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Separating Pairs of Points by Standard Boxes

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Let A be a set of distinct points in $\mathbb{R}^d$. A 2-subset $\{a, b\}$ of A is called separated if there exists a closed box with sides parallel to the axes, containing a and b but no other points of A. Let $s(A)$ denote the number of separated 2-sets of A and put $f(n, d) = \max\{s(A): A \subset \mathbb{R}^d, |A| = n\}$. We show that $f(n, 2) = [n^2/4] + n - 2$ for all $n \geq 2$ and that for each fixed dimension d

$$f(n, d) = (1 - 1/2^{2^{d-1}-1}) \cdot n^2/2 + o(n^2).$$

1. Introduction

Let A be a set of distinct points in the Euclidean d-dimensional space $\mathbb{R}^d$. For $a, b \in \mathbb{R}^d$ let $B(a, b)$ denote the minimal closed box with sides parallel to the axes, containing a and b. A 2-subset $\{a, b\}$ of A is called *separated* (in A) if $A \cap B(a, b) = \{a, b\}$. Let $s(A)$ denote the number of separated 2-sets of A and put

$$f(n, d) = \max\{s(A): A \subset \mathbb{R}^d, |A| = n\}.$$

In this paper we study $f(n, d)$. Obviously $f(n, 1) = n - 1$ for all $n \geq 1$. In Section 2 we observe that $f(n, d) = \binom{n}{2}$ if and only if $n \leq 2^{2^{d-1}}$. In Section 3 we prove:

Theorem 1.1.

$$f(n, 2) = [n^2/4] + n - 2 \text{ for all } n \geq 2.$$

In Section 4 we prove the following theorem, which determines the asymptotic behavior of $f(n, d)$ for every fixed d as $n \to \infty$.

Theorem 1.2. *For every fixed $d \geq 1$,*

$$f(n, d) = \left(1 - \frac{1}{2^{2^{d-1}-1}}\right) \cdot \frac{n^2}{2} + o(n^2).$$

Our proofs use a generalization of de Bruijn to a theorem of Erdös and Szekeres on monotone sequences and several results of Bollobás, Erdös, Straus and Stone in extremal graph theory.

A related problem to the one considered here is discussed by Bárány and Lehel in [1].

2. Preliminaries

For a finite set A of points in $\mathbb{R}^d$, let $G(A)$ denote the *separation* graph of A, i.e., the graph on the set of vertices A in which $a, b \in A$ are joined iff $\{a, b\}$ is separated. A sequence $a_1, a_2, \ldots, a_k$ of points in $\mathbb{R}^d$ is called *monotone* if it is (weakly) monotone in each of its coordinates.

Clearly:

$$a, b \in A \text{ are separated iff there is no } c \in A - \{a, b\} \text{ such that } (a, c, b) \text{ is monotone.} \tag{2,1}$$

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Erdös and Szekeres [5] proved that every sequence of l^2+1 real numbers contains a (weakly) monotone subsequence of $l+1$ terms. Furthermore, the number l^2+1 is best possible. In an unpublished work (cf. [7]), de Bruijn has generalized this result and proved:

LEMMA 2.1 (DE BRUIJN). *Every sequence of $l^{2^d}+1$ vectors in $\mathbb{R}^d$ contains a monotone subsequence of size $l+1$. Furthermore, the number $l^{2^d}+1$ is the best possible.*

De Bruijn's proof is simply an iterated application of Erdös-Szekeres' result. Another proof (of a more general result) is given by Kruskal [7]. To show that $l^{2^d}+1$ is best possible one has to construct a sequence $v_1, v_2, \ldots, v_m$ $(m=l^{2^d})$ of vectors in $\mathbb{R}^d$, which contains no monotone subsequence of $>l$ terms. Since such a construction does not appear in [7] we describe it here. For $d=1$ let the sequence be

$$(l-1)l+1, (l-1)l+2, \ldots, l^2, (l-2)\cdot l+1, (l-2)\cdot l+2, \ldots, (l-1)\cdot l, \ldots, 1, 2, \ldots, l.$$

Assuming we have a sequence $v_1, \ldots, v_m$ $(m=l^{2^d})$ of vectors in $\mathbb{R}^d$ without a monotone subsequence of $>l$ terms and without two vectors sharing a common value at some coordinate let us construct a sequence $u_1, \ldots, u_{m^2}$ of m^2 vectors in $\mathbb{R}^{d+1}$ having the same properties. The $d+1$st coordinate of the u_i-s is given by:

$$(m-1)\cdot m+1, (m-1)m+2, \ldots, m^2, (m-2)\cdot m+1, (m-2)m+2, \ldots, (m-1)m, \ldots, 1, 2, \ldots, m.$$

The first d coordinates are given by:

$$xv_1+v_1, xv_1+v_2, \ldots, xv_1+v_m, xv_2+v_1, xv_2+v_2, \ldots, xv_2+v_m, \ldots, xv_m+v_1, \ldots, xv_m+v_m$$

where x is a large constant. One can easily check that if x is large enough, the sequence $u_1, \ldots, u_{m^2}$ has the desired properties.

An easy consequence of Lemma 2.1 is the following (see also [9]):

PROPOSITION 2.2.

$$f(n,d)=\binom{n}{2} \qquad \text{if and only if} \quad n\le 2^{2^{d-1}}.$$

PROOF. Let A be a set of $n>2^{2^{d-1}}$ points in $\mathbb{R}^d$. Arrange the points of A in a sequence so that the first coordinate is monotone. By Lemma 2.1, applied to the last $d-1$ coordinates of the points, A contains a monotone subsequence of three points. Thus, by (2.1), $s(A)<\binom{n}{2}$.

Conversely, if $n\le 2^{2^{d-1}}$ let $\nu_1, \nu_2, \ldots, \nu_n$ be a sequence of n points in $\mathbb{R}^{d-1}$ with no monotone subsequence of size >2. For $1\le i\le n$ define $u_i\in\mathbb{R}^d$ by $u_i=(i, \nu_i)$ and put $A=\{u_1, u_2, \ldots, u_n\}$. By (2.1) $s(A)=\binom{n}{2}$.

3. THE PLANAR CASE

In this section we prove Theorem 1.1, which determines $f(n, 2)$ precisely for all n.

Let A be a strictly decreasing sequence of $[n/2]$ points in the first quadrant, let B be a strictly decreasing sequence of $[n/2]$ points in the third quadrant and define $A_0=A\cup B$, see Figure 1. For any $a\in A$ and $b\in B$ there is a rectangle with sides parallel to the axes that contains exactly a and b, so that a and b are separated. There are $[n^2/4]$ such pairs of points. In addition there are exactly $n-2$ pairs of consecutive $a-s$ in A or consecutive

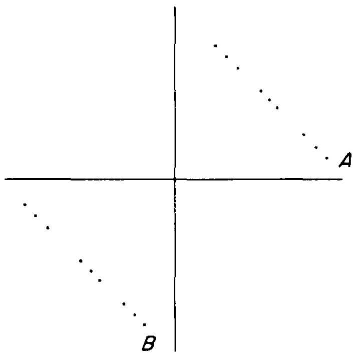

FIGURE 1.

bs in B which are separated. It follows that

$$s(A_0) = \left[\frac{n^2}{4}\right] + n - 2$$

so that

$$f(n, 2) \geq \left[\frac{n^2}{4}\right] + n - 2.$$

It remains to show that for any configuration A of $n \geq 2$ points in the plane

$$s(A) \leq n^2/4 + n - 2.$$

The proof is by induction on n. Since the cases $n = 2, 3$ and 4 are obvious, assume that $n \geq 5$.

Let a be a point of A whose x-coordinate is minimal, let N be all the points of A adjacent to a [in $G(A)$], and let b be the point of N whose x-coordinate is maximal.

Let $N_1(N_2)$ be the points of N with y coordinate not less (less) than the y-coordinate of a. From (2.1) it follows that if the points of N_1 (of N_2) are arranged according to increasing x-coordinate then the y-coordinates form a decreasing (increasing) sequence. This implies by (2.1) that b is adjacent in $G(A)$ to at most two points in N (at most one in each of N_1 and N_2).

Thus the number of edges in $G(A)$ adjacent to either a or b is at most $(n-2)+2+1 = n+1$, so that

$$s(A) \leq s(A \backslash \{a, b\}) + n + 1.$$

By the induction hypothesis $s(A\backslash\{a, b\}) \leq (n-2)^2/4 + (n-2) - 2$. Thus $s(A) \leq (n-2)^2/4 + (n-2) - 2 + n + 1 = n^2/4 + n - 2$. This completes the proof.

4. THE GENERAL CASE

In this section we prove Theorem 1.2, (actually we prove a slightly stronger assertion). For natural numbers $q \leq n$, let $T(q, n)$ denote the complete q-partite graph with $[(n+i)/q]$ vertices in the ith class $(0 \leq i < q)$. Note that $T(q, n)$ is the unique complete q-partite graph of order n whose color classes are as equal as possible. Let $t(q, n) = \sum_{0 \leq i < j < q} [(n+i)/q] \cdot [(n+j)/q]$ be the number of edges of $T(q, n)$. As is easily checked (see, e.g., [2, p. 327])

$$t(q, n) = \tfrac{1}{2}(1 - 1/q)n^2 + O(n).$$

We will show:

LEMMA 4.1. *Put* $q = 2^{2^{d-1}-1}$. *If* $n \geq q$ *then*

$$f(n, d) \geq t(q, n) + n - q.$$

This clearly implies the lower bound in Theorem 1.2. To prove Lemma 4.1 we need a simple consequence of de Bruijn's result (Lemma 2.1). If $u, v \in \mathbb{R}^d$, let $u \leq v (u < v)$ denote that the ith coordinate of v is not smaller (strictly greater, respectively) than the ith coordinate of u for all $1 \leq i \leq d$.

LEMMA 4.2. *Suppose* $l, d \geq 2$ *and put* $s = l^{2^{d-1}-1}$. *Then there is a set A of s points in* $\mathbb{R}^d$, *such that A contains no monotone sequence of size* $> l$ *and no two distinct vectors u, v with* $u \leq v$.

PROOF. By Lemma 2.1 there is a sequence $a_1, a_2, \ldots, a_{s \cdot l}$ of vectors in $\mathbb{R}^{d-1}$ with no monotone subsequence of size $> l$. As in the proof of Proposition 2.2, define $b_i = (i, a_i)$ to get a set B of $s \cdot l$ vectors in $\mathbb{R}^d$ with no monotone sequence of size $> l$. Let D be the directed graph on the set of vertices B in which (b, c) is a directed edge iff $b \leq c$. Since B contains no monotone sequence of size $l+1$, D contains no directed path of length l. Hence, by the result of Gallai [6] and, independently, Roy [8] (see also [2, pp. 225-226]) the chromatic number of D is $\leq l$. Therefore D contains an independent set of $\geq s$ vertices. Take A as a set of s of them, to complete the proof.

PROOF OF LEMMA 4.1. By Lemma 4.2 there exists a set $B = \{b_1, \ldots, b_q\}$ of q points in $\mathbb{R}^d$, containing neither a monotone sequence of 3 terms nor two points u, v with $u \leq v$. Let A be a set of n points obtained from B by replacing b_i by a monotone increasing sequence of $[(n+i-1)/q]$ points all lying in a small neighborhood of b_i. More precisely, put $c = (\varepsilon, \varepsilon, \ldots, \varepsilon) \in \mathbb{R}^d$ for some small $\varepsilon > 0$, and define $A = \bigcup_{i=1}^{q} A_i$ where $A_i = \{b_i + jc;\ 0 \leq j < [(n+i-1)/q]\}$. We claim that if $a_i \in A_i$ and $a_j \in A_j$ where $1 \leq i < j \leq q$ then a_i and a_j are separated. Indeed, if this is false then there is some $a_k \in A_k$ such that (a_i, a_k, a_j) is monotone. If $k \neq i, j$ (and $\varepsilon > 0$ is sufficiently small) this is impossible since (b_i, b_k, b_j) is not monotone. Similarly if $k = i$ or $k = j$ this is impossible since it would imply that either $a_i < a_j$ or $a_j < a_i$, which would imply (for small enough ε) that either $b_i \leq b_j$ or $b_j \leq b_i$ contradicting the construction of B. Therefore the separation graph $G(A)$ contains $T(q, n)$ as a subgraph (on the sets of vertices A_i, $1 \leq i \leq q$). In addition each A_i contains $|A_i| - 1$ separated pairs. Hence

$$f(n, d) \geq t(q, n) + \sum_{i=1}^{q} (|A_i| - 1) = t(q, n) + n - q.$$

To prove the upper bound in Theorem 1.2 we combine Proposition 2.2 with some known results in extremal graph theory. We first state these results as lemmas.

LEMMA 4.3 (Erdös and Stone [4], see also [2, pp. 327-336] for some extensions). *For every natural number q, m and every* $\varepsilon > 0$ *if* $n > n(q, m, \varepsilon)$ *and G is a graph on n vertices having* $\geq t(q, n) + \varepsilon n^2$ *edges then G contains* $T(q+1, (q+1)m)$, *i.e. a complete* $(q+1)$*-partite graph with m vertices in each vertex class.*

LEMMA 4.4 (Bollobás, Erdös and Straus [3], see also [2, p. 317]). *Let G be an r-partite graph with l vertices in each vertex class. If G contains no complete graph on p vertices then the number of edges of G is* $\leq t(p-1, r) \cdot l^2$.

We can now prove the upper bound in Theorem 1.2. Put $q = 2^{2^{d-1}-1}$ and let A be a set of n points in $\mathbb{R}^d$. We must show that $s(A) \leq t(q, n) + o(n^2)$. Assume this is false, i.e. assume $\varepsilon > 0$, $n > n(\varepsilon, d)$ and

$$s(A) \geq t(q, n) + \varepsilon n^2. \tag{4.1}$$

Put $m = (2q^2 + 2q + 1)^{2^{d-1}} + 1$, and let $G = G(A)$ be the separation graph of A. By expression (4.1) and Lemma 4.3 G contains a complete $(q+1)$-partite graph with m vertices in each vertex class. By Lemma 2.1 the ith class $(1 \leq i \leq q+1)$ contains a monotone sequence of points $v(i, 0)$, $v(i, 1), \ldots, v(i, l)$ of size $l + 1 = 2q^2 + 2q + 2$. (Indeed, arrange the points according to their first coordinate and apply Lemma 2.1 to the other $d-1$ coordinates.) For $1 \leq i \leq q+1$ and $1 \leq j \leq l$ let $B(i, j) = B(v(i, j-1), v(i, j))$ be the smallest closed box with sides parallel to the axes containing $v(i, j-1)$ and $v(i, j)$. Note that since $\{v(i, j)\}_{j=0}^{l}$ is monotone the boxes $B(i, j)$ and $B(i, k)$ are disjoint if $k > j+1$ and they share exactly one common point $(= v(i, j))$ if $k = j+1$. Let D be the $(q+1)$-partite directed graph on the classes of vertices $\{B(i, j)\}_{j=1}^{l} (1 \leq i \leq q+1)$ in which $(B(i, j), B(i', j'))$ is a directed edge iff $i \neq i'$ and $(v(i, j-1) \in B(i', j')$ or $v(i, j) \in B(i', j'))$. By the preceding remark about the pairwise disjointness of $\{B(i, j)\}_{j=1}^{l}$, the outdegree of every vertex of D is at most $2 \cdot q$. Let H be the $(q+1)$-partite graph on the classes of vertices $\{B(i, j)\}_{j=1}^{l}$ in which $\{B(i, j), B(i', j')\}$ is an edge iff $i \neq i'$ and $B(i, j)$ and $B(i', j')$ are *not* adjacent in D. The number of edges of H is at least

$$\binom{q+1}{2} l^2 - 2q(q+1) \cdot l > \left(\binom{q+1}{2} - 1\right) l^2 = t(q, q+1) \cdot l^2.$$

Hence, by Lemma 4.4 (with $p = r = q+1$) H contains a complete graph on $q+1$ vertices $\{B(i, j_i)\}_{i=1}^{q+1}$. For $1 \leq i \leq q+1$ define $u(i, 1) = v(i, j_i - 1)$, $u(i, 2) = v(i, j_i)$, and put $C = \bigcup_{i=1}^{q+1} \{u(i, 1), u(i, 2)\}$. We claim that $s(C) = \binom{|C|}{2}$, i.e. that every two points of C are separated in C. Indeed, for $i \neq i' u(i, j)$ and $u(i', j')$ are separated even in the larger set A, and hence certainly in C. To show that $u(i, 1)$ and $u(i, 2)$ are separated in C note that otherwise there is some $i' \neq i$ and some $j' \in \{1, 2\}$ such that $u(i', j') \in B(u(i, 1), u(i, 2))$. But this implies that $(B(i', j_{i'}), B(i, j_i))$ is an edge of D, contradicting the choice of the $B(i, j_i) - s$. Hence $s(C) = \binom{|C|}{2}$. However $|C| = 2q + 2 > 2q = 2^{2^{d-1}}$ and thus by Proposition 2.2. $s(C) > \binom{|C|}{2}$. This contradiction shows that (4.1) is false and establishes the upper bound in Theorem 1.2. This together with Lemma 4.1 completes the proof of Theorem 1.2.

REMARK 4.5. It is worth noting that we actually proved a somewhat stronger assertion than that of Theorem 1.2. The lower bound for $f(n, d)$ given in Lemma 4.1 is asymptotically

$$\left(1 - \frac{1}{2^{2^{d-1}-1}}\right) \frac{n^2}{2} + O(n).$$

Replacing Lemma 4.3 by the extensions of Bollobás, Erdös and Simonovits (see [2, pp. 328-336]) to Erdös-Stone's result one can easily check that our proof implies that for every d there is a $\delta = \delta(d) > 0$ such that

$$f(n, d) \leq \left(1 - \frac{1}{2^{2^{d-1}-1}}\right) \frac{n^2}{2} + O(n^{2-\delta}).$$

It seems likely that our lower bound is the correct value of $f(n, d)$. We conclude our paper with the following conjecture:

CONJECTURE 4.6. Put $q(d) = 2^{2^{d-1}-1}$.
If $n \leq 2q(d)$ then $f(n, d) = \binom{n}{2}$.
If $n > 2q(d)$ then $f(n, d) = t(q, n) + n - q$.

Note that by Theorem 1.1 and Proposition 2.2 the conjecture's assertion holds for $d \leq 2$ and for $\{(d, n): n \leq 2q(d)\}$.

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