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AN EXTREMAL PROBLEM CONCERNING KNESER'S CONJECTURE

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Abstract

It is proved that if $\mathscr{F}_1$ and $\mathscr{F}_2$ are k-uniform, intersecting set-systems over an n-element set ($F\cap F'\neq 0$ for all $F, F'\in\mathscr{F}_i$, $i=1,2$) and $n>6k$ then $|\mathscr{F}_1\cup\mathscr{F}_2|\leqq\binom{n-1}{k-1}+\binom{n-2}{k-1}$. It is to be expected that the same holds for all $n\geqq 2k+2$.

1. Introduction

The well-known Kneser Conjecture [12], proved by Lovász [14] and Bárány [1], is equivalent to the following assertion. If $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_t$ are k-uniform, intersecting set-systems over an n-element set X $\big($i.e. $|F|=k$ for all $F\in\mathscr{F}_i$ and $F\cap F'\neq\emptyset$ for all $F, F'\in\mathscr{F}_i (1\leqq i\leqq t)\big)$ and $n\geqq t+2k-1$ then

(1) $$|\cup\mathscr{F}_i|<\binom{n}{k}.$$

In other words (1) means that Kneser's graph K^k_{t+2k-1}, obtained by connecting two k-element subsets of X whenever they have empty intersection, cannot be coloured by t colours. So the chromatic number $\chi(K^k_{t+2k-1})=t+1$.

However, Lovász's and Bárány's nice proofs of (1), even Schrijver's [16] interesting generalization, which use deep geometrical tools, do not tell the true order of magnitude of the left-hand side of (1). This geometrical method does not seem suitable for estimating the distribution of sizes of colour classes of K^k_{t+2k-1}. The determination of max $|\cup\mathscr{F}_i|$ is in fact a problem belonging to extremal hypergraph theory. More than 10 years ago P. Erdős suggested [6] to attack the Kneser Conjecture in this way, however no "real", i.e. hypergraph theoretical proof is known. Essentially this paper is concerned with the case $t=2$.

2. Results

Let us set $f(n, k, t)=\max\Big\{\Big|\bigcup\limits_{i=1}^{t}\mathscr{F}_i\Big|: \mathscr{F}_i$ is a k-uniform, intersecting set-system over the n-element set $X\Big\}$. Henceforth we shall assume that the elements of X are the integers from 1 to n. If $n\leqq t+2k-2$ then

(2) $$f(n, k, t)=\binom{n}{k}.$$

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Indeed, let $\mathscr{F}_i=\{F\subset X\colon |F|=k,\ i\in F\}$ for $1\leqq i\leqq t-1$ and $\mathscr{F}_t=\{F\subset X\colon |F|=k, F\subset(X-\{1, \dots, t-1\})\}$. As $|X-\{1, \dots, t-1\}|\leqq 2k-1$, the set-system $\mathscr{F}_t$ is intersecting, too. For greater values of n one can replace $\mathscr{F}_t$ by the set-system whose members have the common point t, hence

$$f(n, k, t) \geqq \binom{n-1}{k-1}+\binom{n-2}{k-1}+\dots+\binom{n-t}{k-1}. \tag{3}$$

Erdős [6] conjectured that here equality holds for all $n\geqq t+2k-1$. This would be a substantial improvement of the Kneser Conjecture. The case $k=1$ is of no interest. It is easy to see that the Erdős conjecture holds for $k=2$.

PROPOSITION 1. *If* $n\geqq t+3$ *then*

$$f(n, 2, t) = \binom{n}{2}-\binom{n-t}{2}. \tag{4}$$

The Erdős—Ko—Rado [8] theorem states that the conjecture is true for $t=1$ and each k:

$$\text{If}\quad n\geqq 2k\quad\text{then}\quad f(n, k, 1)=\binom{n-1}{k-1}. \tag{5}$$

However, A. J. W. Hilton [10] observed that the Erdős conjecture is not true for $k=3, t=2$. Here we prove that if $k>2$ and $t>1$ then equality cannot hold in (3) for $n=t+2k-1$, i.e.

PROPOSITION 2. *If* $k\geqq 3,\ t\geqq 2$ *then*

$$f(t+2k-1, k, t) > \sum_{i=1}^{t}\binom{n-i}{k-1}. \tag{6}$$

This disproves the conjecture, but for the smallest admissible value of n, namely $n=t+2k-1$. Nevertheless, we believe that the conjecture is correct for any other value of n:

CONJECTURE. If $n\geqq t+2k$ then $f(n, k, t)=\sum_{i=1}^{t}\binom{n-i}{k-1}$. Here equality holds iff $\cap\mathscr{F}_i$ is nonempty for each i.

For n large enough this conjecture is supported by the following result of P. Erdős [5].

$$\textit{If}\quad n>n_0(k, t)\quad\textit{then}\quad f(n, k, t)=\sum_{i=1}^{t}\binom{n-i}{k-1}. \tag{7}$$

His proof gives an exponentially large value for $n_0(k, t)$. Later he, together with Bollobás and Daykin [3], proved that $n_0(k, t)<2k^3t$. (This bound was improved to $ckt\log t$ by P. Frankl and the author (unpublished) but our c is very large.) Below we consider the case $t=2$.

THEOREM. *If $\mathscr{F}_1$ and $\mathscr{F}_2$ are k-uniform intersecting set-systems over an n-element underlying set X ($n>6k$) and the cardinality of $\mathscr{F}_1\cup\mathscr{F}_2$ is maximal with respect* *to these assumptions, then there exist* $x_1, x_2 \in X$ *such that* $x_i \in \cap \mathcal{F}_i$, *i.e.*,

$$\text{(8)} \qquad \text{if} \quad n > 6k \quad \text{then} \quad f(n, k, 2) = \binom{n-1}{k-1} + \binom{n-2}{k-1}.$$

Thus there are several extremal families, but each of them is incident to two points.

REMARK 1. In the theorem the assumption $n > 6k$ can be replaced by $n > r_0(k)$, where r_0 is the smallest number r satisfying (13). The conjectured value of $r_0(k)$ is $3k-3$ (see [9]).

REMARK 2. However, Proposition 2 does not disprove the following nice conjecture of P. Erdős [5]:

$$g(n, k, t) = \max\left\{\sum_{i=1}^{t}\binom{n-i}{k-1};\ \binom{kt+k-1}{k}\right\},$$

where $g(n, k, t) =: \max\{|\mathcal{F}|: \mathcal{F}$ is k-uniform set-system over an n-element set X, and $\nexists$ $F_1, F_2, \dots F_{t+1}$ pairwise disjoint$\}$. This is a theorem of Erdős and Gallai [7] for $k=2$.

REMARK 3. Having learned Hilton's counterexample Erdős modified his conjecture in the following way: If $n \geqq t+2k-1$ then

$$f(n, k, t) < \binom{n-1}{k-1} + \dots + \binom{n-t}{k-1} + \binom{n-t-1}{k-1}.$$

Here the right-hand side is less than $\binom{n}{k}$, so this conjecture would imply (1), too.

3. Proofs

For the proofs of (1), (2), (3), (5) and (7) see the references. The proof of (4) is extremely easy.

LEMMA 1. *If* $\mathcal{F}$ *is a 2-uniform, intersecting set-system, then either* $\cap \mathcal{F} \neq \emptyset$ *(i.e.,* $\mathcal{F}$ *is a star) or* $\mathcal{F}$ *is a triangle.* □

This lemma contains the case $t=1$. We use induction on t. If in the case $t>1$ each set-system $\mathcal{F}_1, \dots, \mathcal{F}_t$ is a triangle then

$$|\cup \mathcal{F}_i| \leqq \Sigma|\mathcal{F}_i| = 3t < \binom{n}{2} - \binom{n-t}{2}$$

and we are ready. So we can suppose that some $\mathcal{F}_i$ is a star, say $1 \in \cap \mathcal{F}_1$. Then we can use the induction hypothesis for the members of $\mathcal{F}_2, \mathcal{F}_3, \dots, \mathcal{F}_t$ lying in $X-\{1\}$. □

The proof of (6) is a construction It is enough to give this construction for the case $t=2$, because for all n, k and t

$$f(n, k, t) \geqq \binom{n-1}{k-1} + f(n-1, k, t-1).$$

So $|X|=n=2k+1$, and let $X=A\cup B$, $|A|=3$, $|B|=2k-2$ and choose some $b\in B$. Let $\mathscr{F}_1=\{F\subset X\colon |F|=k,\ |F\cap A|\geqq 2\}$, and $\mathscr{F}_2=\{F\subset B\colon |F|=k\}\cup\{F\subset X\colon |F|=k,\ |F\cap B|=k-1,\ b\in F\}$. Then $\mathscr{F}_1$ and $\mathscr{F}_2$ are two disjoint set-systems and each of them is intersecting. Moreover

$$|\mathscr{F}_1|+|\mathscr{F}_2| = 3\binom{n-3}{k-2}+\binom{n-3}{k-3}+\binom{n-3}{k}+3\binom{n-4}{k-2} =$$

$$= \binom{n-1}{k-1}+\binom{n-2}{k-1}+\binom{2k-3}{k} > \binom{n-1}{k-1}+\binom{n-2}{k-1}.$$

Proof of the Theorem. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two k-uniform, intersecting set-systems over X, $X=\{1, 2, \dots, n\}$, without common member ($\mathscr{F}_1\cap\mathscr{F}_2=\emptyset$). Suppose that

$$|\mathscr{F}_1|+|\mathscr{F}_2| \geqq \binom{n-1}{k-1}+\binom{n-2}{k-1}.$$

It is enough to show that $\bigcap\mathscr{F}_1\neq\emptyset$. Use the following operation P_{ij} which was first applied by Erdős, Ko and Rado for the proof of the theorem (5). However, here we are using this P_{ij} for ordering two set-systems at the same time. For $1\leqq i<j\leqq n$ and $F\in\mathscr{F}_1\cup\mathscr{F}_2$ we have

$$P_{ij}(F)=\begin{cases} F\cup\{i\}-\{j\} & \text{if } F\in\mathscr{F}_1,\ i\notin F,\ j\in F \text{ and } F\cup\{i\}-\{j\}\notin\mathscr{F}_1 \\ F\cup\{j\}-\{i\} & \text{if } F\in\mathscr{F}_2,\ j\notin F,\ i\in F \text{ and } F\cup\{j\}-\{i\}\notin\mathscr{F}_2 \\ F & \text{otherwise.}\end{cases}$$

Let

$$P_{ij}(\mathscr{F}_s) = \{P_{ij}(F)\colon F\in\mathscr{F}_s\} \qquad (s = 1, 2).$$

So we have got two new set-systems. Clearly, $|\mathscr{F}_s|=|P_{ij}(\mathscr{F}_s)|$, and with the aid of [8] it can be shown that $P_{ij}(\mathscr{F}_1)$ and $P_{ij}(\mathscr{F}_2)$ are two disjoint, intersecting set-systems, too. Applying each P_{ij} for all $1\leqq i<j\leqq n$ (perhaps several times) after finitely many steps we get two (disjoint and intersecting) set-systems $\mathscr{L}$ and $\mathscr{R}$ which are pushed to the left and to the right, respectively, i.e.,

(9) If $L\in\mathscr{L}$, $i<j$, $i\notin L$, $j\in L$ then $L\cup\{i\}-\{j\}\in\mathscr{L}$.

Similarly,

If $R\in\mathscr{R}$, $i<j$, $i\in R$, $j\notin R$ then $R-\{i\}\cup\{j\}\in\mathscr{R}$. Denote by

$$\mathscr{L}(1, n) =: \{L\in\mathscr{L}\colon 1\in L,\ n\in L\},$$

$$\mathscr{L}(1, \bar{n}) =: \{L\in\mathscr{L}\colon 1\in L,\ n\notin L\}, \quad \text{etc.}$$

Now we give an upper estimation for the cardinalities of these parts of $\mathscr{L}$ and $\mathscr{R}$. Clearly,

(10) $$|\mathscr{L}(1, n)|+|\mathscr{R}(1, n)| \leqq \binom{n-2}{k-2}.$$

LEMMA 2. $|\mathscr{L}-\mathscr{L}(1, n)| \leqq \binom{n-2}{k-1}$, *furthermore if here equality holds, then* $1 \in \bigcap \mathscr{L}$ (*so* $\mathscr{L}(\bar{1}, n)=\emptyset$ *and* $\mathscr{L}(\bar{1}, \bar{n})=\emptyset$).

PROOF of Lemma 2. Let us write $|\mathscr{L}(\bar{1}, \bar{n})|=m$. The case $m=0$ is trivial, so suppose $m \geqq 1$. Then

$$|\mathscr{L}(\bar{1}, n)| \leqq m \frac{k}{n-k+1}. \tag{11}$$

In fact, using (9) to each $L \in \mathscr{L}(\bar{1}, n)$ there corresponds $n-k+1$ members of $\mathscr{L}(\bar{1}, \bar{n})$, namely, the sets $L-\{n\} \cup \{i\}$ $(2 \leqq i \leqq n-1,\ i \notin L)$. Moreover, in this way we can get each member of $\mathscr{L}(\bar{1}, \bar{n})$ at most k times.

$$\text{If} \quad n > 6k \quad \text{then} \quad m \leqq \binom{n-4}{k-2} = \binom{n-4}{n-k-2}. \tag{12}$$

In fact, the set-system $\mathscr{L}(\bar{1}, \bar{n})$ is 2-intersecting, i.e., if $A, B \in \mathscr{L}(\bar{1}, \bar{n})$ then $|A \cap B| \geqq 2$. (This follows from the fact that $A \cap B=\{x\}$ implies $(A-\{x\} \cup \{1\}) \cap B=\emptyset$). So we can apply the sharper form of the Erdős—Ko—Rado theorem which is due to P. Frankl [9]:

If $\mathscr{H}$ is a k-uniform, 2-intersecting set-system over an r-element set and $r \geqq 6k-1$ then

$$|\mathscr{H}| \leqq \binom{r-2}{k-2}. \tag{13}$$

Now the cardinality of $\mathscr{L}(1, \bar{n})$ will be estimated by an upper bound depending on m. The main idea of the following argument is due to Daykin [4], who gave an original proof for theorem (5) in this way. We need the Kruskal—Katona theorem [11], [13]. We use a weaker but much more simple form of this theorem which is due to Lovász [15]:

If $\mathscr{A}$ is an arbitrary a-uniform set-system and

$$|\mathscr{A}| = \binom{x}{a}, \quad \text{then} \quad |\Delta_b(\mathscr{A})| \geqq \binom{x}{b}. \tag{14}$$

Here $a \geqq b > 0$ are integers, $x \geqq a$ is a real number,

$$\binom{x}{a} =: \frac{1}{a!} x(x-1)\dots(x-a+1)$$

and

$$\Delta_b(\mathscr{A}) =: \{B\colon |B| = b,\ \exists A \in \mathscr{A}\ B \subset A\}.$$

Denote by $\mathscr{A}$ the set of the complements of the members of $\mathscr{L}(\bar{1}, \bar{n})$ with respect to $X-\{1, n\}$. $|\mathscr{A}|=m=\binom{x}{n-k-2}$, where $x \geqq n-k-2$. If $|B|=k-1$, $B \subset A \in \mathscr{A}$

then $B\cup\{1\}\notin\mathscr{L}(1,\bar{n})$, because $\mathscr{L}$ is intersecting. Hence

$$\mathscr{L}(1,\bar{n})\leqq\binom{n-2}{k-1}-|\Delta_{k-1}(\mathscr{A})|\leqq\binom{n-2}{k-1}-\binom{x}{k-1}. \tag{15}$$

Now (12) gives that $x\leqq n-4$, so applying (11) we get

$$|\mathscr{L}(\bar{1},n)|+|\mathscr{L}(\bar{1},\bar{n})|\leqq m\frac{n+1}{n-k+1}= \tag{16}$$

$$\binom{x}{n-k-2}\frac{n+1}{n-k+1}\leqq\binom{x}{k-2}\frac{n+1}{n-k+1}<\binom{x}{k-1}.$$

Finally, summing (15) and (16) we obtain Lemma 2. Moreover heie equality can hold only if $m=0$, so $\mathscr{L}(\bar{1},\bar{n})=\emptyset$, $\mathscr{L}(\bar{1},n)=\emptyset$ and $\mathscr{L}(1,\bar{n})=\{L\subset X:\ |L|=k,\ 1\in L,\ n\notin L\}$. $\square$

Returning to the proof of the Theorem we can get an estimation for $\mathscr{R}$, similarly to Lemma 2. Hence together with (10) we have:

$$|\mathscr{L}|+|\mathscr{R}|\leqq 2\binom{n-2}{k-1}+\binom{n-2}{k-2}=\binom{n-1}{k-1}+\binom{n-2}{k-1}.$$

Finally, it is easy to check that if $\cap\mathscr{L}=\{1\}$ and $\cap\mathscr{R}=\{n\}$ then before carrying out the operations P_{ij} $\cap\mathscr{F}_s\neq\emptyset$, too. $\square$

Added in proof. Further new results are in P. Frankl and Z. Füredi „Extremal problems concerning Kneser-graphs" (submitted to *J. Combinatorial Theory Ser. B*). Especially, it is proved that (8) is true for $n>\frac{1}{2}(3+\sqrt{5})k\sim 2.62k$ and it does not hold for $n<2k+o(\sqrt{k})$ disproving my conjecture.

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