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A Ramsey-Sperner Theorem

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Abstract. Let $n \geq k \geq 1$ be integers and let $f(n, k)$ be the smallest integer for which the following holds: If $\mathscr{F}$ is a family of subsets of an n-set X with $|\mathscr{F}| > f(n, k)$ then for every k-coloring of X there exist $A, B \in \mathscr{F}, A \neq B, A \subset B$ such that $B - A$ is monochromatic. Here it is proven that for a fixed k there exist constants c_k and d_k such that $c_k(1 + o(1)) < f(n, k)\sqrt{n}/2^n < d_k(1 + o(1))$ and $c_k = \sqrt{k/2 \log k}(1 + o(1)) = d_k$ as $k \to \infty$. The proofs of both the lower and the upper bounds use probabilistic methods.

1. Introduction, Preliminaries

Let $n \geq k$ be positive integers and X be an n-element set. 2^X denotes the power set of X and $\binom{X}{t}$ is the family of all t-subsets of X. A k-coloring of X is a partition of $X = X_1 \cup \cdots \cup X_k$ into at most k parts. A family $\mathscr{F} \subset 2^X$ has *k-color Sperner* property for the coloring $X_1, \ldots, X_k$ if for all $A, B \in \mathscr{F}$ $A \neq B$, $A \subset B$, $B - A$ is not monochromatic. (I.e., there exist $i \neq j$, $X_i \cap (B - A) \neq \varnothing$, $X_j \cap (B - A) \neq \varnothing$.) Define $f(n, k) =: \max\{|\mathscr{F}|: \mathscr{F} \subset 2^X, \mathscr{F}$ has k-color Sperner property$\}$.

The 1-color Sperner families are just the usual Sperner families, in other words *antichains*, i.e., $\mathscr{F} \subset 2^X, \forall F, F' \in \mathscr{F}$ one has $F \not\subset F'$. An antichain has k-color Sperner property for every k and every k-coloring, whence $f(n, k) \geq \left|\binom{|X|}{\lfloor n/2 \rfloor}\right| = \binom{n}{\lfloor n/2 \rfloor}$. By Sperner's theorem [16] no antichain of subsets of X contains more than $\binom{n}{\lfloor n/2 \rfloor}$ members, whence

$$f(n, 1) = \binom{n}{\lfloor n/2 \rfloor}. \tag{1}$$

For $k = 2$ Katona [8] and Kleitman [10] proved independently that $f(n, 2) = \binom{n}{\lfloor n/2 \rfloor}$ although there are several 2-color Sperner families which are not antichains. Very recently P.L. Erdös and Katona [3] described all the extremal families (i.e., having $f(n, 2)$ members).

For $k = 3$ Katona [9] gave a simple example $\mathscr{F}$ having more than $\binom{n}{\lfloor n/2 \rfloor}$ members.

Example 1.1. Let $X = X_1 \cup X_2 \cup X_3$, $X_i \approx n/3$ (i.e., $|X_i| = \lfloor n/3 \rfloor$ or $\lceil n/3 \rceil$) and $\mathscr{F} = \binom{X}{\lfloor n/2 \rfloor} \cup \binom{X}{\lfloor n/6 \rfloor - 1} \cup \binom{X}{\lceil 5n/6 \rceil + 1}$. Then $\mathscr{F}$ is a 3-color Sperner family.

Griggs and Kleitman [4], Griggs [5] and Katona [9] have found additional conditions which imply that a 3-color Sperner family $\mathscr{F}$ fulfils $|\mathscr{F}| \le \binom{n}{\lfloor n/2 \rfloor}$.

For fixed k it is not immediately clear that for all n $f(n,k)$ is at most a constant (depending only on k) times $\binom{n}{\lfloor n/2 \rfloor}$. This was proved by Griggs [6] showing that $f(n,k)/\binom{n}{\lfloor n/2 \rfloor} \le 2^{k-2}$ for $k \ge 2$. Here the right hand side was decreased by Sali [3] to k and later to $3\sqrt{k}$ [14]. Here we present a short proof due to Graham and Fan Chung [2] which gives

$$f(n,k) < (2^n/\sqrt{n})\sqrt{2k/\pi} = (1 + o(1))\binom{n}{\lfloor n/2 \rfloor}\sqrt{k}. \tag{2}$$

Proposition 1.2. *Let $\mathscr{F}$ be a k-color Sperner family with parts $X_1, X_2, \dots, X_k$, $k \ge 2$ and let $|X_i| = n_i$. Then*

$$|\mathscr{F}| < (2^n/\sqrt{n_k})\sqrt{2/\pi}. \tag{3}$$

Proof. For all $T \subset X_1 \cup \cdots \cup X_{k-1}$ denote by $\mathscr{F}(T) =: \{F \in \mathscr{F}: F \cap (X_1 \cup \cdots \cup X_{k-1}) = T\}$. Then $\mathscr{F}(T)$ is an antichain hence by (1) its cardinality is at most $\binom{n_k}{\lfloor n_k/2 \rfloor}$. Using $(1 + o(1))(2^a/\sqrt{a})\sqrt{2/\pi} = \binom{a}{\lfloor a/2 \rfloor} < (2^a/\sqrt{a})\sqrt{2/\pi}$ we have $|\mathscr{F}| \le 2^{|X_1 \cup \cdots \cup X_{k-1}|}\binom{n_k}{\lfloor n_k/2 \rfloor} < 2^{n-n_k}(2^{n_k}/\sqrt{n_k})\sqrt{2/\pi} = (2^n/\sqrt{n})\sqrt{2/\pi}\sqrt{n/n_k} = (1 + o(1))\binom{n}{\lfloor n/2 \rfloor}\sqrt{n/n_k}$. We can choose $n_k \ge n/k$, hence (3) implies (2). □

The constant $\sqrt{k}$ cannot be replaced by e.g. $0.6\sqrt{k}$ without any further restriction because $f(k,k) = 2^{k-1}$ (as it can be seen considering $\mathscr{F}_{\text{even}} =: \{F \subset X: |F| = \text{even}\}$ or $\mathscr{F}_{\text{odd}} =: \{F \subset X: |F| = \text{odd}\}$), hence $f(k,k) = (1 + o(1))\binom{k}{\lfloor k/2 \rfloor}\sqrt{k}\sqrt{\pi/8}$. But it can be improved whenever n is large compared to k. This paper is devoted to obtain estimations for $f(n,k)$ when $k \ge 3$ and n is large enough.

2. Results

Theorem 2.1. *For every positive integer k there exist constants c_k and d_k depending only on k such that $c_k 2^n/\sqrt{n}(1 + o(1)) \le f(n,k) \le d_k(2^n/\sqrt{n})(1 + o(1))$ if n tends to infinity. Moreover*

$$c_k > \sqrt{k/2\log k}(1 - o(\log\log k/\log k)), \tag{4}$$

$$d_k < \sqrt{k/2\log k}(1 + o(\log\log\log k/\log k)). \tag{5}$$

Obviously, $c_{k+1} \geq c_k$ and the exact form of (4) (cf. (7)) implies $f(n,4) > 1.001\binom{n}{\lfloor n/2 \rfloor}(1 + o(1))$. The following result disproves a conjecture about $f(n,3)$.

Theorem 2.2. *If n is large enough then* $f(n,3) > 1.018\binom{n}{\lfloor n/2 \rfloor}$.

We have to remark that similar results were obtained independently by Griggs, Odlyzko and Shearer [7]. Their proofs are different. The upper bound in (5) is slightly better. But they can prove that for fixed k $\lim f(n,k)/(2^n/\sqrt{n})$ exists and $c_3 > 1.036$.

3. Constructions

Now k is fixed and n tends to infinity. We are going to apply some wellknown properties of binomial coefficients (see e.g. Spencer [15]). For every integer m and real $x \geq 0$ we have

$$\sum_{j \geq m/2+x} \binom{m}{j} < 2^m \exp(-2x^2/m). \tag{6}$$

Proof of (4). We improve the idea of Example 1.1. Let $|X| = n$, $X = X_1 \cup \cdots \cup X_k$ a partition into almost equal parts ($|X_i| = \lfloor n/k \rfloor$ or $\lceil n/k \rceil$) and t a positive integer. Set $\mathscr{F}^t =: \{F \subset X: ||F \cap X_i| - |X_i|/2| < t/2 \text{ for all } 1 \leq i \leq k\}$, and $\mathscr{F}_r^t =: \{F \in \mathscr{F}^t: |F| \equiv r (\bmod t)\}$ where $0 \leq r < t$. Obviously, $\mathscr{F}_r^t$ is a k-color Sperner family.

Proposition 3.1. *For* $t \approx \sqrt{n}\sqrt{2(1+\varepsilon)\log k/k}$ ($\varepsilon > 0$ is an arbitrary constant) *we have* $|\mathscr{F}^t| > 2^n(1 - 2k^{-\varepsilon})(1 + o(1))$.

Proof. Choose $F \subset 2^X$ randomly. Then by (6) we have $\mathrm{Prob}(|F \cap X_i| \geq |X_i|/2 + t/2) = 2^{-|X_i|}\sum_{j \geq |X_i|/2+t/2}\binom{|X_i|}{j} < \exp(-t^2/2|X_i|) = (1 + o(1))k^{-1-\varepsilon}$. Similar inequality holds for $\mathrm{Prob}(|F \cap X_i| \leq |X_i|/2 - t/2)$. Because of the independence of the events we have $|\mathscr{F}^t|2^{-n} = \prod_{1 \leq i \leq k} \mathrm{Prob}(||F \cap X_i| - |X_i|/2| < t/2) \geq (1 + o(1))(1 - 2k^{-1-\varepsilon})^k \geq (1 + o(1))(1 - 2k^{-\varepsilon})$. □

Proposition 3.2. *We can choose t and r such that for the family* $\mathscr{F}_r^t$ *the inequality* (4) *holds.*

Proof. Let $0 \leq r < t$ such that $|\mathscr{F}_r^t|$ is maximal. Clearly $|\mathscr{F}_r^t| \geq |\mathscr{F}^t|/t$. Choose t as in Proposition 3.1 then we have $|\mathscr{F}_r^t| \geq (1 + o(1))(2^n/\sqrt{n})\sqrt{k/2\log k}(1 - 2k^{-\varepsilon})/\sqrt{1+\varepsilon}$. For $k \geq 20$ one can choose $0 < \varepsilon \leq 1$ such that $(1 - 2k^{-\varepsilon})/\sqrt{1+\varepsilon} > 1 - \log\log k/\log k$. ($\varepsilon \sim (\log\log k + o(1))/\log k$.) □

Proof of Theorem 2.2. Instead of (6) we can use the Moivre-Laplace formula (see [11]) to improve Proposition 3.1. This yields that for $t = \sqrt{n}\sqrt{2(1+\varepsilon)\log k/k}$ we have

$$\frac{1}{t}\sum_r |\mathscr{F}_r^t| \geq (1+o(1))(2^n/\sqrt{n})\sqrt{k/2\log k} \times (-1+2\varphi(2(1+\varepsilon)\log k))^k/\sqrt{1+\varepsilon}. \tag{7}$$

We remark that for large k (7) does not give an essentially better lower bound for c_k than (4). Optimizing (7) for $k = 3$ we obtain $\frac{1}{t}\sum|\mathscr{F}_r^t| > 0.97\binom{n}{\lfloor n/2\rfloor}$ for n sufficiently large. If the average of $|\mathscr{F}_r^t|$ is so large we can hope that $\max_r|\mathscr{F}_r^t| > 1.01\binom{n}{\lfloor n/2\rfloor}$. This is true. To prove this let $t \approx 1.2\sqrt{n}$ and $r \equiv \lfloor n/2\rfloor \pmod t$. Then $\mathscr{F}_r^t$ consists of 3 levels of 2^X, more precisely if $F \in \mathscr{F}_r^t$ then $|F| = \lfloor n/2\rfloor + t$, $\lfloor n/2\rfloor$ or $\lfloor n/2\rfloor - t$. More exactly

$$|\mathscr{F}_r^t| = \sum_{\substack{x_1+x_2+x_3=0\\ |x_i|<t/2,\text{ integers}}} \binom{n_1}{\lfloor n_1/2\rfloor + x_1}\binom{n_2}{\lfloor n_2/2\rfloor + x_2}\binom{n_3}{\lfloor n_3/2\rfloor + x_3}$$
$$+ 2\sum_{\substack{x_1+x_2+x_3=t\\ |x_i|<t/2,\text{ integers}}} \binom{n_1}{\lfloor n_1/2\rfloor + x_1}\binom{n_2}{\lfloor n_2/2\rfloor + x_2}\binom{n_3}{\lfloor n_3/2\rfloor + x_3}.$$

Use the following equality which holds for $|x| < C\sqrt{m}$ (see [11]).

$$\binom{m}{m/2+x} = (2^m/\sqrt{\pi m/2})\exp(-2x^2/m)(1+O(1/m)).$$

We obtain

$$|\mathscr{F}_r^t|\Big/\binom{n}{\lfloor n/2\rfloor}(1+o(1)) = \frac{6\sqrt{3}}{\pi}\iint\limits_{\substack{|x|\leq a\\ |y|\leq a\\ |x+y|\leq a}} \exp(-12(x^2+xy+y^2))dx\,dy$$
$$+ 2\frac{6\sqrt{3}}{\pi}\iint\limits_{\substack{|x|\leq a\\ |y|\leq a\\ |2a-x-y|\leq a}} \exp(-6(x^2+y^2+(2a-x-y)^2))dx\,dy.$$

Here $a = 0.6\ (\sim t/2\sqrt{n})$. Using a computer one can show that for this value of a the right hand side equals to $1.0189\cdots$. □

4. The Proof of the Upper Bound

We begin with a technical lemma.

Lemma 4.1. $(1-\exp(-1/x))(1-\exp(-1/y)) \leq (1-\exp(-2/(x+y)))^2$ *holds in the following cases*

(a) $0 < x, y \leq c$, *where* $0 < c < 1$ *such that* $e^{-1/c} = 1 - 1/2c$. ($c = 0.627\cdots$)

(b) $0 < y < 0.251$, $c \leq x \leq 4/\pi$.

Proof. (a) The function $f(x) = \log(1 - \exp(-1/x))$ is concave (convex) if $0 < x \le c$ ($x > c$, resp.) as it can be shown by derivations. The case (b) follows from (a) and the fact that $(f(y_0) + f(4/\pi))/2 = f((y_0 + 4/\pi)/2)$ for $y_0 = 0.2513\cdots$. □

Moreover we will use the following estimation which holds for *every m* and *t*. (See [11] pp. 151–152.)

$$\left(\binom{m}{\lfloor m/2\rfloor} - \binom{m}{\lfloor m/2\rfloor - t/2}\right)\binom{m}{\lfloor m/2\rfloor}^{-1} = (1 + o(1/m))(1 - \exp(-t^2/2m)). \quad (8)$$

Now let $\mathscr{F} \subset 2^X$ be a k-color Sperner family with respect to the coloring $X_1, \ldots, X_k$, $|X_i| = n_i$. Let $t \approx \sqrt{n2(1-\varepsilon)\log k/k}$ where $0 \le \varepsilon \le 1$ is a fixed small real ($\varepsilon = 0(\log\log\log k/\log k)$)

Lemma 4.2. *If some $n_i > (4(1-\varepsilon)\log k/\pi k)n$ then $|\mathscr{F}| \le d_k\, 2^n/\sqrt{n}$, where d_k is given by* (5).

Proof. It is a trivial consequence of (3). □

From now on we can suppose that for each i $(2n_i/t^2) \le 4/\pi$.

A family of sets $\mathscr{C} = \{C_i, C_{i+1}, \ldots, C_m\} \subset 2^Y$ is called a *symmetric chain* if it is linearly ordered by inclusion and $|C_j| = j$, $m = |Y| - i$, $|\mathscr{C}| = m - i + 1$. deBruijn, Kruijswijk and Tengbergen [1] proved that there exists a chain decomposition of $2^Y = \bigcup \mathscr{C}_i$ into $\binom{y}{\lfloor y/2\rfloor}$ pairwise disjoint symmetric chains ($|Y| = y$). The number of chains of length t in this decomposition equals to $\binom{y}{\lfloor (y-t)/2\rfloor} - \binom{y}{\lfloor (y-t)/2\rfloor}$. Fix this decompositon and permute the elements of Y. Then every $F \in 2^Y$ belongs to a chain of length at least t at least $y!\binom{y}{\lfloor (y-t)/2\rfloor}\Big/\binom{y}{\lfloor y/2\rfloor}$ times. In other words, using (8) we get

$$\begin{aligned}\operatorname{Prob}(F \in \mathscr{C}, |\mathscr{C}| < t) &\le 1 - \binom{y}{\lfloor (y-t)/2\rfloor}\binom{y}{\lfloor y/2\rfloor}^{-1} \\ &= (1 + o(1/y))(1 - \exp(-t^2/2y)).\end{aligned} \quad (9)$$

Now fix a chain decomposition of 2^{X_i} for all $1 \le i \le k$. Choose a chain $\mathscr{C}_i \subset 2^{X_i}$ for all $1 \le i \le k$. The family $\mathscr{B} \subset 2^X$, $\mathscr{B} = \mathscr{C}_1 \times \mathscr{C}_2 \times \cdots \times \mathscr{C}_k =: \{C_1 \cup \cdots \cup C_k: C_i \in \mathscr{C}_i\}$ is called a *block*.

The chain decompositions of 2^{X_i} define a block decomposition of 2^X.

Lemma 4.3. $|\mathscr{F} \cap \mathscr{B}| \le |\mathscr{B}|/\max_i |\mathscr{C}_i|$.

Proof. Suppose $|\mathscr{C}_1| = \max_i |\mathscr{C}_i|$. Split $\mathscr{B}$ according to the traces on $X_2 \cup \cdots \cup X_k$, i.e., $\mathscr{B} = \bigcup \{\mathscr{B}(T): T \in \mathscr{C}_2 \times \cdots \times \mathscr{C}_k\}$ where $\mathscr{B}(T) =: \{B \in \mathscr{B}: B - X_1 = T\}$. Then each $\mathscr{B}(T)$ is a chain in 2^X of length $|\mathscr{C}_1|$ and $|\mathscr{F} \cap \mathscr{B}(T)| \le 1$ because $\mathscr{F}$ is a k-color Sperner family. □

A block $\mathscr{B} = \mathscr{C}_1 \times \cdots \times \mathscr{C}_k$ is *small* if $\max_i |\mathscr{C}_i| < t$. Fix a member $F \in 2^X$ and consider all the permutations of X which are the products of the permutations of the X_i's. The inequality (9) implies

$$\begin{aligned}\text{Prob}(F \in \text{small block}) &= \prod_{1 \le i \le k} \text{Prob}(F \cap X_i \text{ belongs to a small chain in } 2^{X_i}) \\ &\le (1 + o(1/\sqrt{n})) \prod_{1 \le i \le k} (1 - \exp(-t^2/2n_i)).\end{aligned} \tag{10}$$

Lemma 4.4. If $k \ge 50$ then $\prod_{1 \le i \le k}(1 - \exp(-t^2/2n_i)) \le (1 - \exp(-t^2k/2n))^k$.

Proof. If each $(2n_i/t^2) \le c$ $(\sim 0.6275 \cdots)$ then we can apply Lemma 4.1 (a) and the Jensen's inequality. If for some i $c < (2n_i/t^2)$ $(\le 4/\pi)$ then there exists a j such that $2n_j/t^2$ is at most 0.251 and first we can apply (possibly repeatedly) Lemma 4.1(b). □

Hence the right hand side of (10) is at most $(1 - k^{-1+\varepsilon})^k < \exp(-k^\varepsilon)$. Thus we have obtained an upper bound for the mean value

$$E\,(\# F \in \mathscr{F}\colon F \text{ belongs to a small block}) \le |\mathscr{F}|/\exp(k^\varepsilon).$$

This implies that there exists a block decomposition in which $\ge(1 - \exp(-k^\varepsilon))$ proportion of $\mathscr{F}$ belongs to a large (that is not a small) block.

Apply Lemma 4.3

$$\begin{aligned}|\mathscr{F}|(1 - \exp(-k^\varepsilon)) &\le \sum_{\mathscr{B} \text{ large block}} |\mathscr{F} \cap \mathscr{B}| \le \sum_{\mathscr{B} \text{ large block}} |\mathscr{B}|/\max|\mathscr{C}_{\mathscr{B}}| \\ &\le \sum |\mathscr{B}|/t \le 2^n/t.\end{aligned} \tag{11}$$

Rearranging (11) we get (5). □

Finally we remark that an argument similar to the one giving (5) can be found in Rödl [12].

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