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SET SYSTEMS WITH THREE INTERSECTIONS

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Let X be a finite set of n elements and $\mathcal{F}$ a family of $4a+5$-element subsets, $a\geqq 6$. Suppose that all the pairwise intersections of members of $\mathcal{F}$ have cardinality 0, a or $2a+1$. We show that $c_1 n^{4/3} < \max|F| < c_2 n^{4/3}$ for some positive c_i's. This answers a question of P. Frankl.

1. Introduction

Let $0\leqq l_1<l_2<\ldots<l_s<k<n$ be integers, and X be a finite set of cardinality n. Denote by $\binom{X}{k}$ the system of all k-subsets of X. We say that the family $\mathcal{F}\subset\binom{X}{k}$ is an $(n, k, \{l_1, \ldots, l_s\})$-system if for every $F_1, F_2\in\mathcal{F}$, $F_1\neq F_2$ we have $|F_1\cap F_2|\in \in\{l_1, \ldots, l_s\}=L$. Let us denote by $m(n, k, L)$ the maximum cardinality of an (n, k, L)-system. This function has been investigated by many authors, but to determine its exact value or even its correct order of magnitude appears to be hopeless.

Ray-Chaudhuri and Wilson [8] proved that

$$m(n, k, \{l_1, \ldots, l_s\}) \leqq \binom{n}{s}. \tag{1}$$

Deza, Erdös and Singhi [2] proved that

$$m(n, k, \{0, a\}) \leqq \frac{n}{k}\frac{n-a}{k-a}, \tag{2}$$

moreover if a does not divide k then

$$m(n, k, \{0, a\}) \leqq n. \tag{3}$$

The next results of Babai and Frankl [1] generalize (3): If g.c.d $(l_1, \ldots, l_s)$ does not divide k then

$$m(n, k, \{l_1, \ldots, l_s\}) \leqq n. \tag{4}$$

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However if $l_1=0$ and there exist non-negative integers $\alpha_1, \ldots, \alpha_s$ such that $k= \sum_{i=1}^{s} \alpha_i l_i$ then

(5) $$m(n, k, \{l_1, \ldots, l_s\}) \geqq n^2/4k^2.$$

Generalizing an earlier result of Frankl [6] the author [7] gave necessary and sufficient conditions for $m(n, k, L)=O(n)$. We need a definition to recall this. A set-system $\mathcal{M}$ is *closed under intersection* if $M \cap M' \in \mathcal{M}$ holds for all $M, M' \in \mathcal{M}$. We say that the numbers $l_1<\ldots<l_s<k$ satisfy the condition (*) if

(*) *there exists a set-system* $\mathcal{M}$ *which is closed under intersection,* $|\bigcup \mathcal{M}|=k$ *and* $|M| \in \{l_1, \ldots, l_s\}$ *for all* $M \in \mathcal{M}$.

Now, the following statement is proved in [7]:

(6) *If the numbers* $l_1, \ldots, l_s$ *and* k *satisfy the condition* (*) *then* $m(n, k, L)>c_k n^{k/(k-1)}$, *otherwise* $m(n, k, L) \leqq c_k' n$.

2. Set-systems with $\leqq 3$ intersections

We write $f(n) \approx g(n)$ if there exist positive constants c_1, c_2 such that $f(n) \leqq c_1 g(n)$ and $g(n) \leqq c_2 f(n)$ hold for $n>n_0$. It is easy to prove that (see [5])

$$m(n, k, \{l_1, \ldots, l_s\}) \approx m(n, k-l_1, \{0, l_2-l_1, \ldots, l_s-l_1\}).$$

Therefore from now on we always assume $l_1=0$. Trivially $m(n, k, \{0\})=\lfloor n/k \rfloor \approx n$.
For $s=2$ from (2), (3) and (5) we deduce

$$m(n, k, \{0, a\}) \approx \begin{cases} n^2 & \text{if } a \mid k \\ n & \text{if } a \nmid k. \end{cases}$$

In [6] Frankl investigated the case $s=3$. He proved the following theorem:

(7a) *If either there exist non-negative integers* α, β *such that* $\alpha a+\beta b=k$ *or* $b-a$ *divides* $k-a$ *then* $m(n, k, \{0, a, b\}) \geqq n^2/4k^2$.

(7b) *If (7a) does not hold but* (*) *holds then* $n^{k/(k-1)}<m(n, k, \{0, a, b\})<c_k n^{3/2}$.

(7c) *If* (*) *does not hold then* $m(n, k, \{0, a, b\}) \approx n$.

Suppose that the numbers $0, a, b$ and k satisfy the condition (*), i.e. there exists a set-system $\{A_1, \ldots, A_f, B_1, \ldots, B_g\}=\mathcal{M}$ such that $|A_i|=a$, $|B_i|=b$, $A_i \cap A_j=0$, $|A_i \cap B_j|=0$ or a and $|\bigcup \mathcal{M}|=k$. Let $I=\{i\colon 1 \leqq i \leqq f,\ A_i$ is contained in at least two B_j's$\}$ and $B_j'=\{i \in I\colon A_i \subset B_j\}$, finally define $\mathcal{C}=\{B_j'\colon 1 \leqq j \leqq g\}$. We say that the numbers a, b and k satisfy the condition (**) if

(**) *there exists a set-system* $\mathcal{M}$ *on points* $\{1, 2, \ldots, k\}$ *satisfying the condition* (*) *such that* $\mathcal{C}$ *is a 2-design on* I. *(I.e., each pair of* I *is contained in exactly one* $C \in \mathcal{C}$.)

In [6] Frankl gave a better lower bound than (7b):
(8) *If* (**) *holds for the numbers* a, b *and* k *and there exists an embedding* φ *of* $\mathcal{C}$

into the system of lines of a projective plane over a finite field then $m(n, k, \{0, a, b\}) \geqq$ $\geqq c'_k n^{3/2}$.

In [6] Frankl and Frost posed the question whether $m(n, k, \{0, a, b\}) \approx n^{3/2}$ holds in the case (7b) or not. We give a negative answer by showing that this problem is rather complicated.

3. Results and constructions

Theorem 1. *If* (**) *does not hold (consequently* (7a) *does not hold either) then* $m(n, k, \{0, a, b\}) \leqq c''_k n^{4/3}$.

Example 1. Let a, d, k be positive integers with $k = 4a + 5d$. For n large enough we have $m(n, 4a+5d, \{0, a, 2a+d\}) \geqq n^{4/3}/10d^2$. Let t be a positive integer (the value of t will be about $\sqrt[3]{n/4d}$) and A^1_p, A^2_q $(0 \leqq p, q < t)$, $A^3_{r,s}, A^4_{u,v}$ $(0 \leqq r, s, u, v < t)$ pairwise disjoint a-sets and $D^{12}_{u,v}$, $D^{13}_{u,v,w}$, $D^{14}_{u,v,w}$, $D^{23}_{u,v,w}$ and $D^{24}_{u,v,w}$ disjoint d-sets $(0 \leqq u, v, w < t)$. The ground-set X of $\mathscr{F}$ is the union of all A's and D's. Hence $|X| = 2at + 2at^2 + dt^2 + 4dt^3$. For integers $0 \leqq p, q, r, s < t$ let us denote by

$$F(p, q, r, s)$$
$$= A^1_p \cup A^2_q \cup A^3_{r,s} \cup A^4_{p+r,q+s} \cup D^{12}_{p,q} \cup D^{13}_{p,r,s} \cup D^{14}_{p,p+r,q+s} \cup D^{23}_{q,r,s} \cup D^{24}_{q,p+r,q+s}.$$

Here the indices are considered mod t. Clearly $|\mathscr{F}| = t^4 > n^{4/3}/10d^2$ if $t = \lfloor \sqrt[3]{n/4d} \rfloor$ and n is large enough. It is easy to check that $F(p, q, r, s) \cap F(p', q', r', s') = \emptyset$ or A^i or $A^i \cup A^j \cup D^{ij}$ i.e. $\mathscr{F}$ is a $\{0, a, 2a+d\}$-system.

Example 2. For n large enough and $k = 5a + 8d$ we have $m(n, 5a+8d, \{0, a, 2a+d\}) \geqq$ $\geqq n^{4/3}/20d^2$. Let t be a positive odd integer $(t \approx \lfloor \sqrt[3]{n/7d} \rfloor)$ and $A^1_i, A^2_i, A^3_{i,j}, A^4_{i,j}$ and $A^5_{i,j}$ disjoint a-sets $(0 \leqq i, j < t)$. Define eight sequences $D^{\alpha\beta}$ $(1 \leqq \alpha < \beta \leqq 5$ except $\alpha\beta \neq 35, 45)$ of d-sets, $D^{12}_{i,j}, D^{13}_{i,j,k}, D^{14}_{i,j,k}, D^{15}_{i,j,k}, D^{23}_{i,j,k}, D^{24}_{i,j,k}, D^{25}_{i,j,k}$ and $D^{34}_{i,j,k}$ $(0 \leqq i, j, k < t)$. Each $D^{\alpha\beta}_{i,j,k}$ corresponds to the pair $A^{\alpha}_i, A^{\beta}_{j,k}$. The ground-set of $\mathscr{F}$ consists of the A^{α}-s and $D^{\alpha\beta}$-s. (So $|X| = 2at + 3at^2 + dt^2 + 7dt^3$.)

A^1	A^2	A^3	A^4	D^{12}	D^{13}	D^{23}	D^{14}	D^{24}
				p	p	q	p	q
p	q	r	$p+r$	q	r	r	$p+r$	$p+r$
		s	$q+s$		s	s	$q+s$	$q+s$

Example 1

A^5	A^1	A^2	A^3	A^4	D^{12}	D^{15}	D^{25}	D^{13}	D^{14}	D^{23}	D^{24}	D^{34}
α			$u+v$	$u+v$		α	α	$u+v$	$u+v$	$u+v$	$u+v$	$u+v$
β			$u+\alpha$	$v+\alpha$	$u+\beta$	β	β	$u+\alpha$	$u+\beta$	$u+\alpha$	$v+\alpha$	$u+\alpha$
	$u+\beta$	$v+\beta$			$v+\beta$	$u+\beta$	$v+\beta$	$u+\beta$	$v+\alpha$	$v+\beta$	$v+\beta$	$v+\alpha$

Example 2

Let

$$F(\alpha, \beta, u, v) = A^1_{u+\beta} \cup A^2_{v+\beta} \cup A^3_{u+v,\alpha+u} \cup A^4_{u+v,\alpha+v} \cup A^5_{\alpha,\beta} \cup D^{12}_{u+\beta,v+\beta} \cup D^{13}_{u+\beta,u+v,\alpha+u}$$
$$\cup D^{14}_{u+\beta,u+v,\alpha+v} \cup D^{15}_{u+\beta,\alpha,\beta} \cup D^{23}_{v+\beta,u+v,\alpha+u} \cup D^{24}_{v+\beta,u+v,\alpha+v} \cup D^{25}_{v+\beta,\alpha,\beta} \cup D^{34}_{u+v,\alpha+u,\alpha+v}.$$

Theorem 1 and Example 1 yield (with $d=1$)

Corollary. *If* $a \geqq 6$ *then* $m(n, 4a+5, \{0, a, 2a+1\}) \approx n^{4/3}$, *e.g.* $m(n, 29, \{0, 6, 13\}) \approx \approx n^{4/3}$. ∎

4. Proof of the upper bound

4.1. Lemmas and definitions. The sets $F_1, \ldots, F_t$ form a *t-star* with kernel A if $F_i \cap F_j = A$ for all $1 \leqq i < j \leqq t$. The k-uniform set-system $\mathcal{G}$ is *k-partite* with parts $X_1, \ldots, X_k$ if these sets are disjoint and $|G \cap X_i| = 1$ holds for every $G \in \mathcal{G}$, $1 \leqq i \leqq k$. Erdős and Kleitman [3] proved that one can choose a k-partite subgraph $\mathcal{G}$ from any k-uniform set-system $\mathcal{F}$ such that $|\mathcal{G}| \geqq (k!/k^k)|\mathcal{F}|$. The following theorem (see [7]) is a generalization of the theorem of Erdős and Kleitman and a theorem of Erdős and Rado [4] about t-stars.

Lemma. *For any positive integers k and t, there exists a positive real number* $c = c(k, t)$ *with the following property: If* $\mathcal{F}$ *is a k-graph then we can choose a subsystem* $\mathcal{F}^* \subset \mathcal{F}$ *such that*

(i) $|\mathcal{F}^*| > c|\mathcal{F}|$

(ii) $\mathcal{F}^*$ *is k-partite with parts* $X_1, \ldots, X_k$

(iii) *every intersection is a kernel of a t-star in* $\mathcal{F}^*$ *(i.e.,* $\forall F, F' \in \mathcal{F}^* \exists F_1, \ldots, \ldots, F_t \in \mathcal{F}^*$ *such that* $F \cap F' = F_i \cap F_j$ *for all* $1 \leqq i < j \leqq t$*).*

(iv) *there exists a set-system* $\mathcal{M}$ *on the elements* $\{1, 2, \ldots, k\}$ *such that* $\mathcal{M}$ *is isomorphic (in the natural way) to the intersection-system of each* $F \in \mathcal{F}^*$ *(i.e.* $\mathcal{M} \cong \mathcal{M}(F, \mathcal{F}^*) =: \{F \cap F' \colon F' \in \mathcal{F}^*\}$ *for each* $F \in \mathcal{F}^*$*).*

(v) *For* $t \geqq k+1$ $\mathcal{M}$ *is closed under intersection.* ∎

4.2. Proof of Theorem 1. Suppose first $a|b$. Since (7a) does not hold, $a \nmid k$. Then (4) yields $|\mathcal{F}| \leqq n$. From now on we may suppose $a \nmid b$. Let $\mathcal{F}$ be an $(n, k, \{0, a, b\})$-system, and let $\mathcal{F}^* \subset \mathcal{F}$ be chosen according to the Lemma with $t = k+1$. We are going to estimate $|\mathcal{F}^*|$. Let $\mathcal{A} = \{A \colon |A| = a,\ \exists F, F' \in \mathcal{F}^*\ F \cap F' = A\}$ and $\mathcal{B} = \{B \colon |B| = b,\ \exists F, F' \in \mathcal{F}^*\ F \cap F' = B\}$. By the Lemma we have $A \cap A' = \emptyset$ for every $A, A' \in \mathcal{A}$, hence $|\mathcal{A}| \leqq \lfloor n/a \rfloor \leqq n$. Similarly, $B \cap B' \in \mathcal{A} \cup \{\emptyset\}$ for every $B, B' \in \mathcal{B}$. So $\mathcal{B}$ is an $(n, b, \{0, a\})$-system. Hence $|\mathcal{B}| \leqq n$ by (3).

Let $\mathcal{F}_0 = \mathcal{F}^*$. Let us define sub-systems $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \ldots \supset \mathcal{F}_i$ and subsets $C_1, C_2, \ldots, C_i \in \mathcal{A} \cup \mathcal{B}$ in the following way. If there exists a $C_{i+1} \in \mathcal{A} \cup \mathcal{B} - \{C_1, \ldots, \ldots, C_i\}$ such that $|\{F \in \mathcal{F}_i \colon C_{i+1} \subset F\}| < \sqrt[3]{n}$ then let $\mathcal{F}_{i+1} = \mathcal{F}_i - \{F \in \mathcal{F}_i \colon C_{i+1} \subset F\}$. When our procedure stops we get $\mathcal{F}_r$. Clearly

$$|\mathcal{F}^* - \mathcal{F}_r| \leqq |\mathcal{A} \cup \mathcal{B}| \sqrt[3]{n} < 2n^{4/3}. \tag{9}$$

The number of members of $\mathcal{F}_r$ containing a given $C \in \mathcal{A} \cup \mathcal{B}$ is either at least $\sqrt[3]{n}$ or 0.

Now we show that $|\mathcal{F}_r|\leq b^2n^{4/3}$. Let us denote by $\mathcal{A}_0$ the set of members of $\mathcal{A}$ which are contained in at least $\sqrt[3]{n}$ members of $\mathcal{B}$. Obviously, we have

(10)

$$|\mathcal{A}_0| \leq \frac{1}{\sqrt[3]{n}} \sum_{A\in\mathcal{A}} |\{B\in\mathcal{B}\colon A\subset B\}| = \frac{1}{\sqrt[3]{n}} \sum_{B\in\mathcal{B}} |\{A\in\mathcal{A}\colon A\subset B\}| \leq \frac{1}{\sqrt[3]{n}} n\lfloor b/a\rfloor \leq bn^{2/3}.$$

Let $F_0\in\mathcal{F}_r$ be chosen arbitrarily, and $\mathcal{M}_{F_0}=\{C\in\mathcal{A}\cup\mathcal{B}\colon C\subset F_0\}$. (If $\mathcal{F}_r=\emptyset$ then we are ready.) The condition (**) does not hold, hence there exist two distinct a-sets A_1 and A_2 in $\mathcal{M}_{F_0}$ which are contained in at least two b-sets, $A_1=B_1\cap B_1'$ and $A_2=B_2\cap B_2'$ ($B_1, B_2, B_1', B_2'\in\mathcal{M}_{F_0}$), but there is no $B\in\mathcal{B}$ such that $A_1\cup A_2\subset B$.

B_1 is contained in at least $\sqrt[3]{n}$ members of $\mathcal{F}_r$. The set-systems $\mathcal{M}_{F'}$ ($F'\in\mathcal{F}_r$, $B_1\subset F'$) are isomorphic to $\mathcal{M}_{F_0}$. Hence each of them contains a set $B_1'(F')\subset F'$ such that $B_1\cap B_1'(F')=A_1$, $B_1'(F')\in\mathcal{B}$. So we have $A_1\in\mathcal{A}_0$. Similarly, $A_2\in\mathcal{A}_0$ holds. The union of A_1 and A_2 is contained only in F_0 from the members of $\mathcal{F}_r$ (if $F_1\in F_r$, $(A_1\cup A_0)\subset F_1\cap F_0$ then $F_1\cap F_0\in\mathcal{B}$, but $A_1\cup A_2$ is not contained in any $B\in\mathcal{B}$). So the pair $\{A_1, A_2\}$ uniquely determines F_0. Hence

$$|\mathcal{F}_r| \leq \binom{|\mathcal{A}_0|}{2} < b^2n^{4/3},$$

by (10). This and (9) yield

$$|\mathcal{F}| \leq (1/c(k, k+1))|\mathcal{F}^*| \leq (1/c(k, k+1))(2+b^2)n^{4/3}. \quad \blacksquare$$

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