A LOWER BOUND FOR THE CARDINALITY OF A MAXIMAL FAMILY OF MUTUALLY INTERSECTING SETS OF EQUAL SIZE

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Let $\mathcal{A}$ be a collection of k-subsets (called *lines*) of a set V (of *points*). If every point lies on at least one line and any two lines intersect in at least one point, then we call $\Sigma = (V, \mathcal{A})$ a *k-clique*. A k-clique is said to be *maximal* if it cannot be extended to another k-clique by adding a new line (and possibly new points). A subset B of V is called a *blocking set* of Σ if $\emptyset \neq B \cap A \neq A$ for every line A. Thus a k-clique is maximal if and only if it contains no blocking set of k or fewer points.

Erdös and Lovász [1] have given bounds for the minimum number $m(k)$ and the maximum number $M(k)$ of lines in a maximal k-clique. In particular, Theorem 10 of [1] states that $m(k) \geq (8k/3)-3$. The purpose of this note is to improve this lower bound by proving the following theorem.

THEOREM. *For all $k \geq 4$, $\quad m(k) \geq 3k$.*

J.C. Meyer [3] has observed that $m(1) = 1$, $m(2) = 3$ and $m(3) = 7$; so the restriction $k \geq 4$ is essential. Füredi [2, Theorem 1] has proved that $m(k) \leq 3k^2/4$ whenever $k = 2n$ for an integer n that is the order of a projective plane. Thus our theorem yields $m(4) = 12$. The value of $m(k)$ is still not known for $k > 4$.

PROOF OF THEOREM: Let Σ be a maximal k-clique with $k \geq 4$ and let A be a line of Σ. For each $x \in A$, the set $A\backslash\{x\}$ is not a blocking set, so there is a line B such that $A \cap B = \{x\}$. If there is only one line B such that $A \cap B = \{x\}$, then $(A\backslash\{x\}) \cup \{y\}$ is a line for every $y \in B\backslash\{x\}$. Thus for each $x \in A$, either

(1) there are exactly two lines B such that $A \cap B = \{x\}$, or
(2) there are at least three lines B such that $A \cap B = \{x\}$ or $A \cap B = A\backslash\{x\}$.

Let S be the set of points of A satisfying (1), $|S| = s$. Then there are at least $3k - s$ lines B such that $|A \cap B| = 1$ or $k-1$. If $s \leq 1$ then we are done, so assume that $s \geq 2$ and let x and y be distinct points of S. Let B_1 and B_2 be the lines that meet A in x alone, C_1 and C_2 be the lines that meet A in y alone, and let $x_i \in B_i \cap C_i$ for $i = 1$ and 2. Either there is a line B such that $A \cap B = \{x, y\}$ or $(A\backslash\{x, y\}) \cup \{x_1, x_2\}$

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is a line, since otherwise the latter set is a blocking set of size k or less. Unless $k = s = 4$, the pairs of points of S give rise to $\binom{s}{2}$ distinct lines B such that $|A \cap B| = 2$ or $k - 2$, and the total number of lines of Σ is at least $3k - s + \binom{s}{2} + 1 \geq 3k$. Finally, in the case $k = s = 4$, there must be at least three distinct lines B such that $|A \cap B| = 2$, so the total number of lines is at least $3k - s + 3 + 1 = 12$.

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