Discrete Mathematics 55 (1985) 317–321
North-Holland

NOTE

HYPERGRAPHS WITHOUT A LARGE STAR

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Received 18 January 1985

Let r, t be positive integers, $\mathcal{F}$ a set-system of rank r (i.e., $|F| \leqslant r$ for every $F \in \mathcal{F}$). If $|\mathcal{F}| > \binom{r+t}{t}$, then there exist $F_0, F_1, \ldots, F_{t+1} \in \mathcal{F}$ and points $p_1, \ldots, p_{t+1}$ forming a 'star', i.e., $p_i \in F_i$ but $p_i \in F_j$ for $i \neq j$.

1. Introduction and results

1.1. *Hypergraphs without a large star*

Let $\mathcal{H}$ be a hypergraph (i.e., a finite set-system, $\emptyset \in \mathcal{H}$ is allowed). The *rank* of $\mathcal{H}$ is r if $\max_{H \in \mathcal{H}} |H| = r$. The hypergraph $\mathcal{H}$ is an r-graph (an r-uniform hypergraph) if each member of $\mathcal{H}$ has exactly r elements. The complete hypergraph consisting of all the l-element (all the at most l-element) subsets of an n-element set is denoted by K_l^n ($K_{\leqslant l}^n$).

A set-system $\{A_1, \ldots, A_t\}$ is *strongly representable* if every A_i has an own point (its strong representative), i.e., there exist $a_1, \ldots, a_t$ such that $a_i \in A_i \setminus (\bigcup_{j \neq i} A_j)$, $1 \leqslant i \leqslant t$. (The other name of a strongly representable system is t-star.)

Frankl and Pach [8] proved that if an r-graph $\mathcal{F}$ does not contain a 3-star as subsystem then $|\mathcal{F}| \leqslant [r^2/4] + r + 1$. (The extremal $\mathcal{F}$ can be obtained from the complement of the Turán graph on $r+2$ points.) They conjecture that an r-graph without a t-star can have at most $T(r+t-1, t, t-1)$ members, where $T(n, k, l)$ is the Turán number, i.e., $\max\{|\mathcal{H}| : \mathcal{H} \subset K_l^n, K_l^k \not\subset \mathcal{H}\}$. This conjecture seems to be hopeless now because there is almost nothing known on the Turán numbers if $l \geqslant 3$. The following result, concerning hypergraphs of rank r instead of r-uniform ones, settles the problem apart from a constant factor.

Theorem 1. *Let $\mathcal{F}$ be a set-system of rank r without a $(t+1)$-star. Then $|\mathcal{F}| \leqslant \binom{r+t}{t}$.*

Remark. Since $T(n, t+1, t) > \frac{1}{2}\binom{n}{t}$ (see [10]), for the maximal number $m = m(r, t)$

of edges in an r-uniform hypergraph without a $(t+1)$-star we have $\frac{1}{2}\binom{r+t}{t} < m \leqslant \binom{r+t}{t}$.

1.2. *Extremal families*

There is a great number of extremal families (except for the trivial cases when $r=1$ or $t=1$). That is, if $r, t>1$, there exist a lot of non-isomorphic set-systems $\mathscr{F}$ of rank r without a $(t+1)$-star and having $\binom{r+t}{t}$ members. Now we give two examples. The notation $\binom{X}{i}$ stands for the set-system consisting of all the i-element subsets of X.

Example 1. Let $X=\{x_1, \ldots, x_{r+t-1}\}$, $X_i=\{x_1, \ldots, x_i\}$, $X_0=\emptyset$. Set $\mathscr{F}_i=\binom{X_{i+t-1}}{i}$ and $\mathscr{F}=\mathscr{F}_0 \cup \cdots \cup \mathscr{F}_r$. Then $\mathscr{F}$ is a set-system of rank r without a $(t+1)$-star, and $|\mathscr{F}|=\binom{r+t}{t}$.

Example 2. Let $t \geqslant 2$, $Y=\{y_1, \ldots, y_r\}$ and X_i as in Example 1 $(0 \leqslant i \leqslant r+t-2)$. Set

$$\mathscr{F}_{i,j}=\left\{F: F=\{y_1, \ldots, y_i\} \cup J,\ J \in \binom{X_{j+t-2}}{j}\right\}$$

and $\mathscr{F}=\bigcup \mathscr{F}_{i,j}$ where $i, j \geqslant 0$ and $i+j \leqslant r$.

These examples can be found in [9] concerning a related problem. We could not describe all the extremal families, even for $t=2$, but the following holds.

Theorem 2. *If $\mathscr{F}$ is a set-system of rank r without a $(t+1)$-star, and $|\mathscr{F}|=\binom{r+t}{t}$, then $|\mathscr{F}(i)|=\binom{i+t-1}{t-1}$ where $\mathscr{F}(i)=\{F \in \mathscr{F}: |F|=i\}$ $(0 \leqslant i \leqslant r)$.*

Another property of extremal families will be given later in Lemma 2.

1.3. *Traces of finite sets*

Let $\mathscr{F}$ and $\mathscr{H}=\{H_1, \ldots, H_m\}$ be two set-systems. We write $\mathscr{F} \to \mathscr{H}$ ($\mathscr{F}$ implies $\mathscr{H}$) if there exist $F_1, \ldots, F_m \in \mathscr{F}$ and a set Y such that the set-system $\{F_i \cap Y: 1 \leqslant i \leqslant m\}$ is isomorphic to H: otherwise $\mathscr{F} \not\to \mathscr{H}$. E.g., Theorem 1 can be formulated as follows. If $\mathscr{F}$ has rank r and $|\mathscr{F}|>\binom{r+t}{t}$ then $\mathscr{F} \to K_1^{t+1}$.

One of the first results of this type was given by Sauer [11] who proved that if $\mathscr{F} \subset K_{\leqslant n}^n$, $|\mathscr{F}|>\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{t}$, then $\mathscr{F} \to K_{\leqslant t+1}^{t+1}$. One could think that $\mathscr{F} \to K_1^{t+1}$ even if $\mathscr{F}$ is smaller. However, for $t=2$ Frankl (unpublished) and Anstee [1], and later Füredi [9] showed the existence of set-systems

$$\mathscr{F}_{n,t} \subset K_{\leqslant n}^n \quad \text{with } |\mathscr{F}_{n,t}|=\binom{n}{0}+\cdots+\binom{n}{t} \text{ and } \mathscr{F}_{n,t} \not\to K_1^{t+1}.$$

Corollary 1 ([9], for $t=2$ [1]). *If $\mathscr{F} \subset K_{\leqslant n}^n$, $|\mathscr{F}|=\binom{n}{0}+\cdots+\binom{n}{t}$ and $\mathscr{F} \not\to K_1^{t+1}$, then $|\mathscr{F}(i)|=\binom{i+t-1}{t-1}$ whenever $0 \leqslant i \leqslant n-t+1$.*

For further results see Bollobás [4], Bondy [5], Frankl [7]. Instead of Theorem 1, we prove the following slightly stronger result.

Theorem 3. *Let $\mathcal{F}$ be a set-system of rank r and $|\mathcal{F}|>\binom{r+t}{t}$. Then $\mathcal{F}\to K_{\leq 1}^{t+1}$. (That is, there exist $F_0, F_1, \ldots, F_{t+1}\in\mathcal{F}$ and a set $Y=\{y_1,\ldots,y_{t+1}\}$ such that $Y\cap F_0=\emptyset$ and $Y\cap F_i=\{y_i\}$ if $1\leq i\leq t+1$.)*

We mention that, for K_1^{t+1}-extremal families, one cannot expect a statement similar to Theorem 2 because e.g. K_r^{r+t} is $K_{\leq 1}^{t+1}$-extremal.

1.4. *More examples for $t=2$*

Let C_k denote the cycle of length k (i.e., a 2-uniform connected hypergraph with k points and k edges). Anstee [1] proved that if $\mathcal{F}$ is K_2^3-extremal on n points, then $\mathcal{F}\not\to C_k$ (whenever $k\geq 3$) and $|\mathcal{F}(i)|=n-i+1$, $1\leq i\leq n$. (Then $|\mathcal{F}|=1+n+\binom{n}{2}$.) On the other hand, he proved that if $\mathcal{F}\not\to C_k$ for any $k\geq 3$, then $\mathcal{F}$ can be completed to a K_2^3-extremal family (see [2]). From these results we obtain:

Example 3. Let $\mathcal{F}$ be a K_2^3-extremal family on an n-element set X ($n\geq r+1$). Set $\mathcal{G}=\overline{\mathcal{F}(n)}\cup\cdots\cup\overline{\mathcal{F}(n-r)}$ where $\overline{\mathcal{F}(i)}=\{X\setminus F\colon F\in\mathcal{F},\ |F|=i\}$. Then $\mathcal{G}$ has rank r, $\mathcal{G}=\binom{r+2}{2}$, $\mathcal{G}\not\to K_1^3$.

These examples show that the number of K_1^3-extremal families is not smaller than the number of trees on r vertices, that is, at least exponential in r.

2. Proof of Theorem 3

The following result of Frankl [6] is an improvement of a theorem of Bollobás [3]. (For other developments and applications see [12] and [13].)

Theorem 4 (Frankl [6]). *Let $A_1,\ldots,A_m$ be at most r-element, $B_1,\ldots,B_m$ at most t-element sets with $A_i\cap B_i=\emptyset$. Suppose that $A_i\cap B_j\neq\emptyset$ for $i>j$. Then $m\leq\binom{r+t}{t}$.*

For a set-system $\mathcal{H}$ and a set X define $\mathcal{H}|_X=\{H\cap X\colon H\in\mathcal{H}\}$ and $\mathcal{H}\setminus X=\{H\setminus X\colon H\in\mathcal{H}\}$. The set B is called *transversal* of $\mathcal{H}$ if $B\cap H\neq\emptyset$ holds for every $H\in\mathcal{H}$, $H\neq\emptyset$. Set $\tau(\mathcal{H})=\min\{|B|\colon B$ is transversal of $\mathcal{H}\}$. Finally, let $t(\mathcal{H})=\max\{t\colon$ there exist $H_1,\ldots,H_t\in\mathcal{H}$ strongly representable subsystem$\}$.

The following lemma can be found in [8] as well.

Lemma 1. *For every set-system $\mathcal{F}$, $\tau(\mathcal{F})\leq t(\mathcal{F})$. Therefore $\tau(\mathcal{F}\setminus X)\leq t(\mathcal{F})$ and $\tau(\mathcal{F}|_X)\leq t(\mathcal{F})$.*

Proof. Let B be a minimal transversal of $\mathcal{F}$, then $|B| \geq \tau(\mathcal{F})$. For every $b \in B$ there exists an $F_b \in \mathcal{F}$ such that $F_b \cap B = \{b\}$. Therefore $t(\mathcal{F}) \geq |B|$. □

Corollary 2. *If $\mathcal{F} \not\to K_{\leq 1}^{t+1}$ and $F \in \mathcal{F}$, then $\tau(\mathcal{F} \setminus F) \leq t$.*

Proof. Indeed, $\tau(\mathcal{F} \setminus F) \leq t(\mathcal{F} \setminus F) \leq t(\mathcal{F}) \leq t$. □

Now we are ready to prove Theorem 3. Let $\mathcal{F}$ be a set-system of rank r and $\mathcal{F} \not\to K_{\leq 1}^{t+1}$. Using Corollary 2, for every $F \in \mathcal{F}$ choose a $B(F)$ such that $B(F) \cap F = \emptyset$, $|B(F)| \leq t$ and $B(F) \cap F' \neq \emptyset$ whenever $F' \in \mathcal{F}$ and $F' \not\subset F$.

Without loss of generality we can suppose that $\mathcal{F} = \{F_1, \ldots, F_m\}$ and $|F_i| \leq |F_j|$ if $i \leq j$. In this case Theorem 4 can be applied for $\{F_i, B(F_i)\}$, implying $m \leq \binom{r+t}{t}$. □

3. Proof of Theorem 2

Lemma 2. *Let $\mathcal{F}$ be a set-system of rank r such that $\mathcal{F} \not\to K_{\leq 1}^{t+1}$, $|\mathcal{F}| = \binom{r+t}{t}$, and let $F \in \mathcal{F}$, $|F| < r$. Then there exist exactly t different sets $F' \in \mathcal{F}$ for which $F \subset F'$ and $|F'| = |F| + 1$.*

Proof. Clearly, the number of such sets F' is at most t. We prove that if $F \in \mathcal{F}$, $|F| < r$, then $F \cup \{x\} \in \mathcal{F}$ holds for every $x \in B(F)$, where $B(F)$ is as defined in the proof of Theorem 3.

Suppose that this is not true for $F_i \in \mathcal{F}$. We can assume $|F_i| < |F_j|$ for every $j > i$. Let $F' = F_i \cup \{x\} \notin \mathcal{F}$ for some $x \in B(F_i)$. Lemma 1 guarantees the existence of $B(F')$, a transversal of $\mathcal{F} \setminus F'$, $|B(F')| \leq t$. Now $B(F') \cap F_j \neq \emptyset$ if $j > i$, $F_j \in \mathcal{F}$, because $|F_j| \geq |F'|$. Let $\mathcal{F}' = \mathcal{F} \cup \{F'\}$. Now the sets $F_1, \ldots, F_i, F', F_{i+1}, \ldots, F_m$ and $B(F_1), \ldots, B(F_i), B(F')$, $B(F_{i+1}), \ldots, B(F_m)$ satisfy the assumptions of Theorem 4, therefore $|\mathcal{F}| < |\mathcal{F}'| \leq \binom{r+t}{t}$, contradicting the maximality of $\mathcal{F}$. □

Corollary 3. *Let $\mathcal{F}$ be a set-system of rank r, $\mathcal{F} \not\to K_{\leq 1}^{t+1}$. If $B(F)$ is not uniquely determined for some $F \in \mathcal{F}$, $|F| < r$, then $|\mathcal{F}| < \binom{r+t}{t}$.*

Proof. If $B(F)$ and $B'(F)$ are two different t-element transversals of $\mathcal{F} \setminus F$, then the number of sets $F' \in \mathcal{F}$, $F \subset F'$, $|F'| = |F| + 1$, is at least $|B(F) \cup B'(F)| > t$. □

To prove Theorem 2, we proceed by induction on r and t. The statement is trivially true when $r = 1$ or $t = 1$.

If $\mathcal{F} \not\to K_{\leq 1}^{t+1}$ and $\mathcal{F}$ is maximal then $\emptyset \in \mathcal{F}$. It follows from Lemma 2 that $\{x\} \in \mathcal{F}$ for some point x. (There are exactly t such points.) Set

$$\mathcal{F}_1 = \{F \setminus \{x\} : x \in F \in \mathcal{F}\}, \qquad \mathcal{F}_2 = \{F \in \mathcal{F} : x \notin F\}.$$

Then $\mathcal{F}_1$ has rank $r-1$ and $t(\mathcal{F}_1) \leq t$, therefore $|\mathcal{F}_1| \leq \binom{r+t-1}{t}$. Also, $\mathcal{F}_2$ has rank

r and $t(\mathscr{F}_2) \leq t-1$, so $|\mathscr{F}_2| \leq \binom{r+t-1}{t-1}$. Since $|\mathscr{F}_1| + |\mathscr{F}_2| = \binom{r+t}{t}$, $\mathscr{F}_1$ is $(r-1, t)$-extremal and $\mathscr{F}_2$ is $(r, t-1)$-extremal. Consequently, the inductive hypothesis can be used for $\mathscr{F}_1$ and $\mathscr{F}_2$, therefore

$$|\mathscr{F}(i)| = |\mathscr{F}_1(i-1)| + |\mathscr{F}_2(i)| = \binom{i-1+t-1}{t-1} + \binom{i+t-2}{t-2} = \binom{i+t-1}{t-1}. \quad \square$$

Acknowledgment

The authors would like to express their thanks to J. Lehel for the fruitful discussions.

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