 or 6/3 are possible) we have that at least one of the K_3's remains separated. This contradicts the connectivity of $\mathscr{H}$.

Finally in the case 6/3 we have $\mathscr{H} - H_1 \approx \mathscr{D} + K_3$. Suppose that there is an edge H_2 with 3 elements such that $|V(K_3) \cap H_2| \geq 2$. E.g., let $H_1 = \{1, 2, 3\}$, $K_3 = \{\{11, 12\}, \{12, 13\}, \{13, 11\}\}$ and $H_2 = \{1, 11, 12\}$. Then $\mathscr{H} - H_2 \approx \mathscr{D} + K_3$ hence $\mathscr{H} - \{1\}$ has two odd components ($\{2, 3, 11, 12, 13\}$ and $\{4, \ldots, 10\}$).

This contradicts Proposition 5.14. So we can suppose that $|H \cap \{11, 12, 13\}| \leq 1$ for $H \in \mathscr{H}(3)$. Then the following m shows that $\mu(\mathscr{H}) \geq 55/12 > 4.5$. $m(1) = m(2) = m(3) = 1/4$, $m(11) = m(12) = m(13) = 1/2$ and $m(i) = 1/3$ otherwise. □

Conjecture 5.16. Suppose that $\mathscr{H}$ is connected and t-expansive. Then $f(\mathscr{H}) \geq (t/6) - (1/9) + (1/9)(-1)^t 2^{-t}$.

This is best possible (if true) as the following example shows.

Example 5.17. Let $\mathscr{H}_t = \{\{1, 2\}, \{2t - 1, 2t + 1\}, \{2t, 2t + 1\}\} \cup \{\{i, 2j - 1, 2j\} : i = 2j - 2 \text{ or } 2j - 3,\ 2 \leq j \leq t\}$. Then $\mathscr{H}_t$ is critical t-expansive, $\mu(\mathscr{H}_t) = \frac{2}{3}t + \frac{8}{9} + \frac{1}{9}(-1)^t \frac{1}{2^t}$.

Define $\mu_t = \inf\{\mu(\mathscr{H}) : \mathscr{H} \text{ is } t\text{-expansive, connected}\}$. Then we can prove the following proposition which can help for the proof of Proposition 5.16.

Proposition 5.18. *There exists a t-expansive, connected hypergraph $\mathscr{H}$ with $\mu(\mathscr{H}) = \mu_t$ and $|\mathscr{H}| \leq |V(\mathscr{H})|$.*

6. The Proof of Theorem 3.8

First we need some definitions.

We call *edge-contraction* the following operation on a family $\mathscr{H}$: we replace an edge $E \in \mathscr{H}$ by a smaller, non-empty set $E' \subsetneq E$, and thus we get the family $\mathscr{H} -$

$\{E\} \cup \{E'\}$. A t-wise intersecting family is *$\hat{t}$-critical* if it has no multiple edges and the hypergraph obtained by contracting any of its edges is not t-wise intersecting. That is:

$$\begin{aligned}&\text{For all } E \in \mathscr{H},\ x \in E \text{ there exists } H_1, \ldots, H_{t-1} \in \mathscr{H}\\ &\text{such that } E \cap H_1 \cap \cdots \cap H_{t-1} = \{x\}.\end{aligned} \tag{6.1}$$

We can get a $\hat{t}$-critical system from any t-wise intersecting set-system by contracting its edges as far as possible and deleting all but one copy of the appearing multiple edges. Of course, a $\hat{t}$-critical hypergraph does not contain two edges, E, F such that $E \subset F$.

Remark 6.2. If $\mathscr{H}$ is $\hat{t}$-critical hypergraph of rank r then $|\mathscr{H}| \leq r^{2r}$.

Proof. We say that the sets $E_1, E_2, \ldots, E_k$ form a *k-star* with kernel A if $E_i \cap E_j = A$ holds for all $1 \leq i < j \leq k$. Obviously, $\mathscr{H}$ does not contain an $(r+1)$-star as subsystem. Hence we can apply the following theorem of Erdös and Rado [8]: If the family $\mathscr{H}$ of rank r does not contain a k-star then $|\mathscr{H}| \leq r!(k-1)^r$. □

For $t = 2$ Erdös and Lovász [7] proved that $[r!e] \leq \max\{|\mathscr{H}|: \mathscr{H}$ is $\hat{2}$-critical of rank $r\} \leq r^r$.

Problem 6.3. Determine or estimate $\max|\mathscr{H}|$ where $\mathscr{H}$ is $\hat{t}$-critical hypergraph of rank r.

N. Alon and the author [1] have more results on this problem.

Now we are ready to prove Theorem 3.8. Let $\mathscr{H}$ be a t-wise intersecting hypergraph of rank r such that $\tau^*(\mathscr{H}) > \tau^*(r-1, t)$. Contracting its edges we get a $\hat{t}$-critical hypergraph $\mathscr{H}_0$. Our next aim is:

Proposition 6.4. *$\mathscr{H}_0$ is isomorphic to one of the Examples* 3.2–3.7.

This Proposition is the crucial point of the proof. Its proof consists of investigating several cases.

$\tau^*(\mathscr{H}_0) \geq \tau^*(\mathscr{H})$ because a fractional cover $t: (\bigcup \mathscr{H}_0) \to \mathbb{R}$ covers $\mathscr{H}$, too. Hence $\tau^*(\mathscr{H}_0) > \tau^*(r-1, t)$, thus there exists an edge $F \in \mathscr{H}_0$, $|F| = r$. Let us define $V = \bigcup \mathscr{H}_0$ and set $\mathscr{H}_1 = \{F - F': F' \in \mathscr{H}_0, F' \neq F\}$.

Proposition 6.5. *$\mathscr{H}_1$ is a critical $(r-t)$-expansive hypergraph.*

Proof. Let $H_1, H_2, \ldots, H_{t-1} \in \mathscr{H}_1$ and let F_i be an edge from $\mathscr{H}_0$ such that $F - F_i = H_i$. $\mathscr{H}_0$ is t-wise intersecting, hence $F \cap F_1 \cap \cdots \cap F_{t-1} \neq \varnothing$ holds, thus we get $H_1 \cup \cdots \cup H_{t-1} \subsetneq F$, i.e., $\mathscr{H}_1$ is at most $(r-t)$-expansive. Now let $x \in F$ arbitrary. By (6.1) there exist edges $F_1, \ldots, F_{t-1}$ such that $F \cap F_1 \cap \cdots \cap F_{t-1} = \{x\}$. Then $\bigcup(F - F_i) = F - \{x\}$, i.e., $\{F - F_i: 1 \leq i \leq t-1\}$ is an $(r-t)$-expansive subsystem of $\mathscr{H}_1$ outside the point x. □

Apply Corollary 2.2 for the hypergraph $\mathscr{H}_1$. We get

$$\left|\bigcup\{H \in \mathscr{H}_1: |H| \geq 2\}\right| \leq 3(r-t). \tag{6.6}$$

$3(r-t) < r$, so we get that $\mathscr{H}_1$ contains isolated point. We distinguish two cases.

Case 1. $\mathscr{H}_1$ contains only one isolated point. Denote it by p. We get $r - 3(r-t) \leq 1$, i.e., $r = (3t-1)/2$. Moreover equality holds in (6.6) thus $\mathscr{H}_1$ consists of $(t-1)/2$ triangles and the isolated point p by Corollary 2.2.

Let us denote by $F_1, F_2, \ldots, F_l$ the edges of $\mathscr{H}_0$ such that $F - F_j = \{p\}$. Let $F_j = (F \cap F_j) \cup \{x_j\}$. We claim that the point x_j is covered by all of the edges of $\mathscr{H}_0 - \{F_1, \ldots, F_l\}$. Indeed, knowing the structure of $\mathscr{H}_1$, we can see that for each edge $F^1 \in \mathscr{H}_0 - \{F_1, \ldots, F_l\}$ we can choose edges $F^2, F^3, \ldots, F^{t-1}$ such that

$$\bigcap_{1 \le i \le t-1} F^i \cap F = \{p\}.$$

Replace F by F_j. We have $\bigcap_{1 \le i \le t-1} F^i \cap F_j \ne \varnothing$, hence it is $\{x_j\}$. So $x_j \in F^1$ follows. Hence we get $l \le 2$, and we obtain Example 3.2 or 3.3 according to $l = 1$ or 2.

Case 2. $\mathscr{H}_1$ contains at least two isolated points. We claim that

$$|\bigcup \mathscr{H}_0| = |V| = r + 1. \tag{6.7}$$

Denote two of the isolated points by p_1 and p_2, and let $\mathscr{F}_i = \{H \in \mathscr{H}_0 \colon H \cap F = F - \{p_i\}\}, i = 1, 2$. We prove that $|\mathscr{F}_2| = 1$. (Then, similarly $|\mathscr{F}_1| = 1$ holds.) Indeed, let $F_1 = F - \{p_1\} \cup \{x\}$. Apply (6.1) for the edge F_1 at the point x. We get $F^2, \ldots, F^t$ such that $F_1 \cap F^2 \cap \cdots \cap F^t = \{x\}$. Among the sets $F^2, \ldots, F^t$ there exists an edge, e.g. F^2, such that $p_2 \notin F^2$. Then $F^2 \in \mathscr{F}_2$. Suppose for contradiction, that there exists an edge $\mathscr{G}^2 \in \mathscr{F}_2 - \{F^2\}$. We get that $F_1 \cap G^2 \cap F^3 \cap \cdots \cap F^t = \varnothing$, which is a contradiction.

So we have $|\mathscr{F}_2| = 1$ holds, i.e., all edge of $\mathscr{H}_0$ contains the point p_2, except F^2. Because $\mathscr{H}_0$ is $\hat{t}$-critical, this implies that $\bigcup \mathscr{H}_0 = V = F^2 \cup \{p_2\}$. I.e., $|V| = r + 1$. □

Let $V = F \cup \{p\}$. Now we prove that $\mathscr{H}_0$ is isomorphic to one of the Example 3.2 or 3.4–3.7.

Consider the complement of $\mathscr{H}_0$, $\mathscr{H}^c = \{V - F \colon F \in \mathscr{H}_0\}$. Let $\mathscr{H}^c = \mathscr{H}_1 + \cdots + \mathscr{H}_s$. $\mathscr{H}^c$ is $(r-t)$-expansive and by Proposition 5.1 we have

$$\tau^*(\mathscr{H}) = \textstyle\sum/(\sum - 1)$$

where $\sum = \sum_i \mu(\mathscr{H}_i)$.

Hence we have

$$\begin{aligned} \tau^*(\mathscr{H}) &= 1 + 1\Big/\left(\sum_i \left(\mu(\mathscr{H}_i) - |V(\mathscr{H}_i)| + \frac{3}{2}t_i\right) + |V(\mathscr{H})| - \frac{3}{2}(r-t)\right) \\ &= 1 + 2/\left(3t - r + 2\textstyle\sum f(\mathscr{H}_i)\right). \end{aligned} \tag{6.8}$$

Lemma 5.3 immediately implies that $\tau^*(\mathscr{H}) \le 1 + 2/(3t - r)$, i.e. $\tau^*(r,t) = 1 + 2/(3t - r)$, proving Theorem 3.1 in the case $r < (3t-1)/2$. Now suppose that $\tau^*(\mathscr{H}) > \tau^*(r-1,t) = 1 + 2/(3t - r + 1)$. Then (6.8) implies that

$$\sum_{i=1}^{s} f(\mathscr{H}_i) < 1/2.$$

Using Lemma 5.4 we have that every but at most one $\mathscr{H}_i$ is isomorphic to K_1 or K_3 and the exceptional is isomorphic to one of $\mathscr{A}$, $\mathscr{B}$, $\mathscr{F}$ (where $\mathscr{C} \subseteq \mathscr{F} \subseteq \mathscr{C}'$), or $\mathscr{D}$. Hence $\mathscr{H}_0$ is isomorphic to one of the Examples 3.2, 3.4–7. Finally, Theorem 3.8 follows from the following fact:

Proposition 6.9. Let $\mathscr{H}$ be a t-wise intersecting hypergraph of rank r given by one of the Examples 3.2–7. Let $E \in \mathscr{H}$, $|E| < r$, $x \in V(\mathscr{H}) - E$ and replace the edge E by $E \cup \{x\}$. Then for the obtained hypergraph $\mathscr{H}' = \mathscr{H} - \{E\} \cup \{E \cup \{x\}\}$ we have $\tau^*(\mathscr{H}') \leq \tau^*(r-1, t)$.

Proof. It is an easy calculation. □

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