Traces of Finite Sets

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ABSTRACT

Let X be an n-element set, $n \geq t \geq l$ non-negative integers. Sauer proved that if $\mathbf{F} \subset 2^X$ is a set-system with cardinality bigger than $\sum_{i \leq t-1}\binom{n}{i}$ then there exists a subset $T \subset X$, $|T| = t$ such that $\mathbf{F}|T = \{F \cap T: F \in \mathbf{F}\}$ contains all subsets of T, in particular all l-element subsets of T.

In this paper it is proved that this value is best in a stronger sense. That is, there exists a set-system $\mathbf{G} \subset 2^X$, $|\mathbf{G}| = \sum_{i \leq t-1}\binom{n}{i}$ such that for any $T \subset X$, $|T| = t$ the set-system $\mathbf{G}|T$ does not contain all l-subsets of T.

1. Introduction, Results.

Let $n \geq t \geq l$ be non-negative integers, X an n-element set. 2^X denotes the set-system of all subsets of X, $\binom{X}{l}$ denotes the set-system of all l-element subsets of X. The complete hypergraphs 2^X and $\binom{X}{l}$ are denoted by $\mathbf{P}_n$, $\mathbf{K}_l^n$ respectively.

Let $\mathbf{H}$ be a hypergraph with edge-set $E(\mathbf{H})$, vertex-set $V(\mathbf{H})$ and let $\mathbf{F}$ be a set-system on X. We say that $\mathbf{F} \to \mathbf{H}$ (i.e. $\mathbf{F}$ *induces* $\mathbf{H}$) if $\mathbf{H}$ can be obtained as traces of members of $\mathbf{F}$. That is, there exist $F_1,\dots,F_m \in \mathbf{F}$ $(m = |E(\mathbf{H})|)$ and $T \subset X$, $|T| = |V(\mathbf{H})|$ such that $F_1 \cap T,\dots,F_m \cap T$ form a hypergraph on T which is isomorphic to $\mathbf{H}$. Otherwise, $\mathbf{F} \not\to \mathbf{H}$. Sauer [13] proved the following conjecture of Erdös: If $\mathbf{F} \subset 2^X$, $|\mathbf{F}| > \sum_{i \leq t-1}\binom{n}{i}$ then there exists a $T \subset X$, $|T| = t$ such that $\mathbf{F} \to 2^T$. The set-system $\mathbf{F} = \bigcup_{i \leq t-1}\binom{X}{i}$ shows that this theorem cannot be improved without any further restriction.

Theorem 1. *For all $n \geq t \geq l \geq 0$ there exists a set-system $\mathbf{F} = \mathbf{F}(n,t,l)$ on an n-element set such that $|\mathbf{F}| = \sum_{i \leq t-1}\binom{n}{i}$ but $\mathbf{F} \not\to \mathbf{K}_l^t$.*

This statement is trivial for $l = 0$ or $l = t$ (see above). The case $t = 2$, $l = 2$ was proved by Anstee [1,2] and by Frankl (unpublished).

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Proof. We have to give constructions for $n \geq t > l > 0$. Order the elements of X, e.g. $X = \{1,2,...,n\}$. For $x_1,x_2,...,x_i \in X$, $x_1 < x_2 < \cdots < x_i$, let $E(x_1,...,x_i) = \{x \in X: x = x_j$ for $j \leq l\} \cup \{x \in X: x > x_l$ but $x \neq x_j$ for any $j > l\}$ (see Fig. 1), $E(\emptyset) = \emptyset$. Let $\mathbf{F}(n,t,l)$ consist of all $E(x_1,...,x_i)$ where $i \leq t-1$. It is easy to check that if $T \subset X$, $T = \{y_1,...,y_t\}$, $y_1 < y_2 < \cdots < y_t$ then the subset $\{y_1, \ldots, y_l\}$ cannot be obtained as trace of a member of $\mathbf{F}$ on T. Thus, $\mathbf{F}(n,t,l) \not\to \mathbf{K}_l^t$. Q.E.D.

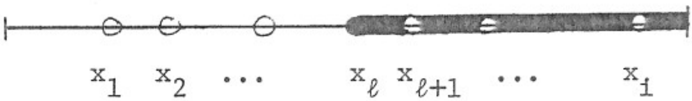

Figure 1

2. Problems, Remarks.

Let $F(n,\mathbf{H}) = \max\{|\mathbf{F}| : \mathbf{F} \subset 2^X, |X| = n, \mathbf{F} \not\to \mathbf{H}\}$ and call a set-system $\mathbf{F}$ $\mathbf{H}$-*extremal* if $|\mathbf{F}| = F(n,\mathbf{H})$ and $\mathbf{F} \not\to \mathbf{H}$. E.g. $F(n,\mathbf{K}_l^t) = \sum_{i \leq t-1}\binom{n}{i}$ and the set-system $\mathbf{F}(n,t,l)$ is $\mathbf{K}_l^t$-extremal by the Theorem 1 and the theorem of Sauer. It is natural to ask how one can characterize all $\mathbf{H}$-extremal set-systems. We deal with the case $\mathbf{H} = \mathbf{K}_l^t$ only. The cases $l = 0$ or t are trivial. If $0 < l < t \geq 3$ one can construct $\mathbf{K}_l^t$-extremal families different from the example of §1. However, the following conjecture seems to be true:

Conjecture 1. *If $\mathbf{F}$ is a $\mathbf{K}_l^t$-extremal family and $\mathbf{F}_j$ denotes the set of its j-element members, then $|\mathbf{F}_j| = |\mathbf{F}(n,t,l)_j|$.*

That is $|\mathbf{F}_j| = \binom{n}{j}$ for $j < l$ or $j > n-t+l$, otherwise $|\mathbf{F}_j| = \sum_a \binom{j+t-2l}{j-l+1+2a}\binom{n-j-t+2l}{l-1-a}$. We can prove Conjecture 1 for $l = 1$ and (taking the complements) for $l = t-1$.

Theorem 2. *If* $\mathbf{F} \subset 2^X$, $|X| = n$, $|\mathbf{F}| = \sum_{i \le t-1}\binom{n}{i}$ *and* $\mathbf{F} \not\to \mathbf{K}_l^t$ *then* $|\mathbf{F}_j| = \binom{j+t-2}{t-2}$ *for* $j \le n-t+1$ *and* $\binom{n}{j}$ *otherwise.*

(The proof of this statement is given in §3.) Anstee [1,3] proved this statement for $t = 3$, $l = 2$ together with the following structure theorem [4]: If $\mathbf{F}$ is $\mathbf{K}_2^3$-extremal then $\mathbf{F} \not\to \mathbf{C}_k$ for any k ($\mathbf{C}_k$ denotes the circle of length k, that is a hypergraph on k points with k 2-element edges). Further, if $\mathbf{F}_0 \subset 2^X$ and $\mathbf{F}_0 \not\to \mathbf{C}_k$ for every k, then $\mathbf{F}_0$ can be extended to be $\mathbf{K}_2^3$-extremal. (That is, there exists a set-system $\mathbf{G} \subset 2^X$, $\mathbf{F}_0 \subset \mathbf{G}$, $|\mathbf{G}| = 1+n+\binom{n}{2}$, $\mathbf{G} \not\to \mathbf{K}_2^3$). Hence the number of the $\mathbf{K}_2^3$ (and $\mathbf{K}_1^3$) -extremal families is greater than the number of trees, i.e. its order of magnitude is exponential.

Remark 1. Following Ryser [12], Anstee used linear algebraic tools. However, it seems to be likely that linear spaces play an important role in this topic, e.g. a theorem due to Frankl and Pach [10] says: If $\mathbf{F} \subset 2^X$, $|X| = n$ and the rows of the matrix $M(\mathbf{F}, \le t-1)$ are linearly dependent then $\mathbf{F} \to \mathbf{P}_t$. Here the definition of $M(\mathbf{F}, \le t-1)$ is the following. Let $\mathbf{F} = \{F_1,...,F_m\}$ and let $A_1,...,A_r$ denote all at most $(t-1)$-element subsets of X. Then $M = (M_{ij})$ $1 \le i \le m$, $1 \le j \le r$ an m by r $0-1$ matrix where $M_{ij} = 1$ if $A_j \subset F_i$ and 0 otherwise.

Problem 1. If $\mathbf{F}$ is a $\mathbf{K}_1^t$-extremal set-system then $|\mathbf{F}_j|$ is independent from n by Theorem 2. Let $f_k(n)$ be the greatest integer m such that there exists a k-uniform set-system $\mathbf{E} = \{E_1,...,E_m\}$ which $\mathbf{E} \not\to \mathbf{K}_1^t$. Clearly, $\binom{k+t-2}{t-2} \le f_k(t)$. This problem was posed by Frankl and Pach, too. They pointed out that $(k/t)^{t-1} \le T(k+t-1,t,t-1) \le f_k(t) \le \binom{k(t-2)}{t-1}$, where $T(n,t,l)$ is the Turán-number, i.e. $T(n,t,l) =: \max\{|\mathbf{H}| : \mathbf{H} \subset \binom{X}{l}, \quad |X| = n, \mathbf{H} \not\supset \mathbf{K}_l^t\}$. Moreover, $f_2(t) = [(t^2-1)/2] = T(t+1,t,t-1)$ holds.

Remark 2. Let n,k,m,s denote integers. The symbol $(n,k) \to (m,s)$ means that whenever $\mathbf{F} \subset 2^X$, $|X| = n$, $|\mathbf{F}| \ge k$ then we can find an m-element subset Y of X such that among the intersections $Y \cap F$, $F \in \mathbf{F}$ there are at least s different sets. So Sauer's theorem says that $(n,1+\sum_{i \le t-1}\binom{n}{i}) \to (t,2^t)$. It is clear that $(n,\sum_{i \le t-2}\binom{n}{i}+T(n,t,t-1)) \not\to (t,2^t-1)$, and Frankl [9] proved that $(n,1+\sum_{i \le t-2}\binom{n}{i}+T(n,t,t-1)) \to (t,2^t-1)$. Here the left hand side is much smaller than $\binom{n}{t-1}$ for great values of n. However, this result suggested that Theorem 1 may not be true (which turned out to be wrong).

More results can be found in Bondy [6] ($(n,k) \to (n-1,k)$ for $k \le n$), Bollobás [5] ($(n,k) \to (n-1,k-1)$ for $k \le [3n/2]$) and Frankl [8,9].

Problem 2. The *degree* of a point x in a set-system $\mathbf{F}$ is denoted by $d_{\mathbf{F}}(x)$ or simply $d(x) =: |\{F: x \in F \in \mathbf{F}\}|$. Let a hypergraph $\mathbf{H}$ be given and let $d(n,\mathbf{H}) = \max\{\min_{x \in X} d_{\mathbf{F}}(x): \mathbf{F} \subset 2^X, |X| = n, \mathbf{F} \not\to \mathbf{H}\}$. This means (in human language) that if the degree of every point is greater than $d(n,\mathbf{H})$ then $\mathbf{F} \to \mathbf{H}$. This problem was posed (for special $\mathbf{H}$) by Cunningham [7]. The set-system $\bigcup_{i \ge n-t+1} \binom{X}{i}$ shows that $d(n,\mathbf{P}_t) = \sum_{i \le t-1} \binom{n-1}{i}$. So $d(n,\mathbf{P}_t) = F(n-1,\mathbf{P}_t)$, and in general $[F(n,\mathbf{H})/n] \le d(n,\mathbf{H}) \le F(n-1,\mathbf{H})$. Here equality can hold in left hand side, too, e.g. $d(n,\mathbf{K}_1^2) = 1 = [(n+1)/n]$ $(n \ge 2)$.

The hypergraph $\mathbf{H} = (V(\mathbf{H}),E(\mathbf{H}))$ is called *r-partite* if there exist $Y_1,Y_2,\ldots,Y_r \subset V(\mathbf{H})$, $\cup Y_j = V(\mathbf{H})$ such that $|E \cup Y_j| = 1$ for all $e \in E(\mathbf{H})$, $1 \le j \le r$.

Proposition 1. *Given a hypergraph* $\mathbf{H}$, *let* $\mathbf{H}' = \{\bar{E}: E \in E(\mathbf{H})\}$. *If* $\mathbf{H}'$ *is not* $(t-1)$*-partite, then* $d(n,\mathbf{H}) > n^{t-1}/(t-1)^{t-1} - 0(n^{t-2})$.

It is enough to give a construction. Let X be the disjoint union of the sets $X_1,\ldots,X_{t-1}$ where $|X_i| \sim n/(t-1)$ and $\mathbf{F} = \{F \subset X: F = \bigcup_{i \le t-1} (X_i - \{x_i\}),\ x_i \in X_i\}$. The condition of Proposition 1 is satisfied by almost all hypergraph, e.g. $d(n,\mathbf{K}_l^t) \sim n^{t-1}$ for all $l \le t-2$.

Proposition 2. *If* $l \le t-1$ *then* $d(n,\mathbf{K}_l^t) \sim n^{t-1}$ *except for* $t = 2$, $l = 1$. *Moreover* $d(n,\mathbf{K}_t^t) = \sum_{i \le t-2} \binom{n-1}{i}$.

The second part of the statement is obvious. The only missing case is $l = t-1 \ge 2$. Arrange the elements of X along a circle and let $G(x_1,\ldots,x_{t-1}) = \{x \in X: x = x_j\} \cup \{x \in X: x_{t-2} -\!< x -\!< x_{t-1}\}$. (The sign $-\!<$ stands for a direction, say, for the clockwise one.) Finally, let $\mathbf{G} = \{G(x_1,\ldots,x_{t-1}): x_1 -\!< x_2 -\!< \cdots x_{t-1}$ and $x_1,\ldots,x_{t-1}$ can be covered by an arc of length at most $n/3\}$. Then $\mathbf{G} \not\to \mathbf{K}_{t-1}^t$ and $d_{\mathbf{G}}(x) > n^{t-1}/3^{t-1}(t-1)^{t-1} + 0(n^{t-2})$ for all $x \in X$. Q.E.D.

This example, similarly to Theorem 1, can be extended to all l. Hence the coefficient in Proposition 1 can be improved a little. Remark, that the idea of arranging the points along a circle is due to Frankl.

3. Proof of Theorem 2.

We use induction on t and, for a fixed t, induction on n. In the case $t = 2$, if $F_1, F_2 \in \mathbf{F}$ and $\mathbf{F} \not\to \mathbf{K}_1^2$, then one of F_1 and F_2 contains the other. Thus, there exists only one $\mathbf{K}_1^2$-extremal family $\mathbf{G}$ and $|\mathbf{G}_i| = 1$. Hence, we may suppose $t \geq 3$. For any $\mathbf{K}_1^t$-extremal $\mathbf{F}$ $|\mathbf{F}_j| = \binom{n}{j}$ $(j \leq n-t+2)$ holds, obviously. So we get $|\mathbf{F}_{\leq n-t+1}| = \binom{n}{t-1}$. It is easy to check the case $n = t$. From now on we follow the idea of Sauer.

Let $x \in X$ and $\mathbf{F}(x) =: \{F \in \mathbf{F}: x \notin \mathbf{F}\}$. The traces of the sets belonging to $\mathbf{F} - \mathbf{F}(x)$ are pairwise different on $(X - \{x\})$. Since $(\mathbf{F} - \mathbf{F}(x))|(X - \{x\}) \not\to \mathbf{P}_t$ we have $|\mathbf{F} - \mathbf{F}(x)| \leq \sum_{i \leq t-1}\binom{n-1}{i}$ by the theorem of Sauer. Moreover $\mathbf{F}(x) \not\to \mathbf{P}_{t-1}$ ($\mathbf{F}(x) \to \mathbf{P}_{t-1}$ would imply $\mathbf{F} \to \mathbf{P}_t$) and we obtain $|\mathbf{F}(x)| \leq \sum_{i \leq t-2}\binom{n-1}{i}$. The sum of the right-hand sides of these two inequalities is just $\sum_{i \leq t-1}\binom{n}{i} = |\mathbf{F}|$. So we have $|\mathbf{F}(x)| = \sum_{i \leq t-2}\binom{n-1}{i}$. Since $\mathbf{F}$ contains all at most $(n-t+2)$-element subsets we get $|\mathbf{F}(x)_{\leq n-t+1}| = \binom{n-1}{t-2}$.

Next we show that $|\mathbf{F}_1| = t-1$. We clearly have $|\mathbf{F}_0| = 1$, because $\emptyset \in \mathbf{F}$. Count up the pairs (F,F') where $F \in \mathbf{F}_{\leq n-t+1}$, $F' \in \mathbf{F}$, $|F'| = |F| + 1$, $F \subset F'$. Since $\mathbf{F} \not\to K_1^t$, there are at most $(t-1)$ pairs for any fixed F. Hence

$$n\binom{n-1}{t-2} = \sum_{x \in X}|\mathbf{F}(x)_{\leq n-t+1}| = \#(F,F') \leq (t-1)|\mathbf{F}_{\leq n-t+1}|$$

$$= (t-1)\binom{n}{t-1}.$$

Since the left-hand side and right-hand side are equal, we have that each $F \in \mathbf{F}_{\leq n-t+1}$ belongs to exactly $(t-1)$ (F,F') pairs. In particular, for $F = \emptyset$, this yields $|\mathbf{F}_1| = t-1$.

Now, we can suppose that $\{y\} \in \mathbf{F}$. Then $\{F \in \mathbf{F}: y \notin F\} \not\to \mathbf{K}_1^{t-1}$. Using the fact that $|\mathbf{F}(y)| = F(n-1, \mathbf{K}_1^{t-1})$ and $\mathbf{F}(y) \subset \{F \in \mathbf{F}: y \notin F\}$, we get that $\mathbf{F}(y) = \{F \in \mathbf{F}: y \notin F\}$. In other words, this means that $y \in F$ for every $F \in \mathbf{F} - \mathbf{F}(y)$. So, applying the induction hypothesis for $\mathbf{F}(y)$ and $(\mathbf{F} - \mathbf{F}(y))|(X - \{y\})$ we have $|\mathbf{F}_j| = |\mathbf{F}(n-1,t-1,1)_j| + |\mathbf{F}(n-1,t,1)_{j-1}| = \binom{j+t-3}{j} + \binom{j-1+t-2}{j-1} = \binom{j+t-2}{j}$. Q.E.D.

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