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# Forbidding Just One Intersection

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Following a conjecture of P. Erdös, we show that if $\mathscr{F}$ is a family of $k$-subsets of an $n$-set no two of which intersect in exactly $l$ elements then for $k \geqslant 2l+2$ and $n$ sufficiently large $|\mathscr{F}| \leqslant \binom{n-l-1}{k-l-1}$ with equality holding if and only if $\mathscr{F}$ consists of all the $k$-sets containing a fixed $(l+1)$-set. In general we show $|\mathscr{F}| \leqslant d_k n^{\max\{l, k-l-1\}}$, where $d_k$ is a constant depending only on $k$. These results are special cases of more general theorems (Theorem 2.1–2.3). © 1985 Academic Press, Inc.

## 1. Introduction

Let $X$ be a finite set, $|X| = n$. By a family of subsets $\mathscr{F}$ we just mean $\mathscr{F} \subset 2^X$. We call $\mathscr{F}$ a *multi-family* if it may have repeated members. $\binom{X}{k}$ denotes the family of all $k$-subsets of $X$. Let $a(n)$, $b(n)$ be two positive real functions over the positive integers. If there are positive reals $c$ and $c'$ such that $ca(n) \geqslant b(n) \geqslant c'a(n)$ hold for $n > n_0$ then we shall write $a(n) \approx b(n)$. One of the most important intersection theorems concerning finite sets is

THEOREM 1.1 (Erdös, Ko, and Rado [4]). *Suppose $k, t$ are integers, $k \geqslant t \geqslant 1$, and $\mathscr{F}$ is a family of $k$-subsets of $X$, i.e., $\mathscr{F} \subset \binom{X}{k}$. Suppose further that for all $F, F' \in \mathscr{F}$ we have*

$$|F \cap F'| \geqslant t. \tag{1}$$

*Then for $n > n_0(k, t)$*

$$|\mathscr{F}| \leqslant \binom{n-t}{k-t} \tag{2}$$



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*holds, moreover equality holds in* (2) *if and only if for some* $T \subset X$, $|T| = t$ *we have* $\mathscr{F} = \{F \in \binom{X}{k}: T \subset F\}$.

In 1975 Erdös [2] raised the problem of what happens if we weaken the condition (1) to

$$|F \cap F'| \neq t - 1. \tag{3}$$

In this case one can easily construct a family of $k$-subsets $\mathscr{F}$, $|\mathscr{F}| = ((1 + o(1))(n/k)^{t-1} \approx n^{t-1}$ such that for all $F, F' \in \mathscr{F}$, $|F \cap F'| < t - 1$, in particular, (3) holds. Therefore if $k < 2t - 1$ (and in the case $k = 2t - 1$, see later), one cannot hope to have a bound like (2). Erdös [2] conjectured that for $k \geqslant 2t$ the condition (3) implies (2) if $n > n_0(k)$. Here we prove this conjecture.

## 2. Results

For a subset $L = \{l_1, ..., l_s\}$ of the integers satisfying $0 \leqslant l_1 < \cdots < l_s < k$, we call a family $\mathscr{F} \subset \binom{X}{k}$ an $(n, k, L)$-system if $|F \cap F'| \in L$ holds for all distinct $F, F' \in \mathscr{F}$. The maximum cardinality of an $(n, k, L)$-system is denoted by $m(n, k, L)$. Suppose $l, l'$ are nonnegative integers satisfying $l + l' < k$. Let us define $L(l, l') = \{0, 1, ..., l-1,\ k - l',\ k - l' + 1, ..., k-1\}$. Abusing of notation we shall call an $(n, k, L(l, l'))$-system an $(l, l')$-system, i.e., either $|F \cap F'| < l$ or $|F' \cap F| \geqslant k - l'$ hold for all distinct members $F, F'$ of an $(l, l')$-system. Our main results are

THEOREM 2.1. *There exists a positive constant* $d_k$ *such that* $m(n, k, L(l, l')) < d_k n^{\max\{l, l'\}}$ *holds. Consequently,* $m(n, k, L(l, l')) \approx n^{\max\{l, l'\}}$.

THEOREM 2.2. *If* $n > n_0(k)$ *and* $l' > l$ *then*

$$m(n, k, L(l, l')) = \binom{n - k + l'}{l'} \tag{4}$$

*holds. Moreover the* $(l, l')$*-system* $\mathscr{F}$ *attains equality in* (4) *if and only if for some* $(k - l')$*-element subset* $T$ *we have* $\mathscr{F} = \{F \in \binom{X}{k}: T \subset F\}$.

Note that the problem of Erdös is the special case $k = l + l' + 1$, $l = t - 1$. Also, for these values, Theorem 2.2 is the up-to-date strongest version of the Erdös–Ko–Rado theorem.

*The Case $l \geqslant l'$.*

THEOREM 2.3. *Suppose $l \geqslant l'$, moreover $k-l$ has a primepower divisor $q$ satisfying $q > l'$. Then*

$$m(n, k, L(l, l')) = (1+o(1)) \binom{n}{l} \left( \binom{k+l'}{l'} \Big/ \binom{k+l'}{l} \right). \tag{5}$$

In the case of $l \geqslant l'$ the right-hand side of (5) is always a lower bound for $m(n, k, L(l, l'))$. (See examples in Chap. 7)

*Conjecture* 2.4. *In* (5) *equality holds for all $l \geqslant l'$.*

Note that our conjecture holds by Theorem 2.3 whenever $l' = 1, 2$ or $k > l + 3^{l'}$.

## 3. REMARKS

Concerning Theorems 2.1 and 2.2 the best results were due to the first author. In [5] he solved the case $l = 1$, and in [6] he proved that $m(n, k, L(l, l')) \leqslant c(k) \cdot n^{\max\{l', l + [l'/(k-l-l')]\}}$. In the nonuniform case the following holds.

THEOREM 3.1 (Katona [13]). *Suppose $t \geqslant 1$ and for all $F, F' \in \mathscr{F} \subset 2^X$* (1) *holds (i.e., $|F \cap F'| \geqslant t$). Then one of the following* 2 *cases occurs*

(a) *$n + t$ is even,*

$$|\mathscr{F}| \leqslant \sum_{i \geqslant (n+t)/2} \binom{n}{i},$$

*and for $t \geqslant 2$ equality holds if and only if $\mathscr{F} = \{F \subset X\colon |F| \geqslant (n+t)/2\}$.*

(b) *$n + t - 1$ is even,*

$$|\mathscr{F}| \leqslant \sum_{i \geqslant (n+t+1)/2} \binom{n}{i} + \binom{n-1}{(n+t-1)/2},$$

*and equality holds for $t \geqslant 2$ if and only if for some $x \in X$ we have $\mathscr{F} = \{F \subset X\colon |F \cap (X - \{x\})| \geqslant (n+t-1)/2\}$.*

In the nonuniform case to any family satisfying (1), one can add $\binom{X}{0} \cup \binom{X}{1} \cup \cdots \cup \binom{X}{t-2}$ without contradicting (3). In [8] we have shown that for $n > n_0(t)$ one cannot do better, ($n_0(t) < 3^t$).

*Conjecture* 3.2[1] (Erdös [3]). Suppose that $\mathscr{F} \subset 2^X$, $|F \cap F'| \neq t$ for $F, F' \in \mathscr{F}$, and $\varepsilon n < t < (\frac{1}{2} - \varepsilon) n$. Then there exists a $c = c(\varepsilon) > 0$ such that $|\mathscr{F}| < (2-c)^n$.

[1] This conjecture was proved recently by Frankl and Rödl.

## 4. Tools of Proofs

*Set System with Lots of Stars*

The main tool in proving Theorems 2.1 and 2.2 is a recent result of the second author. To state it we need some definitions. We call the family of sets $\mathscr{A}$ an *s-star* with *center* $K$ if $|\mathscr{A}| = s$ and for all distinct $A, A' \in \mathscr{A}$, $A \cap A' = K$ holds. We say that $\mathscr{B} \subset 2^X$ is closed under intersection if for all $B, B' \in \mathscr{B}$, $(B \cap B') \in \mathscr{B}$ holds. If $B \in \mathscr{B} \subset 2^X$, we define $M(B, \mathscr{B}) = (B \cap B'$: $B \neq B' \in \mathscr{B}\}$.

Theorem 4.1 (Füredi [12]). *For any two positive integers $k, s$ there exists a positive constant $c(k, s)$ such that every $\mathscr{F} \subset \binom{X}{k}$ contains some $\mathscr{F}^* \subset \mathscr{F}$, satisfying*

(6.1) $|\mathscr{F}^*| \geqslant c(k, s)\, |\mathscr{F}|$

(6.2) *all the families $M(F, \mathscr{F}^*)$ are isomorphic, $F \in \mathscr{F}^*$,*

(6.3) *every $A \in M(F, \mathscr{F}^*)$ is the center of an s-star $\mathscr{A}$, $A \subset \mathscr{F}^*$,*

(6.4) *$M(F, \mathscr{F}^*)$ is closed under intersection, i.e., $A, A' \in M(F, \mathscr{F}^*)$ implies $A \cap A' \in M(F, \mathscr{F}^*)$.*

When we refer to Theorem 4.1 we always mean the case $s = k + 1$, and we set $c_k = c(k, k+1)$. The reason for this is given by

Proposition 4.2. *Suppose $\mathscr{F}$ is an $(n, k, L)$-system, $F, F' \in \mathscr{F}^*$, $A \in M(F, \mathscr{F}^*)$, $A' \in M(F', \mathscr{F}^*)$, $F'' \in \mathscr{F}$. Then we have*

$$|A| \in L, \qquad |A \cap A'| \in L, \qquad |A \cap F''| \in L. \tag{7}$$

Let us mention that the idea of using $(k+1)$-stars for investigating $(n, k, L)$-systems is due to M. Deza. Proposition 4.2 can be verified easily by using that if $A$ is the center of the $(k+1)$-star $\{F_1, ..., F_{k+1}\}$, then the sets $F_i - A$ are pairwise disjoint. For a proof see [1].

*Set Systems with Many Intersection Conditions*

We will also use

Theorem 4.3 (Frankl and Katona [9]). *Suppose $\mathscr{D} = \{D_1, ..., D_m\}$ is a collection of not necessarily distinct subsets of $Y$, $|Y| = r$. Suppose further $s$ is a positive integer such that for all $t$, $1 \leqslant t \leqslant m$, and all $1 \leqslant i_1 < \cdots < i_t \leqslant m$,*

$$|D_{i_1} \cap \cdots \cap D_{i_t}| \neq t - s$$

*holds. Then we have $|\mathscr{D}| = m \leqslant r + (s-1)$.*

We shall need the following strengthening of this theorem.

PROPOSITION 4.4. *Suppose that $\mathscr{D}$ satisfies the assumptions of Theorem* 4.3. *If $|\mathscr{D}| = r+s-1$, then for every $y \in D \in \mathscr{D}$ the number of sets in $\mathscr{D}$ containing $y$ is $|D|+s-1$. (Moreover, for $s \geqslant 2$ $\mathscr{D}$ consists of $r+s-1$ copies is $Y$.) If $|\mathscr{D}| = r+s-2$ then there exists at most one set $D'$ such that $\mathscr{D}' = \mathscr{D} \cup \{D'\}$ satisfies the assumptions, too.*

To prove Proposition 4.4 we shall give a new proof for Theorem 4.3. We present the proof of these statements at the end of the paper in Chapter 8.

*Shadows of Set Systems*

For $\mathscr{F} \subset \binom{X}{a}$, $0 \leqslant s \leqslant a$, let us define $\Delta_s(\mathscr{F}) = \{G \in \binom{X}{s} : \exists F \in \mathscr{F}, G \subset F\}$. Given $|\mathscr{F}|$ what is the minimum of $|\Delta_s(\mathscr{F})|$? This problem was completely solved by Kruskal [15] and Katona [14], however their formula for $\min |\Delta_s(\mathscr{F})|$ is not convenient for computation. We will rather use the following version of the Kruskal–Katona theorem.

THEOREM 4.5 (Lovász [16]). *Suppose that the real number $x$, $x \geqslant a$ is defined by $|\mathscr{F}| = \binom{x}{a} = x(x-1)\cdots(x-a+1)/a!$. Then*

$$|\Delta_s(\mathscr{F})| \geqslant \binom{x}{s}$$

*holds for all $0 \leqslant s \leqslant a$.*

(Cf. [19] for a unified, simple proof of the Kruskal–Katona theorem and Theorem 4.5.)

*A General Bound for $m(n, k, L)$*

For the proof of Theorem 2.3 we need

THEOREM 4.6 (Frankl and Wilson [11]). *Suppose that for some integer valued polynomial of degree $d$ and a prime $p$ for all $l \in L$ $p \mid g(l)$ holds but $p \nmid g(k)$. Then we have*

$$m(n, k, L) \leqslant \binom{n}{d}.$$

*Steiner-Systems and Quasi-Steiner-Systems*

Suppose $r > t \geqslant 1$. We say that $\mathscr{S} \subset \binom{X}{r}$ is a Steiner-system, $S(n, t, r)$ if for every $T \in \binom{X}{t}$ there exists exactly one $S \in \mathscr{S}$, containing $T$. Of course, we have $|\mathscr{S}| = \binom{n}{t}/\binom{r}{t}$, and $\mathscr{S}$ is a maximal $(n, r, \{0, 1, \ldots, t-1\})$-system.

For $t = 1$ a Steiner-system is just a partition of $X$ into $r$ subsets, it exists if and only if $r \mid n$. For $t=2$, $n>n_0(r)$ Wilson [18] proved that the trivial necessary conditions $\binom{r}{2} \mid \binom{n}{2}$, $(r-1) \mid (n-1)$ are sufficient for the existence of Steiner-systems. However, very little is known about the existence of Steiner-systems for $r \geqslant 3$. We shall use

THEOREM 4.7 (Rödl [17]). *For all* $r>t\geqslant 1$

$$m(n, r, \{0, 1, ..., t-1\}) = (1+o(1)) \binom{n}{t} \Big/ \binom{r}{t}$$

*holds.*

## 5. THE PROOF OF THEOREM 2.1 AND SOME LEMMAS

Actually we prove the following stronger statement:

THEOREM 5.1. *Suppose* $\mathscr{F}$ *is an* $(l, l')$*-system,* $c_k = c(k, k+1)$ *is the constant from Theorem* 4.1, *then we have*

$$|\Delta_{\max\{l,l'\}}(\mathscr{F})| \geqslant c_k\, |\mathscr{F}|.$$

*Proof.* Apply Theorem 4.1 to $\mathscr{F}$. We obtain a family $\mathscr{F}^* \subset \mathscr{F}$ satisfying (6.1)–(6.4), i.e., $|\mathscr{F}^*| > c_k\, |\mathscr{F}|$, all the families $M(F, \mathscr{F}^*)$ are isomorphic for $F \in \mathscr{F}^*$, each $A \in M(F, \mathscr{F}^*)$ is a center of a $(k+1)$-star, $M(F, \mathscr{F}^*)$ is closed under intersection.

In view of (6.1) it will be sufficient to deal with $\mathscr{F}^*$. We say that $B \subset F$ is an *own* subset of $F \in \mathscr{F}^*$ if $B \subset F' \in \mathscr{F}^*$ implies $F' = F$.

LEMMA 5.2. *Each* $F \in \mathscr{F}^*$ *has an own subset* $B$ *satisfying* $|B| \leqslant \max\{l, l'\}$.

First we finish the proof of the Theorem 5.1 using this Lemma. Let us note that $B \subset F$ is an own subset of $F$ if and only if $B \not\subset A$ holds for all $A \in M(F, \mathscr{F}^*)$.

If $B$ is an own subset of $F$ and $B \subset B' \subset F$ then $B'$ is an own subset of $F$ as well. Thus by Lemma 5.2 for each $F \in \mathscr{F}^*$ we may choose an own subset $B(F)$ of $F$, having $|B(F)| = \max\{l, l'\}$. Consequently $\mathscr{B} = \{B(F) \colon F \in \mathscr{F}^*\}$ satisfies $\mathscr{B} \subset \Delta_{\max\{l,l'\}}(\mathscr{F}^*)$ and $|\mathscr{B}| = |\mathscr{F}^*| \geqslant c_k\, |\mathscr{F}|$, yielding the statement of Theorem 5.1. Q.E.D.

*Remark.* Theorem 5.1 is related to the following theorem due to Frankl and Singhi [10].

THEOREM 5.3. *If* $\mathscr{F} \subset \binom{X}{k}$ *is an* $(n, k, L(l, l'))$*-system with* $k = l + l' + 1$ *and* $l > 3^{l'}$ *then* $|\Delta_l(\mathscr{F})| \geqslant |\mathscr{F}|$.

They conjecture that Theorem 5.3 holds for all $l \geqslant l'$. The above proof shows that it is useful to investigate $M(F, \mathscr{F}^*)$, i.e., the intersection structure of $F$.

Let $F$ be a $k$-element set and let $\mathscr{M} \subset 2^F - \{F\}$. Suppose $\mathscr{M}$ is closed under intersection and for all $M \in \mathscr{M}$ we have $|M| < l$ or $|M| \geqslant k - l'$. We say that $B \subset F$ is *covered* (by $\mathscr{M}$) if there exists an $A \in \mathscr{M}$ such that $B \subseteq A$. Clearly Lemma 5.2 is a consequence of the following

LEMMA 5.4 (Main Lemma). *There exists a subset* $B \subset F$ *satisfying* $|B| \leqslant \max\{l, l'\}$ which is not covered by $\mathscr{M}$.

LEMMA 5.5. *Suppose now* $l' > l$ *and all* $(l'-1)$*-element subsets of* $F$ *are covered by* $\mathscr{M}$. *Then* (c) *and one of* (a) *and* (b) *hold.*

(a) *There exists a* $(k - l')$*-element subset* $A(F)$ *of* $F$ *such that* $\mathscr{M}$ *consists of all supersets of* $A(F)$ *and eventually some at most* $(l-1)$*-element subsets.*

(b) $l' = l + 1$, $k = l' + l + 1$ *and there are at least two* $l'$*-element subsets of* $F$ *which are not covered by* $\mathscr{M}$.

(c) *If* $B$ *is an uncovered* $l'$*-element subset of* $F$ *and* $B \subseteq C \subsetneqq F$ *then* $l \leqslant |A \cap C| < k - l'$ *holds for some* $A \in \mathscr{M}$.

Lemma 5.5 says that in the cases $l' \geqslant l + 2$, $l' = l + 1 < (k/2)$ there exists only one $\mathscr{M}$ which covers all $(l'-1)$-element subsets. However the description of such $\mathscr{M}$s seems to be very hard in the case $l' = l + 1$, $k = 2l + 2$. In fact, an $S(2l+2, l+1, l)$ Steiner-system extended with all subsets of size less than $l$ satisfies the assumptions of Lemma 5.5, and the existence of these designs is an old unsolved problem.

*Proof of Lemmas* 5.4 *and* 5.5. We prove these lemmas together. Choose a minimal subset $B$ of $F$ which is uncovered by $\mathscr{M}$. It is possible because $F$ is not covered. We may suppose $|B| = b > l$ holds. Let $B = \{x_1, x_2, \ldots, x_b\}$. As $B - \{x_i\}$ is covered, there exists an $A_i \in \mathscr{M}$ for which $B \cap A_i = B - \{x_i\}$ holds. First we show that $b \leqslant l + l'$. Indeed, let $A = A_1 \cap A_2 \cap \cdots \cap A_{l'+1}$, then $A \in \mathscr{M}$, $x_i \notin A$ for $1 \leqslant i \leqslant l' + 1$ which implies $|A| < k - l'$, i.e., $|A| < l$. But $|A \cap B| = b - l' + 1$, whence $b \leqslant l + l'$.

Fix an arbitrary $(l + l' + 1 - b)$-element subset $Y$ of $F - B$. Define $D_i = Y \cap A_i$, $i = 1, \ldots, b$. We claim that for $1 \leqslant i_1 < \cdots < i_t \leqslant b$ we have

$$|D_{i_1} \cap \cdots \cap D_{i_t}| \neq t - (b - l). \tag{$*$}$$

Suppose the contrary and consider the set $A = (A_{i_1} \cap \cdots \cap A_{i_t}) \in \mathscr{M}$. By definition $|A \cap B| = b - t$ and $0 \leqslant |A \cap (F - B - Y)| \leqslant k - l - l' - 1$. Using $|A| = |A \cap B| + |A \cap Y| + |A \cap (F - B - Y)|$ we infer $l \leqslant |A| < k - l'$, a contradiction.

Now, let us apply Theorem 4.3 to the multi-family $\mathscr{D} = \{D_1, ..., D_b\}$. We conclude $b = |\mathscr{D}| \leqslant |Y| + (b - l) - 1 = (l + l' + 1 - b) + (b - l) - 1 = l'$, i.e., $b \leqslant l'$ proving Lemma 5.4.

For the proof of Lemma 5.5 we suppose that $|B| = l'$, whence $|Y| = l + 1$. When we apply Theorem 4.3 for the family $\mathscr{D}$ with $s = l' - l$ we get equality $|\mathscr{D}| = |Y| + (l' - l) - 1$. Thus we can use Proposition 4.4. Hence we get in the case $l' - l \geqslant 2$ that $\mathscr{D}$ consists of $l'$ copies of $Y$. The choice of $Y$ was arbitrary so we get $A_i = F_i - \{x_i\}$ for all $1 \leqslant i \leqslant l'$, $A(F) = F - \{x_1, ..., x_{l'}\}$. Now we prove this in the case $l' - l = 1$, $|Y| < |F - B|$ (i.e., $k > 2l + 2$).

The arbitrary choice of $Y$ and Proposition 4.4 yields $\bigcup A_i = F$. Since $|F - B| > |B| = l'$, we may choose an $A_j$ satisfying $|A_j - B| \geqslant 2$. If $|A_j| = k - 1$, that is $A_j - B = F - B$, then again by Proposition 4.4, $A_i - B = F - B$ follows for all $i$.

Since $\mathscr{M}$ is closed under intersection, we gain the assertion of the lemma with $A(F) = F - B$.

To complete the proof, we derive a contradiction from $2 \leqslant |A_i - B| < |F - B|$. Choose $u, v \in A_j - B$, $w \in (F - B) - A_j$, and let $Y$, $Y'$ chosen such that $v \in Y \cap Y'$ and $Y' = Y - \{u\} \cup \{w\}$. Denote $\mathscr{D} = \{A_i \cap Y: 1 \leqslant i \leqslant l'\}$, $\mathscr{D}' = \{A_i \cap Y': 1 \leqslant i \leqslant l'\}$ and $D = A_j \cap Y$, $D' = A_j \cap Y'$. We have $D' = D - \{u\}$ so using Proposition 4.4 we get the contradiction $|D| = d_{\mathscr{D}}(v) = d_{\mathscr{D}'}(v) = |D'|$.

Now investigate the case $l' = l + 1$, $k = l' + l + 1$. Then $Y = F - B$. If $\mathscr{M} \not\supset \{A: Y \subseteq A \subsetneq F\}$ then there exists an $A_i$ such that $|A_i \cap Y| < |Y|$. Let $y \in Y - A_i$. We claim that $B - \{x_i\} \cup \{y\}$ is not covered by $\mathscr{M}$ either. Suppose on the contrary that there exists an $A_i' \supset B - \{x_i\} \cup \{y\}$, $A_i' \in \mathscr{M}$. $x_i \notin A_i'$ because $B$ is not covered. Hence $B \cap A_i' = B \cap A_i$, i.e., in the system $\{D_1, ..., D_{l'}\}$ we can replace $D_i$ by $D_i' = A_i' \cap Y$, But this is impossible by Proposition 4.4. This finishes the proof of (a) and (b).

The subset $A(F)$ is unique. (If there were two such $A(F)$, e.g., $A$ and $A'$ then $A \cup (A' - \{x\})$ would intersect $A'$ in $k - l' - 1$ elements $(x \in A' - A)$.)

The proof of (c) in the case (a) is similar to the proof of uniqueness of $A(F)$. If $l' = l + 1 = (k/2)$ then set $|C - B| = t$. Now $|C \cap (A_1 \cap \cdots \cap A_t \cap A_{t+1})| = l$, proving (c).

## 6. The Proof of Theorem 2.2

Let $A_0$ be a fixed $(k - l')$-subset of $X$ and let $\mathscr{F}_0 = \{F \in \binom{X}{k}: A_0 \subset F\}$. In this family $M(F, \mathscr{F}_0) = \{A: A_0 \subseteq A \subsetneq F\}$ holds for all $F \in \mathscr{F}_0$. This motivates our procedure of proof.

Assume $|\mathscr{F}| \geqslant \binom{n-k+l'}{l'}$ holds. First, as in the proof of Theorem 2.1 we apply Theorem 4.1 to $\mathscr{F}$ and obtain $\mathscr{F}_1 = \mathscr{F}^*$ satisfying (6.1) – (6.4). Then we apply Theorem 4.1 to $\mathscr{F} - \mathscr{F}_1$ to obtain $\mathscr{F}_2 = (\mathscr{F} - \mathscr{F}_1)^*$, in the $m$th step we obtain $\mathscr{F}_m = (\mathscr{F} - (\mathscr{F}_1 \cup \cdots \cup \mathscr{F}_{m-1}))^*$. We stop either if there are no more sets or if for $F \in \mathscr{F}_m$ there is no $A \in \binom{F}{k-l'}$ such that $M(F, \mathscr{F}_m) \supset \{B: A \subset B \subsetneqq F\}$.

LEMMA 6.1. *$|\mathscr{F} - (\mathscr{F}_1 \cup \cdots \cup \mathscr{F}_{m-1})| \leqslant c'_k \binom{n}{l'-1}$ holds for some constant $c'_k$.*

We obtain Lemma 6.1 by proving a series of propositions. First we continue applying Theorem 4.1 to obtain $\mathscr{F}_{m+1} = (\mathscr{F} - (\mathscr{F}_1 \cup \cdots \cup \mathscr{F}_m))^*$,..., until we get an $\mathscr{F}_{m'}$ with the property that for some $F \in \mathscr{F}_{m'}$, $F$ has an own subset of size strictly less than $l'$. Then by Theorem 4.1 (6.2) all $F \in \mathscr{F}_{m'}$ share this property, yielding

PROPOSITION 6.2. *$|\mathscr{F} - (\mathscr{F}_1 \cup \cdots \cup \mathscr{F}_{m'-1})| \leqslant (1/c_k)\, |\mathscr{F}_{m'}| \leqslant (1/c_k) \binom{n}{l'-1}$ holds for $n > 2l'$. (Note that eventually $m' = m$ holds.)*

By Lemma 5.4 we know that all $F \in \mathscr{F}_i$, $1 \leqslant i < m'$, have an own subset of size $l'$, i.e., which is not contained in any other member of $\mathscr{F}_i$. Lemma 5.5(c) yields that these sets are not contained in any member of $\mathscr{F} - \mathscr{F}_i$ either. We infer

PROPOSITION 6.3. *Suppose $B \subset F \in \mathscr{F}_i$, $1 \leqslant i < m'$, $|B| = l'$, and $B$ is an own subset of $F$ in $\mathscr{F}_i$. Then $B$ is an own subset of $F$ in $\mathscr{F}$, too.*

Similarly, Lemma 5.5 (a) and (b) give

PROPOSITION 6.4. *If $m \leqslant i < m'$ then every $F \in \mathscr{F}_i$ has at least 2 own subsets of size $l'$.*

PROPOSITION 6.5. $\sum_{1 \leqslant i < m} |\mathscr{F}_i| + \sum_{m \leqslant i < m'} 2\, |\mathscr{F}_i| \leqslant \binom{n}{l'}$.

*Proof.* It is a direct consequence of Proposition 6.3, 6.4., and Lemma 5.2. ∎

Now $|\mathscr{F}| \geqslant \binom{n-k+l'}{l'} > \binom{n}{l'} - (k-l')\binom{n}{l'-1}$, Proposition 6.2, and Proposition 6.5 imply

$$\sum_{m \leqslant i < m'} |\mathscr{F}_i| < \left(\frac{1}{c_k} + k - l'\right)\binom{n}{l'-1}.$$

We infer by Proposition 6.2,

$$|\mathscr{F}-(\mathscr{F}_1\cup\cdots\cup\mathscr{F}_{m-1})|\leqslant\left(\frac{2}{c_k}+k-l'\right)\binom{n}{l'-1},$$

proving Lemma 6.1.

For $F\in\mathscr{F}_i$, $1\leqslant i\leqslant m-1$, let us denote by $A(F)$ the $(k-l')$-subset of $F$ for which $M(F,\mathscr{F}_i)\supset\{B\colon A(F)\subset B\subsetneqq F\}$ holds. (It is easy to see that $A(F)$ is uniquely determined and it is the only $(k-l')$-element set in $M(F,\mathscr{F})$.)

PROPOSITION 6.6. *If* $F\in(\mathscr{F}_1\cup\cdots\cup\mathscr{F}_{m-1})$, $F'\in\mathscr{F}$, *and* $|F\cap F'|\geqslant l$ *then* $A(F)\subset F'$ *holds. Moreover if* $F'\in(\mathscr{F}_1\cup\cdots\cup\mathscr{F}_{m-1})$ *then* $A(F)=A(F')$ *holds.*

*Proof.* Suppose the contrary. Then $|A(F)\cap F'|<k-l'$ holds. Consider an arbitrary chain of subsets $A_0=A(F)\subsetneqq A_1\subsetneqq\cdots\subsetneqq A_{l'}=F$. Let $i$ be the last index in this chain for which $|A_i\cap F'|<k-l'$ holds. Then $|F\cap F'|\geqslant l$ and $l<k-l'$ imply $l\leqslant|A_i\cap F'|<k-l'$, contradicting (7).

For the case $F'\in(\mathscr{F}_1\cup\cdots\cup\mathscr{F}_{m-1})$ we infer $A(F)\in M(F',\mathscr{F})$. Since $|A(F)|=k-l'$, $A(F)=A(F')$ follows. ∎

Let $A_1, A_2,\ldots, A_h$ be the list of $(k-l')$-sets for which $A_i=A(F)$ holds for some $F\in(\mathscr{F}_1\cup\cdots\cup\mathscr{F}_{m-1})$. Define $\mathscr{G}_i=\{G\in(\mathscr{F}_1\cup\cdots\cup\mathscr{F}_{m-1})\colon A_i\subset G\}$ and $\tilde{\mathscr{G}}_i=\{G-A_i\colon G\in\mathscr{G}_i\}$. Assume $|\mathscr{G}_1|\geqslant|\mathscr{G}_2|\geqslant\cdots\geqslant|\mathscr{G}_h|$.

PROPOSITION 6.7. *The sets* $\Delta_l(\mathscr{G}_1),\ldots,\Delta_l(\mathscr{G}_h)$ *are pairwise disjoint.*

*Proof.* It is a direct consequence of the preceding proposition. ∎

Let us define the real number $x_i$ by $|\mathscr{G}_i|=\binom{x_i}{l'}$, $x_i\geqslant l'$, $i=1,\ldots,h$.

PROPOSITION 6.8. $\Delta_l(\mathscr{G}_i)\geqslant\binom{x_i}{l}$, $i=1,\ldots,h$.

*Proof.* Since $\Delta_l(\mathscr{G}_i)\supset\Delta_l(\tilde{\mathscr{G}}_i)$, this follows from $|\tilde{\mathscr{G}}_i|=|\mathscr{G}_i|$ and Theorem 4.5. ∎

Note that in view of Lemma 6.1 we may assume

$$\sum_{1\leqslant i\leqslant h}|\mathscr{G}_i|\geqslant\binom{n-k+l'}{l'}-c_k'\binom{n}{l'-1}>\left(1-\frac{c_k''}{n}\right)\binom{n}{l'},\tag{8}$$

where $c_k''$ is a constant. From Proposition 6.7. and 6.8 we have

$$\sum_{1\leqslant i\leqslant h}\binom{x_i}{l}\leqslant\binom{n}{l}.\tag{9}$$

Using that $\binom{x_1}{l'}/\binom{x_1}{l} \geqslant \binom{x_2}{l'}/\binom{x_2}{l} \geqslant \cdots \geqslant \binom{x_h}{l'}/\binom{x_h}{l}$, we infer

$$\sum_{1 \leqslant i \leqslant h} |\mathscr{G}_i| = \sum_i \binom{x_i}{l}\binom{x_i}{l'}\Big/\binom{x_i}{l} \leqslant \sum_i \binom{x_i}{l} \frac{\binom{x_1}{l'}}{\binom{x_1}{l}} \leqslant \binom{n}{l} \frac{\binom{x_1}{l'}}{\binom{x_1}{l}}.$$

Now, (8) and (9) yield

$$x_1 > n - c_k''',$$

and consequently $\binom{x_1}{l} > \binom{n}{l} - c_k'''\binom{n}{l-1}$. Using (9) we obtain $\sum_{2 \leqslant i \leqslant h} \binom{x_i}{l} \leqslant c_k'''\binom{n}{l-1}$ and consequently

$$\sum_{2 \leqslant i \leqslant h} |\mathscr{G}_i| = \sum_{2 \leqslant i \leqslant h} \binom{x_i}{l'} < c_k''' \binom{n}{l'-1}. \tag{10}$$

Using Lemma 6.1 and (10) we get

$$|\mathscr{F} - \mathscr{G}_1| < c_k^{(\mathrm{iv})} \binom{n}{l'-1}. \tag{11}$$

Let us define $\mathscr{K} = \{F \in \mathscr{F} : A_1 \subset F$ and for each $A_1 \subset B \subsetneqq F$ there is a $(k+1)$-star in $\mathscr{F}$ with center $B\}$. Of course, $\mathscr{G}_1 \subset \mathscr{K}$. Let us set $\mathscr{A} = \{F \in \mathscr{F} : A_1 \subset F,\ \ F \notin \mathscr{K}\}$, and $\mathscr{B} = \mathscr{F} - \mathscr{K} - \mathscr{A}$, i.e., $\mathscr{B} = \{F \in \mathscr{F} : A_1 \not\subset F\}$.

For the family $\mathscr{F}_0 = \{F \in \binom{X}{k} : A_1 \subset F\}$, all its members would be in $\mathscr{K}$, i.e., $\mathscr{K}$ consists of the "regular" elements of $\mathscr{F}$. Our aim is to show $\mathscr{F} = \mathscr{K}$. For $\mathscr{F}_0$ one has $\Delta_{l'}(\mathscr{F}_0) = \binom{X}{l'}$. On the other hand $\Delta_{l'}(\mathscr{K}) \supset \binom{X - A_1}{l'}$ is equivalent to $\mathscr{F} = \mathscr{K}$. We will derive a contradiction from $\mathscr{F} - \mathscr{K} \neq \emptyset$ by showing that $\Delta_l(\mathscr{K})$ and consequently $\Delta_{l'}(\mathscr{K})$ miss too many subsets of $X$. We distinguish two cases according to which is larger $|\mathscr{A}|$ or $|\mathscr{B}|$.

(a) If $|\mathscr{A}| \leqslant |\mathscr{B}|$. It can be proved in the same way as Proposition 6.6 that $\Delta_l(\mathscr{K}) \cap \Delta_l(\mathscr{B}) = \emptyset$. Let $|\Delta_{l'}(\mathscr{B})| = \binom{x}{l'}$. Apply Theorem 4.5 to $\Delta_{l'}(\mathscr{B})$ and use Theorem 5.1 for $\mathscr{B}$:

$$|\Delta_l(\mathscr{B})| = |\Delta_l(\Delta_{l'}(\mathscr{B}))| \geqslant \binom{x}{l} = \frac{\binom{x}{l}}{\binom{x}{l'}} |\Delta_l(\mathscr{B})|$$

$$\geqslant \frac{\binom{x}{l}}{\binom{x}{l'}} c_k\, |\mathscr{B}| \geqslant \frac{\binom{x}{l}}{\binom{x}{l'}} \frac{c_k}{2} (|\mathscr{A}| + |\mathscr{B}|).$$

$|\Delta_{l'}(\mathscr{B})| \leqslant \binom{k}{l'}|\mathscr{B}| \leqslant \binom{k}{l'} c_k^{(\mathrm{iv})}\binom{n}{l'-1}$ by (11). Hence $x < c_k^{(\mathrm{v})} n^{1-1/l'}$ so we get $\binom{x}{l}/\binom{x}{l'} \geqslant \binom{n}{l}/\binom{n}{l'}$, i.e.,

$$|\Delta_l(\mathscr{B})| > \frac{\binom{n}{l}}{\binom{n-k+l'}{l'}}(|\mathscr{A}|+|\mathscr{B}|). \tag{12}$$

Denote by $\tilde{\mathscr{K}} = \{F - A_1 : F \in \mathscr{K}\}$. Obviously $|\Delta_l(\mathscr{K})| = \sum_{0 \leqslant i \leqslant l} |\Delta_{l-i}(\tilde{\mathscr{K}})| \binom{k-l'}{i}$. If $|\tilde{\mathscr{K}}| = \binom{y}{l'}$, then Theorem 4.5 yields

$$|\Delta_l(\mathscr{K})| \leqslant \sum_{0 \leqslant i \leqslant l} \binom{y}{l-i}\binom{k-l'}{i} = \binom{y+k-l'}{l} \geqslant |\mathscr{K}| \frac{\binom{n}{l}}{\binom{n-k+l'}{l'}}. \tag{13}$$

Adding (12) and (13) we obtain

$$\binom{n}{l} \geqslant |\Delta_l(\mathscr{F})| \geqslant |\Delta_l(\mathscr{K})| + |\Delta_l(\mathscr{B})| > |\mathscr{F}| \binom{n}{l} \Big/ \binom{n-k+l'}{l'},$$

i.e., $|\mathscr{F}| < \binom{n-k+l'}{l'}$, as desired.

(b) $|\mathscr{A}| > |\mathscr{B}|$. Apply Theorem 4.1 to $\mathscr{A}$ to obtain $\mathscr{A}^*$. By definition of $\mathscr{K}$ for $F \in \mathscr{A}^*$ we have $M(F, \mathscr{A}^*) \not\supset \{H: A_1 \subset H \subsetneqq F\}$. By (6.4) we can find missing $(k-1)$ sets, i.e., $A_1 \subset H \subsetneqq F$, $|H| = k-1$, $H \notin M(F, \mathscr{A}^*)$.

PROPOSITION 6.9. *We can find an* $H, A_1 \subseteq H \subsetneqq F$, $|H| = k-1$, $H \notin M(F, \mathscr{A}^*)$ *such that* $H \notin \Delta_{k-1}(\mathscr{K})$.

*Proof.* As $F \notin \mathscr{K}$ we can find a $H'$ such that $A_1 \subset H' \subset F$, $H'$ is not the center of any $(k+1)$-star consisting of members of $\mathscr{F}$. Then, again by the definition of $\mathscr{K}$, $H' \not\subset K$ holds for all $K \in \mathscr{K}$.

Let $H_1, ..., H_r$ be the $(k-1)$-sets satisfying $H' \subset H_i \subset F$, $1 \leqslant i \leqslant r$, $r = k - |H'|$. By the choice of $H'$, $H_i \not\subset K$ holds for all $K \in \mathscr{K}$. Since $H' = H_1 \cap \cdots \cap H_r$, by Theorem 4.1 (6.4) we may pick an $i$ $(1 \leqslant i \leqslant r)$ such that $H = H_i \notin M(F, \mathscr{A}^*)$, proving the proposition. ▮

Now let us choose such an $H = H(F)$ for each $F \in \mathscr{A}^*$. Define $\mathscr{H} = \{H(F) - A_1 : F \in \mathscr{A}^*\}$. As $H(F) \notin M(F, \mathscr{A}^*)$, $|\mathscr{H}| = |\mathscr{A}^*|$ holds. By the choice of $H(F)$ we have $\mathscr{H} \cap \Delta_{l'-1}(\tilde{\mathscr{K}}) = \varnothing$, hence

$$|\mathscr{H}| + |\Delta_{l'-1}(\tilde{\mathscr{K}})| \leqslant \binom{n-k+l'}{l'-1}. \tag{14}$$

Obviously,

$$|\Delta_{l'-1}(\tilde{\mathscr{K}})| \geqslant |\tilde{\mathscr{K}}| \cdot l'/(n-k+1)$$
$$= |\tilde{\mathscr{K}}| \binom{n-k+l'}{l'-1} \Big/ \binom{n-k+l'}{l'} \tag{15}$$

and for large enough $n$,

$$|\mathscr{H}| = |\mathscr{A}^*| \geqslant c_k\, |\mathscr{A}| \geqslant \frac{c_k}{2}\, |\mathscr{A} \cup \mathscr{B}| > \frac{l'}{n-k+1}\, |\mathscr{A} \cup \mathscr{B}| \tag{16}$$

holds. Adding (15) and (16) in view of (14) we obtain

$$|\mathscr{F}| = |\mathscr{K}| + |\mathscr{A} \cup \mathscr{B}| < \binom{n-k+l'}{l'}.$$

## 7. Proof of Theorem 2.3

Suppose $\mathscr{F}$ is an $(l, l')$-system, $|\mathscr{F}| = m(n, k, L(l, l'))$. Let us set $b = l - l'$. For $B \in \binom{X}{b}$ define $\mathscr{F}(B) = \{F - B\colon B \subset F \in \mathscr{F}\}$. Of course we have

$$\sum_{B \in \binom{X}{b}} |\mathscr{F}(B)| = \binom{k}{b} |\mathscr{F}|. \tag{17}$$

$\mathscr{F}(B)$ is an $(n-b, k-b, L(l', l'))$-system. Let $q = p^\alpha$ be a primepower divisor of $k - l = (k-b) - l'$, satisfying $q > l'$. Define $g(x) = \binom{x}{l'}$. Then $g(k-b) \equiv g(l') \equiv 1 \pmod{p}$, i.e., $p \nmid g(k-b)$. On the other hand $p \mid g(r)$ holds for $r = 0, 1, \ldots, l'-1$ because of $g(r) = \binom{r}{l'} = 0$, and for $r = (k-b) - l' = k - l,\ k - l + 1, \ldots, k - b + 1$ because of the exponent of $p$ in $g(r) = r!/l'!(r-l')!$ is $\sum_{\beta \geqslant 1} (\lfloor r/p^\beta \rfloor - \lfloor l'/p^\beta \rfloor - \lfloor (r-l')/p^\beta \rfloor)$ by Legendre formula, and the $\alpha$th member of this sum is positive.

Thus we may apply Theorem 4.6 to $\mathscr{F}(B)$. We infer

$$|\mathscr{F}(B)| \leqslant \binom{n-b}{l'}. \tag{18}$$

Combining (17) and (18) we obtain

$$|\mathscr{F}| \leqslant \binom{n}{b}\binom{n-b}{l'} \Big/ \binom{k}{b} = \binom{n}{l}\binom{k+l'}{l'} \Big/ \binom{k+l'}{l},$$

yielding

$$m(n, k, L(l, l')) \leqslant \binom{n}{l}\binom{k+l'}{l'}\Big/\binom{k+l'}{l},$$

the upper bound part of the theorem.

To prove the lower bound take an $(n, k+l', \{0, 1, ..., l-1\})$-system $\mathscr{S}$ with $|\mathscr{S}| = m(n, k+l', \{0, 1, ..., l-1\})$. Define $\mathscr{F} = \Delta_k(\mathscr{S})$. Obviously, we have $|\mathscr{F}| = \binom{k+l'}{l'}|\mathscr{S}|$, thus Theorem 4.7 yields $|\mathscr{F}| = (1+o(1))\binom{n}{l}\binom{k+l'}{l'}/\binom{k+l'}{l}$.

It remains to show that $\mathscr{F}$ is an $(l, l')$-system. Suppose $F, F' \in \mathscr{F}$. Then there exists $S, S' \in \mathscr{S}$, such that $F \subset S$, $F' \subset S'$. If $S \neq S'$, then $|F \cap F'| \leqslant |S \cap S'| < l$. If $S = S'$, then $|F \cap F'| = |F| + |F'| - |F \cup F'| \geqslant 2k - |S| = k - l'$.

*Remark* 7.1. Our proof shows that if $S(n, k+l', l)$ exists, $k$, $l$, $l'$ as in Theorem 2.3, then $m(n, k, L(l, l')) = \binom{n}{l}\binom{k+l'}{l'}/\binom{k+l'}{l}$ holds. Frankl [7] has shown that in the case $k - l = l' + 1$ a prime, the converse holds, too, i.e., the above equality implies the existence of $S(n, k+l', l)$.

## 8. The Proofs of Theorem 4.3 and Proposition 4.4.

First we show that the case $s \geqslant 2$ is an easy consequence of the case $s = 1$. In fact, take an $(s-1)$-element set $Z$ which is disjoint to $Y$. Define $\tilde{Y} = Y \cup Z$, $\tilde{\mathscr{D}} = \{D_i \cup Z : D_i \in \mathscr{D}\}$. Then $\tilde{Y}$ and $\tilde{\mathscr{D}}$ satisfy the assumptions for $s = 1$, yielding

$$m = |\tilde{\mathscr{D}}| \leqslant |\tilde{Y}| = |Y| + s - 1,$$

as desired.

Note that, any $z \in Z$ satisfies $z \in \tilde{D}$ for all $\tilde{D} \in \tilde{\mathscr{D}}$. Thus Proposition 4.4 applied to $\tilde{\mathscr{D}}$ yields that all the sets of $\tilde{\mathscr{D}}$ have the same size, namely that of $\{\tilde{D} : z \in \tilde{D} \in \tilde{\mathscr{D}}\}$, i.e., $|\tilde{\mathscr{D}}|$. Consequently, $\mathscr{D}$ consists of $r + s - 1$ copies of $Y$.

Now we must deal with the case $s = 1$. We apply induction on $|Y| = r$. Both Theorem 4.3 and Proposition 4.4 are trivial if $r \leqslant 1$. Suppose $r \geqslant 2$. If $y$ is an arbitrary element of $Y$ denote by $d(y)$ its *degree*, i.e., $d(y) = |\{D \in \mathscr{D} : y \in D\}|$.

Proposition 8.1. *For every $y \in D \in \mathscr{D}$ we have*

$$d(y) \leqslant |D|. \tag{19}$$

*Proof.* Define $\bar{Y} = D - \{y\}$, $\bar{\mathscr{D}} = \{(D - \{y\}) \cap D_i : y \in D_i \in \mathscr{D},\ D_i \neq D\}$. Then $\bar{Y}$ and $\bar{\mathscr{D}}$ satisfy the assumptions of Theorem 4.3 (with $s = 1$). By the induction hypothesis we infer $d(y) - 1 = |\bar{\mathscr{D}}| \leqslant |\bar{Y}| = |D| - 1$. ∎

If $d(y)=0$ for some $y\in Y$ then we can use the induction hypothesis for $Y-\{y\}$. Hence we can suppose $d(y)\geqslant 1$ for all $y\in Y$.

If $|\mathscr{D}|<|Y|$ we have nothing to prove. So suppose $|\mathscr{D}|=m\geqslant r=|Y|$ holds. By Proposition 8.1 for all $y\in D\in\mathscr{D}$ we have

$$m-d(y)\geqslant r-|D|.$$

From this, using $|D|\geqslant d(y)>0$ we infer

$$\frac{m-d(y)}{d(y)}\geqslant\frac{r-|D|}{|D|}. \tag{20}$$

Let us sum up (20) for all $y\in D\in\mathscr{D}$:

$$\sum_{y\in Y}\sum_{y\in D\in\mathscr{D}}\frac{m-d(y)}{d(y)}\geqslant\sum_{y\in Y}\sum_{y\in D\in\mathscr{D}}\frac{r-|D|}{|D|}=\sum_{D\in\mathscr{D}}\sum_{y\in D}\frac{r-|D|}{|D|}. \tag{21}$$

On the left-hand size of (21) the interior summation gives $m-d(y)$, while that of the right-hand side is $r-|D|$. Thus (21) reduces to

$$mr-\sum_{y\in Y}d(y)\geqslant mr-\sum_{D\in\mathscr{D}}|D|.$$

However, $\sum_{y\in Y}d(y)=\sum_{D\in\mathscr{D}}|D|$, i.e., the assumption $m\geqslant r$ leads to a contradiction unless equality holds in (19) for all $y\in D\in\mathscr{D}$. In that case obviously $m=r$ holds, proving Theorem 4.3 and the first part of Proposition 4.4.

To prove the second statement of the proposition, suppose that both $\mathscr{D}\cup\{D'\}$ and $\mathscr{D}\cup\{D''\}$ satisfy the assumptions of Theorem 4.3 and $|\mathscr{D}|=r-1=|Y|-1$.

If $|D'|=|D''|=1$ then $|D\cap D'|\neq 1$, $|D\cap D''|\neq 1$ hold for all $D\in\mathscr{D}$. This implies $D\subset(Y-(D'\cup D''))$ for all $D\in\mathscr{D}$. As $\mathscr{D}|=|Y|-1$, Theorem 4.3 implies $|Y-(D'\cup D''')|\geqslant|Y|-1$, that is, $D'=D''$.

Next we assume by symmetry $|D'|\geqslant 2$, $D'\not\subseteq D''$. Let $y$ belong to $D'-D''$. Then $d_{\mathscr{D}\cup\{D'\}}(y)=|D'|\geqslant 2$ by the first statement of the proposition. Thus we may find a $D\in\mathscr{D}$ such that $y\in\mathscr{D}$. Now, the first statement of the proposition yields

$$d_{\mathscr{D}\cup\{D'\}}(y)=|D|=d_{\mathscr{D}\cup\{D''\}}(y).$$

However, by the definition of $y$, we have

$$d_{\mathscr{D}\cup\{D'\}}(y)=d_{\mathscr{D}\cup\{D''\}}(y)+1,$$

a contradiction.

## 9. An Open Problem Concerning Designs

The investigation of the extremal families for Theorem 4.3 led to the following notion. Call the family $\mathscr{D} = \{D_1, D_2, \ldots, D_m\}$ on the underlying set $Y$ a *well-intersecting design* of order $r$ if

(i) $|Y| = m$, $|D_i| = r$ for all $1 \leqslant i \leqslant m$.

(i) $|D_{i_1} \cap \cdots \cap D_{i_t}| \neq t-1$ for all $1 \leqslant i_1 < \cdots < i_t \leqslant m$.

(iii) $\mathscr{D}$ is connected, i.e., for all partitions $\{A, B\}$ of $Y$ there exists a $D \in \mathscr{D}$ such that $A \cap D \neq \emptyset \neq B \cap D$.

Proposition 4.4 implies that $\mathscr{D}$ is a 1-design, $d_{\mathscr{D}}(x) = r$ holds for all $x \in Y$. Some examples

(1) $m = r$, $D_i = Y$ for all $1 \leqslant i \leqslant m$.

(2) $m = r+1$, $r$ is odd, and $D_i = Y - \{y_i\}$ $(Y = \{y_1, y_2, \ldots, y_m\})$.

If $r \leqslant 3$ then these are the only well-intersecting designs. But for $r = 4$ there are exactly four: type 1, the complement of the Fano-plane $(m = 7)$, the extended Hamming code $(m = 8)$, and a simple construction on 6 points. See Fig. 1.

(3) $m = \binom{r}{2} + 1$, the biplanes of order $r$.

(4) $m = q^3 + q^2 + q + 1$, $r = q^2 + q + 1$, the planes of $PG(3, q)$.

(5) Finally, it is easy to prove that: If $A$ is the incidence matrix of a well-intersecting design of order $r$ and $A$ is symmetric then the matrix $B = \begin{bmatrix} A & I \\ I & A \end{bmatrix}$ is the incidence matrix of a well-intersecting design of order $r+1$. (Here $I$ denotes the $m \times m$ identity matrix.)

In this way we can obtain a well-intersecting design of order $r$ with $m = 2^{r-1}$.

It would be interesting to know more about the structure of well-intersecting designs.

*Problem* 9. Is it true that the number of different well-intersecting designs of order $r$ is finite for any fixed $r$?

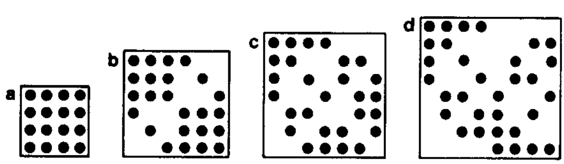


Fig. 1. The only well-intersecting designs of order 4. (c) Biplane of order 4. (d) Extended Hamming code.

*Note Added in Proof.* Cameron, Frankl, and Wilson have shown that any well-intersecting design of order $r$ satisfies $m = n \leqslant 2^{r-1}$. Moreover, the only design with $m = n = 2^{r-1}$ is coming from the $r$-dimensional cube: the incidence matrix of the design is the adjacenty matrix of the cube as a bipartite graph.

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