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ARRANGEMENTS OF LINES WITH A LARGE NUMBER OF TRIANGLES

Z. FÜREDI AND I. PALÁSTI

ABSTRACT. An arrangement of lines is constructed by choosing n diagonals of the regular $2n$-gon. This arrangement is proved to form at least $n(n-3)/3$ triangular cells.

1. Introduction. We shall use the terminology of Grünbaum [5]. By an arrangement $\mathscr{A}$ of lines we mean a finite family of lines $L_1, \ldots, L_n$ in the *real projective* plane Π. The number of lines in $\mathscr{A}$ is denoted by $n(\mathscr{A})$. If no point belongs to more than two of these lines L_i, the arrangement is called *simple*. With an arrangement $\mathscr{A}$ there is associated a 2-dimensional *cell-complex* into which the lines of $\mathscr{A}$ decompose Π. It is well known that in a simple arrangement $\mathscr{A}$ the number of cells (or polygons) of that complex is $(n^2 - n + 2)/2$ $(n = n(\mathscr{A}))$. We shall denote by $p_j(\mathscr{A})$ the number of j-gons among the cells of (the complex associated with) $\mathscr{A}$.

2. Constructions. Let us denote by $P(O)$ a fixed point on the circle $\mathscr{C}$ with centre C. For any real α, let $P(\alpha)$ be the point obtained by rotating $P(O)$ around C, with angle α. Further denote by $L(\alpha)$ the straight line $P(\alpha)\,P(\pi - 2\alpha)$. In case $\alpha \equiv \pi - 2\alpha \pmod{2\pi}$, $L(\alpha)$ is the line tangent to $\mathscr{C}$ at $P(\alpha)$. (See Figure 1.)

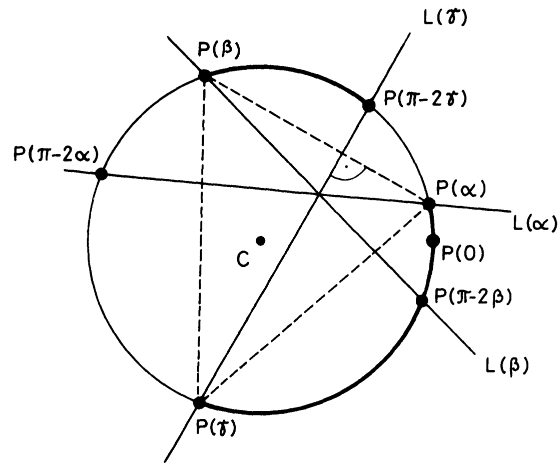

FIGURE 1

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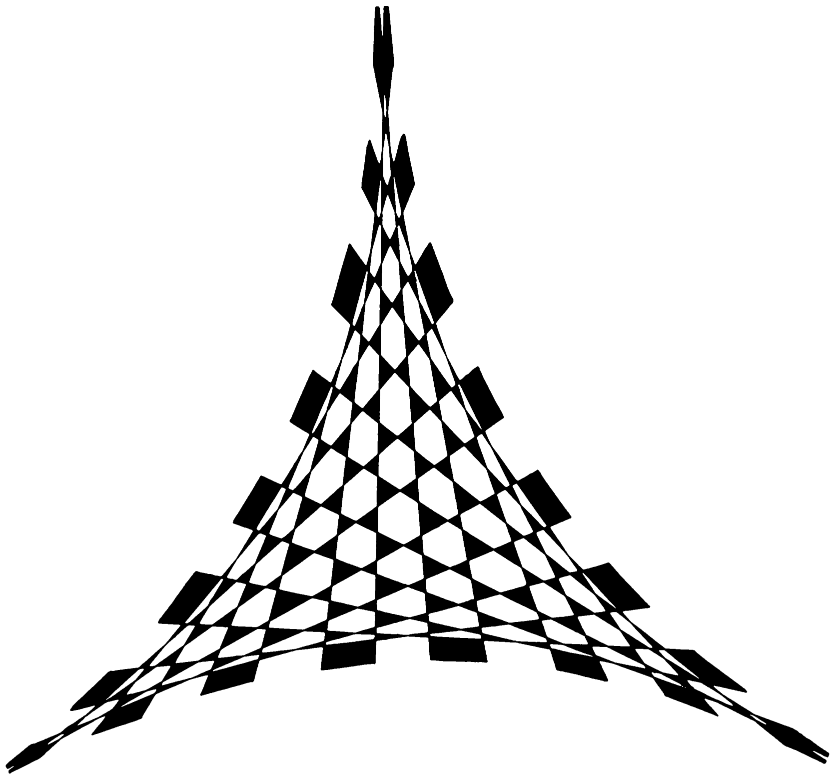

FIGURE 2

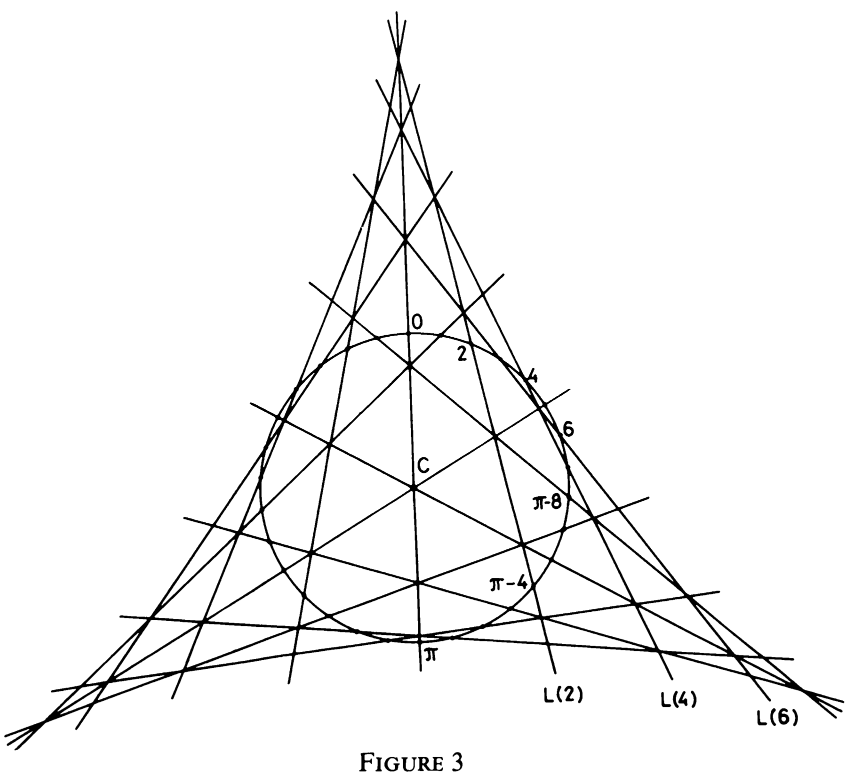

FIGURE 3

EXAMPLE 1. *Given any integer* $n \geqslant 3$, *put* $\mathscr{A}_n = \{L((2i+1)\pi/n)\colon i = 0, 1, \ldots, n-1\}$. (See Figure 2.)

EXAMPLE 2. *Given any integer* $n \geqslant 3$, *put* $\mathscr{B}_n = \{L(2i\pi/n)\colon i = 0, 1, \ldots, n-1\}$. (See Figure 3.)

Remark that our set of lines $\{L(\alpha)\colon 0 \leqslant \alpha < 2\pi\}$ may be regarded as a set of tangents to the arcs of a hypocycloid of third order, drawn in a circle of centre C and radius 3. The line $L(\alpha)$ is tangent to the arc of the cycloid at the αth point. (See Figure 4.) However, we shall not rely upon this fact in what follows.

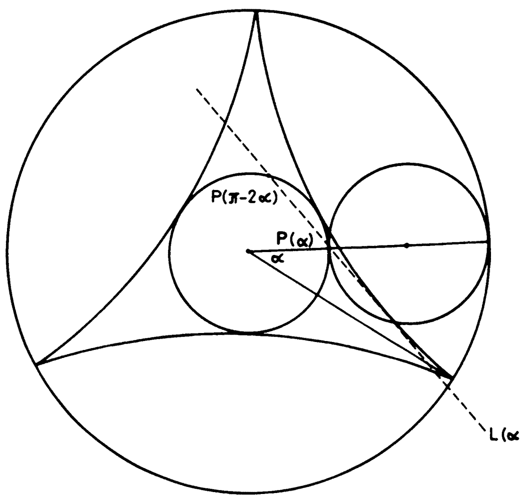

FIGURE 4

LEMMA. *The lines* $L(\alpha)$, $L(\beta)$ *and* $L(\gamma)$ *are concurrent if and only if* $\alpha + \beta + \gamma \equiv 0 \pmod{2\pi}$.

PROOF. If $\alpha + \beta + \gamma \equiv 0 \pmod{2\pi}$, then the sum of lengths of (directed) arcs $(P(\alpha), P(\gamma))$ and $(P(\beta), P(\pi - 2\gamma))$ is equal to π. This implies that $L(\gamma)$ is perpendicular to the line $P(\alpha)\,P(\beta)$. Hence, the lines $L(\alpha)$, $L(\beta)$ and $L(\gamma)$ are the altitudes of the triangle $P(\alpha)\,P(\beta)\,P(\gamma)$ (see Figure 1), consequently they meet at one point.

The reverse can be proved similarly.

3. Triangles in a simple arrangement. Grünbaum [**5**] (cf. Theorem 2.21) pointed out that the maximal number of triangles in a simple arrangement $p_3^s(n) = \max\{p_3(\mathscr{A})\colon n(\mathscr{A}) = n, \mathscr{A} \text{ is simple}\}$ can be estimated by $p_3^s(n) \leqslant \lfloor n(n-1)/3 \rfloor$ for even n, and $p_3^s(n) \leqslant \lfloor n(n-2)/3 \rfloor$ if n is odd. Moreover, he conjectured that this latter inequality holds for all n, $n \not\equiv 4 \pmod 6$. The exact value of $p_3^s(n)$ is known only for some small values of n. (Cf., e.g., Simmons [**15**, **16**] for the case $n = 15$, Grünbaum [**5**] for $n = 20$ and Harborth [**8**] for $n = 17$.)

As far as we know, the best lower bound by now, $p_3^s(n) > (5/16 + o(1))n^2$, was due to Füredi and J. Pach (unpublished). We are now in a position to establish a sharper lower bound.

PROPERTY 1. *$p_3(\mathscr{A}_n) \geq n(n-3)/3$; hence $p_3^s(n) = n^2/3 + \mathcal{O}(n)$.*

PROOF. Let (L_i, L_j) denote the intersection point of lines

$$L_i = L((2i+1)\pi/n) \quad \text{and} \quad L_j = L((2j+1)\pi/n).$$

Using the Lemma, we obtain that only $L((2n-2i-2j-2)\pi/n)$ may cross L_i and L_j at (L_i, L_j). But this line does not belong to $\mathscr{A}_n$, by definition. Therefore, the lines $L_i, L_j, L_{n-i-j-1}$ and $L_i, L_j, L_{n-i-j-2}$, resp., necessarily form triangular cells.

4. Quadrangles in a simple arrangement. Grünbaum conjectured that $p_4(\mathscr{A}) \geq 1$ for any simple arrangement with $n(\mathscr{A}) > 16$. (See 2.12 in **[5]**, cf. **[15, 16]**.) As far as we know one cannot find in the literature any example of a simple arrangement containing $\leq o(n^2)$ quadrangles. In view of this, Grünbaum's conjecture is surprisingly modest. It is easy to prove that

PROPERTY 2. *The simple arrangement $\mathscr{A}_n$ contains only 3 or 5 quadrangles for odd n, and $p_4(\mathscr{A}_n) = \mathcal{O}(n)$ is valid for all values of n.*

The results of §§3 and 4 are collected in Table 1. (We remark that $p_j(\mathscr{A}_n) = 0$ for $j \geq 7$.)

TABLE 1

$n \geq 5$	$p_3(\mathscr{A}_n)$	$p_4(\mathscr{A}_n)$	$p_5(\mathscr{A}_n)$	$p_6(\mathscr{A}_n)$
$n \equiv 0 \pmod 6$	$\frac{1}{3}(n^2-3n)$	$n/2+6$	$n-6$	$\frac{1}{6}(n^2-6n+6)$
$n \equiv \pm 1 \pmod 6$	$\frac{1}{3}(n^2-3n+5)$	5	$2n-9$	$\frac{1}{6}(n^2-9n+20)$
$n \equiv \pm 2 \pmod 6$	$\frac{1}{3}(n^2-3n+8)$	$n/2$	$n-2$	$\frac{1}{6}(n^2-6n+2)$
$n \equiv 3 \pmod 6$	$\frac{1}{3}(n^2-3n+9)$	3	$2n-9$	$\frac{1}{6}(n^2-9n+24)$

5. Triangles in arbitrary arrangements. Grünbaum conjectures that for any arrangement $\mathscr{A}$ of n lines, $p_3(\mathscr{A}) \leq n(n-1)/3$ holds. (See **[5]**.) Let $p_3(n) = \max\{p_3(\mathscr{A}): n(\mathscr{A}) = n\}$. The best upper bound was given by Purdy **[11, 12]**, who proved that

$$p_3(n) \leq \tfrac{7}{18}n(n-1) + \tfrac{1}{3} \quad \text{for } n \geq 6.$$

The best lower bound, $p_3(n) \geq 4 + n(n-3)/3$, is due to Strommer **[18]**. His result uses a construction of Burr, Grünbaum and Sloane **[1]**.

PROPERTY 3. *$p_3(\mathscr{B}_n) \geq 4 + n(n-3)/3$.*

More exactly $p_3(\mathscr{B}_n) = n(n-3)/3 + 6 - 2\varepsilon/3$, where $\varepsilon = 0, 2, 2$ according to whether $n \equiv 0, 1, 2 \pmod 3$. Further, $p_4(\mathscr{B}_n) = n - 6 + \varepsilon$, and $p_j(\mathscr{B}_n) = 0$ for $j \geq 5$. The proof is easy.

6. The orchard problem. Given a vertex V in an arrangement $\mathscr{A}$, denote by $t(V, \mathscr{A})$ the *multiplicity* of V, i.e. the number of lines of $\mathscr{A}$ incident to V. Further, let $t_j(\mathscr{A})$ denote the number of vertices of multiplicity j $(2 \leq j \leq n)$. We use the notation $t_j(n) = \max\{t_j(\mathscr{A})\colon n(\mathscr{A}) = n\}$. The "orchard problem" has been investigated for about 150 years. It can be formulated as follows: find the value of $t_3(n)$. Significant progress has been made by Burr, Grünbaum and Sloane [**1**]. They proved that $t_3(n) \geqslant 1 + \lfloor n(n-3)/6 \rfloor$ by construction using elliptic integrals. Moreover, they conjectured that this result is sharp if n is large enough. (Confer [**1**] for a complete (historical) bibliography of the subject.) Our construction (see Example 2) is much simpler, but this is only a special case of their idea. (The equation of the poles of $L(\alpha)$'s is $(x^2 + y^2)(3x - 1) = 4x^3$. This can be transformed into the form $y^2 = 4x^3 - (1/12)x - (1/216)$. Cf. [**1**].)

PROPERTY 4. *$t_2(\mathscr{B}_n) = n - 3 + \varepsilon$, $t_3(\mathscr{B}_n) = 1 + \lfloor n(n-3)/6 \rfloor$ and $t_j(\mathscr{B}_n) = 0$ for $j \geqslant 4$.*

It should be noted that recently Szemerédi and Trotter [**19**] proved that there exist c and c' positive real numbers such that $cn^2/k^3 < \max_{\mathscr{A}} \sum_{i \geqslant k} t_i(\mathscr{A}) < c'n^2/k^3$ for all $n > k^2$.

7. Two-coloring of arrangements. It is easy to prove by induction that the cells of a (not necessarily simple) arrangement $\mathscr{A}$ in the *Euclidean* plane can be colored by two colors (e.g., black and white) so that any two regions with a common side get different colors. Let $b = b(\mathscr{A})$ and $w = w(\mathscr{A})$ denote the numbers of black and white polygons. Without loss of generality we can assume that $b \geqslant w$. L. Fejes Tóth [**3**] raised the following question: What is the maximum of the ratio b/w? Palásti [**10**] proved that $b/w < 2$ for $n(\mathscr{A}) \leqslant 9$. Upper bounds were given by Grünbaum [**6**], Simmons and Wetzel [**17**] and for higher dimensions by Purdy and Wetzel [**13**]. However, the exact value of max b/w is known only for some small values of n with $n \leqslant 16$.

Grünbaum proved that $b \leqslant 2w - 2$ for all arrangements $\mathscr{A}$ with $n(\mathscr{A}) \geqslant 3$. For $n = 3, 5, 9$ and 15 equality holds. Our Example 1 shows

PROPERTY 5. *The number of black regions $b(\mathscr{A}_n) = (n^2 + \varepsilon)/3$, where $\varepsilon = 0, 2, 2$ if $n \equiv 0, 1, 2 \pmod 3$. So we get $b(\mathscr{A}_n) = 2w(\mathscr{A}_n) - (n + 2 - \varepsilon)$.*

8. Gallai points. Let $\mathscr{A}$ be an arrangement of n lines of the projective plane such that it does not contain a common point (i.e., $t_n(\mathscr{A}) = 0$). T. Gallai [**4**] proved that in this case there exist two lines from $\mathscr{A}$ whose intersection point has multiplicity 2. This statement was improved by Kelly and Moser [**9**] ($t_2(\mathscr{A}) \geqslant 3n/7$) and recently by S. Hansen [**7**] ($t_2(\mathscr{A}) \geqslant \lfloor n/2 \rfloor$).

The following question was posed by P. Erdős [**2**]. Let us suppose that the arrangement $\mathscr{A}$ does not contain a point with multiplicity more than 3. Then does there exist a Gallai triangle, i.e. three lines from $\mathscr{A}$ such that their three intersection points have multiplicity 2, or not? Our construction $\mathscr{B}_n$ shows that the answer is negative for $n \geqslant 4$, $n \not\equiv 0 \pmod 9$. Another problem on Gallai points can be found in [**2**].

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Mathematics Institute, Hungarian Academy of Science, Budapest H-1364 P.O. Box 127, Hungary