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Note

Non-trivial Intersecting Families

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The Erdös–Ko–Rado theorem states that if $\mathbf{F}$ is a family of k-subsets of an n-set no two of which are disjoint, $n \geqslant 2k$, then $|\mathbf{F}| \leqslant \binom{n-1}{k-1}$ holds. Taking all k-subsets through a point shows that this bound is best possible. Hilton and Milner showed that if $\bigcap \mathbf{F} = \varnothing$ then $|\mathbf{F}| \leqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ holds and this is best possible. In this note a new, short proof of this theorem is given. © 1986 Academic Press, Inc.

1. INTRODUCTION

Suppose X is an n-element set and $\mathbf{F}$ is a family of k-subsets of X. The family $\mathbf{F}$ is called *intersecting* if $F \cap F' \neq \varnothing$ holds for all $F, F' \in \mathbf{F}$. For $n < 2k$ every $\mathbf{F}$ is intersecting. From now on assume $n \geqslant 2k$.

If all members of $\mathbf{F}$ contain a fixed element of X then, obviously, $\mathbf{F}$ is intersecting. Such a family is called *trivial*. Clearly, a trivial intersecting family has at most $\binom{n-1}{k-1}$ members.

ERDÖS–KO–RADO THEOREM [1]. *If* $n \geqslant 2k$, $\mathbf{F}$ *is intersecting then* $|\mathbf{F}| \leqslant \binom{n-1}{k-1}$ *holds.*

EXAMPLE 1. Take $F_1 \subset X$, $|F_1| = k$ and $x_1 \in X - F_1$. Define $\mathbf{F}_1 = \{F_1\} \cup \{F \subset X: x_1 \in F, |F| = k, F \cap F_1 \neq \varnothing\}$. It is easily checked that $\mathbf{F}_1$ is intersecting and $|\mathbf{F}_1| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.

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EXAMPLE 2. Take $F_2 \subset X$, $|F_2| = 3$ and define $\mathbf{F}_2 = \{F \subset X; |F| = k, |F \cap F_2| \geqslant 2\}$. Again, $\mathbf{F}_2$ is intersecting. For $k = 2$, $\mathbf{F}_1 = \mathbf{F}_2$ while for $k = 3$, $|\mathbf{F}_1| = |\mathbf{F}_2|$ hold. If $n > 2k$ and $k \geqslant 4$ then $|\mathbf{F}_1| > |\mathbf{F}_2|$.

HILTON–MILNER THEOREM [4]. *If $n > 2k$ and $\mathbf{F}$ is a non-trivial intersecting family then $|\mathbf{F}| \leqslant |\mathbf{F}_1|$ holds. Moreover, equality is possible only for $\mathbf{F} = \mathbf{F}_1$ or $\mathbf{F} = \mathbf{F}_2$, the latter occurs only for $k \leqslant 3$.*

Note that this theorem shows in a strong way that only trivial families attain equality in the Erdös–Ko–Rado theorem. The proof of the Hilton–Milner theorem is rather long and complicated. The aim of this note is to give a more concise argument.

2. THE NEW PROOF OF THE HILTON–MILNER THEOREM

Suppose for simplicity the elements of X are linearly ordered. Let $\mathbf{F}$ be a non-trivial intersecting family of maximal size. We prove the statement by induction on k. If $k = 2$, then $\mathbf{F}$ consists necessarily of the three edges of a triangle. For $x, y \in X$, $x < y$ we define $S_{xy}(\mathbf{F}) = \{S_{xy}(F): F \in \mathbf{F}\}$, where

$$S_{xy}(F) = (F - \{y\}) \cup \{x\} \qquad \text{if } x \notin F,\ y \in F,\ (F - \{y\}) \cup \{x\} \notin \mathbf{F}$$
$$= F \qquad \text{otherwise.}$$

PROPOSITION 2.1 (see [1]). *$|S_{xy}(\mathbf{F})| = |\mathbf{F}|$ and $S_{xy}(\mathbf{F})$ is intersecting.* ∎

Apply repeatedly the operation S_{xy} to $\mathbf{F}$ until we obtain either a family $\mathbf{H}$ such that $S_{xy}(\mathbf{H})$ is trivial or a family $\mathbf{G}$ which is *stable*, i.e., $S_{xy}(\mathbf{G}) = \mathbf{G}$ holds for all $x < y$. In the second case we define $X_0 = \varnothing$ in the first $X_1 = \{x, y\}$. Then $H \cap X_1 \neq \varnothing$ holds for all $H \in \mathbf{H}$. The maximality of $|\mathbf{H}|$ implies that all k-subsets containing X_1 are in $\mathbf{H}$. Now apply repeatedly S_{xy} to $\mathbf{H}$ for $x < y, x, y \in (X - X_1)$. Since the sets containing X_1 stay fixed, finally we obtain a family $\mathbf{G}$, satisfying:

(1) $G \cap X_1 \neq \varnothing$ for all $G \in \mathbf{G}$,

(2) $S_{xy}(\mathbf{G}) = \mathbf{G}$ for $x, y \in (X - X_1)$, $x < y$.

For $i = 0, 1$ let Y_i be the set of first $2k - 2i$ elements of $X - X_i$, $Y = X_i \cup Y_i$.

LEMMA 2.2. *For all $G, G' \in \mathbf{G}$, $G \cap G' \cap Y \neq \varnothing$ holds.*

Proof. Consider first the case $Y = X_1 \cup Y_1$. Suppose for contradiction $G \cap G' \cap Y = \varnothing$ and $G, G' \in \mathbf{G}$ are such that $|G \cap G'|$ is minimal. Now (1) implies that G and G' intersect X_1 in different elements. Thus $G - X_1$,

$G' - X_1$ are $(k-1)$-sets. Since $G \cap G' \cap (X - Y) \neq \emptyset$, we may choose $x \in Y$, $x \notin G \cup G'$, $y \notin Y$, $y \in G \cap G'$. Then (2) implies $(G' - \{y\}) \cup \{x\} =^{\text{def}} G'' \in \mathbf{G}$. However, $G \cap G'' \cap Y = \emptyset$ and $|G \cap G''| < |G \cap G'|$, a contradiction.

The case $Y = X_0 \cup Y_0$ is similar but easier (cf. [2]). ∎

Let us define $\mathbf{A}_i = \{G \cap Y \colon G \in \mathbf{G},\ |G \cap Y| = i\}$.

LEMMA 2.3.

$$|\mathbf{A}_i| \leqslant \binom{2k-1}{i-1} - \binom{k-1}{i-1} \qquad \text{for} \quad 1 \leqslant i \leqslant k-1$$

and

$$|\mathbf{A}_k| \leqslant \binom{2k-1}{k-1} - \binom{k-1}{k-1} + 1 = \binom{2k-1}{k-1}.$$

Proof. Consider first the case $2 \leqslant i \leqslant k-1$. Suppose for contradiction

$$|\mathbf{A}_i| > \binom{2k-1}{i-1} - \binom{k-1}{i-1} \geqslant \binom{2k-1}{i-1} - \binom{2k-i-1}{i-1} + 1.$$

In view of Lemma 2.2, $\mathbf{A}_i$ is intersecting. Thus the induction hypothesis yields that $\mathbf{A}_i$ is trivial, say $x \in \bigcap \mathbf{A}_i$. As $\mathbf{G}$ is nontrivial, we may choose $G \in \mathbf{G}$, $x \notin G$. By Lemma 2.2 $A \cap G \neq \emptyset$ holds for all $A \in \mathbf{A}_i$. Consequently, $|\mathbf{A}_i| \leqslant \binom{2k-1}{i-1} - \binom{k-1}{i-1}$ holds, as desired. The case $i = 1$, i.e, $\mathbf{A}_1 = \emptyset$, is obvious.

$|\mathbf{A}_k| \leqslant \binom{2k-1}{k-1} = \frac{1}{2}\binom{2k}{k}$ follows easily from the fact that $\mathbf{A}_k$ is intersecting and therefore $A \in \mathbf{A}_k$ implies $(Y - A) \notin \mathbf{A}_k$. ∎

Since for a fixed $A \in \mathbf{A}_i$ there are at most $\binom{n-2k}{k-i}$ k-element sets G with $G \cap Y = A$, we infer

$$|\mathbf{G}| \leqslant \sum_{i=1}^{k} |\mathbf{A}_i| \binom{n-2k}{k-i} \leqslant 1 + \sum_{i=1}^{k} \left(\binom{2k-1}{i-1} - \binom{k-1}{i-1} \right) \binom{n-2k}{k-i}$$
$$= 1 + \binom{n-1}{k-1} - \binom{n-k-1}{k-1} = |\mathbf{F}_1|,$$

proving the inequality part of the Theorem.

To have equality we must have equality in Lemma 2.3, in particular $|\mathbf{A}_2| = \binom{2k-1}{1} - \binom{k-1}{1} = k$. As $\mathbf{A}_2$ is intersecting either it is a k-star or $k = 3$ and it is a triangle. In the second case $\mathbf{G} \subseteq \mathbf{F}_2$ is immediate. In the first case let $\mathbf{A}_2 = \{\{x_1, x_2\}, \dots, \{x_1, x_{k+1}\}\}$. If $G \in \mathbf{G}$, $x_1 \notin G$ then necessarily $G = \{x_2, \dots, x_{k+1}\}$, i.e., all other members of $\mathbf{G}$ contain x_1 and intersect G, proving $\mathbf{G} \subseteq \mathbf{F}_1$. Recall that $\mathbf{G}$ was obtained from $\mathbf{F}$ by a series of exchange operations $S_{x,y}$. It is easy to check that if $\mathbf{H}$ is intersecting and $S_{xy}(\mathbf{H}) = \mathbf{F}_i$ then $\mathbf{H}$ is isomorphic to $\mathbf{F}_i$, too ($i = 1, 2$). Consequently, $\mathbf{F}$ is isomorphic to either $\mathbf{F}_1$or $\mathbf{F}_2$. ▌

3. Further Problems

If for $F, F' \in \mathbf{F}$ $|F \cap F'| \geqslant t$ holds then $\mathbf{F}$ is called t-intersecting, $t \geqslant 2$.

Theorem 3.1 (Erdös–Ko–Rado [1]). *Suppose $n \geqslant n_0(k, t)$, $\mathbf{F}$ is t-intersecting then $|\mathbf{F}| \leqslant \binom{n-t}{k-t}$.*

The best possible bound for $n_0(k, t)$ is $(k-t+1)(t+1)$ as was shown by Frankl [2] for $t \geqslant 15$ and very recently by Wilson [5] for all t. They showed that for $n > (k-t+1)(t+1)$ equality holds only if $\mathbf{F}$ consists of all k-subsets containing a fixed t-subset. Again, such an $\mathbf{F}$ is called trivial.

Examples of non-trivial t-intersecting families are $\mathbf{F}_1 = \{F \subset X, |F| = k: (Y_0 \subset F, \; Y_1 \cap F \neq \emptyset) \text{ or } (|Y_0 \cap F| = t-1, Y_1 \subset F)\}$, where $|Y_0| = t$, $|Y_1| = k - t + 1$, $Y_0 \cap Y_1 = \emptyset$, and $\mathbf{F}_2 = \{F \subset X, |F| = k: |F \cap Y_2| \geqslant t+1\}$, where $|Y_2| = t + 2$.

Theorem 3.2 ([3]). *Suppose $\mathbf{F}$ is a non-trivial t-intersecting family, $n > n_1(k, t)$. Then $|\mathbf{F}| \leqslant \max\{|\mathbf{F}_1|, |\mathbf{F}_2|\}$. Moreover, equality holds if and only if either $\mathbf{F} = \mathbf{F}_1$, $k > 2t+1$ or $\mathbf{F} = \mathbf{F}_2$, $k \leqslant 2t+1$.*

It would be interesting to know whether $n_1(k, t) < ckt$ holds.

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