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THE NUMBER OF TRIANGLES COVERING THE CENTER OF AN n-SET

ABSTRACT. Let the points $P_1, P_2, \ldots, P_n$ be given in the plane such that there are no three on a line. Then there exists a point of the plane which is contained in at least $n^3/27$ (open) $P_iP_jP_k$ triangles. This bound is the best possible.

1. Introduction

Let $n \geqslant 3$ be an integer and let $\mathscr{P} := \{P_1, P_2, \ldots, P_n\}$ be a family of points of the Euclidean plane σ such that there are no three of them on a line (i.e. $\mathscr{P}$ is independent). For all points $X \in \sigma$ let $f(\mathscr{P}, X)$ be defined as the number of triangles $P_iP_jP_k$ which contain X as an inner point.

Our problem is to investigate the function $f(\mathscr{P}) := \max_X f(\mathscr{P}, X)$. This problem was posed by Kárteszi [6] in 1955. Many authors (see [7, p. 9] or [4]) have shown that

$$(1) \qquad f(\mathscr{P}) \leqslant \begin{cases} (n^3 - 4n)/24 & \text{if } n \text{ is even,} \\ (n^3 - n)/24 & \text{if } n \text{ is odd} \end{cases}$$

holds for all $\mathscr{P}$, and these bounds are best possible. (In this paper we prove (1) as a by-product.)

Our main result is the determination of $\min f(\mathscr{P})$, where the minimization ranges over all independent n-point sets of the plane.

THEOREM 1. $\min_{\mathscr{P}} f(\mathscr{P}) = n^3/27 + O(n^2)$.

The proof consists of two parts. In Section 5 we prove that for each independent point-family $\mathscr{P}$ one can choose a point $X_0 \in \sigma$ which is contained in at least $n^3/27$ triangles from $\mathscr{P}$. On the other hand, in Section 6 we give an n-point set $\mathscr{P}_n$, such that $f(\mathscr{P}_n, X) < n^3/27 + n^2$ holds for all $X \in \sigma$.

2. Notations and lemmas

Let us denote by $[X, A)$ the closed ray passing through the point A from the point X. Similarly, denote by (X, A) the straight line incident with the point X and $A (X \neq A)$. Let $\sigma(X, A)$ be the open half-plane bounded by the line (X, A) such that for any point $B \in \sigma(X, A)$ the triangle XAB has negative (i.e. clockwise) orientation. Set $\sigma[X, A) = \sigma(X, A) \cup [X, A)$. If C is a convex set $b(C)$ denotes its boundary.

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If the point X lies on some of the lines P_iP_j, $1 \leqslant i \leqslant j \leqslant n$, then moving it inside a small enough circle the value of $f(\mathscr{P}, X)$ can be increased. Our aim is to determine $\max_X f(\mathscr{P}, X)$ so we can suppose that the system $\mathscr{P} \cup \{X\}$ is also independent.

Let η be a fixed half-plane, $\mathscr{P} \cap b(\eta) = \emptyset$, and let $X \in b(\eta)$ be a fixed point.

Suppose that $\{P_1, P_2, \ldots, P_k\} = \mathscr{P} \setminus \eta$ and for any $P_s \in \mathscr{P} \setminus \eta$ define $a_s := |\mathscr{P} \cap \eta \cap \sigma[X, P_s)|$. We may suppose that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k$.

LEMMA 1. $f(\mathscr{P}, X) = \sum_{s=1}^{k} a_s(2s + n - 1 - 2k - a_s)$.

Proof. Any triangle which covers X has one or two vertices belonging to η. The number of triangles $P_sP_tP_i \ni X$ with $s < t$ and $P_s, P_t \in \mathscr{P} \setminus \eta$ is $a_t - a_s$; and the number of triangles $P_sP_iP_j \ni X$ with $P_s \in \mathscr{P} \setminus \eta$ and $P_i, P_j \in \eta$ is $a_s(n - k - a_s)$ by the definition of the numbers a_s. Then

$$f(\mathscr{P}, X) = \sum_{s=1}^{k} a_s(n - k - a_s) + \sum_{s<t} (a_t - a_s) \tag{1}$$

and from this the statement follows by an easy calculation.

For the given $\mathscr{P}$ and X let us define the function $g: (\sigma \setminus \{X\}) \to \{0, 1, \ldots, n\}$ as follows:

$$g(A) := |\mathscr{P} \cap \sigma[X, A)| \quad \text{for all } A \in \sigma \setminus \{X\}.$$

Reflect the points of $\mathscr{P}$ with centre X and denote by $\mathscr{P}'$ its image. List the points of $\mathscr{P} \cup \mathscr{P}'$ in cyclic order around X, say, in clockwise orientation, i.e. $\mathscr{P} \cup \mathscr{P}' = \{S_1, S_2, \ldots, S_{2n}\}$. Then S_i and S_{i+n} are an opposite pair, and one of them belongs to $\mathscr{P}$. This implies

$$g(S_i) + g(S_{i+n}) = n \tag{2}$$

and

$$g(S_{i+1}) - g(S_i) = \begin{cases} 1, & \text{if } S_i \in \mathscr{P}', \\ -1, & \text{if } S_i \in \mathscr{P}. \end{cases} \tag{3}$$

LEMMA 2. $f(\mathscr{P}, X) = \frac{1}{24}(n^3 + 2n) - \frac{1}{4} \times \sum_{i=1}^{2n} \left(g(S_i) - \frac{n}{2} \right)^2$

Proof. A triangle T with vertices from $\mathscr{P}$ contains, or does not contain, X. In the second case T has exactly one vertex $P \in \mathscr{P}$ such that $T \subseteq \sigma[X, P)$. For fixed $P \in \mathscr{P}$ the number of such triangles is $\binom{|\mathscr{P} \cap \sigma[X, P)| - 1}{2}$, and from this follows:

$$(4)\qquad f(\mathscr{P},X)=\binom{n}{3}-\sum_{P\in\mathscr{P}}\binom{g(P)-1}{2}.$$

Clearly, $f(\mathscr{P},X)=f(\mathscr{P}',X)$; thus

$$f(\mathscr{P},X)=\tfrac{1}{2}(f(\mathscr{P},X)+f(\mathscr{P}',X)).$$

Hence the lemma follows from (4) by a simple calculation.

3. THE THICKNESS OF TRIANGLES IN THE pTH CORE OF THE CONVEX HULL

Let $p\geqslant 0$ be an integer. Denote by $\mathrm{Conv}_p(\mathscr{P})$ the *pth core* of the convex hull of $\mathscr{P}$, which is the intersection of the closed half-planes containing exactly $|\mathscr{P}|-p$ points of $\mathscr{P}$. It is clear that $\mathrm{Conv}_0(\mathscr{P})$ is just the convex hull of the pointset $\mathscr{P}$.

PROPOSITION 1. *If* $p\leqslant(n-1)/3$, *then* $\mathrm{Conv}_p(\mathscr{P})\neq\varnothing$.

Proof. Consider the family of closed half-planes containing $|\mathscr{P}|-p$ points of $\mathscr{P}$. Any three of them cover $3(n-p)\geqslant 2n+1$ times the points of $\mathscr{P}$, hence they have a common point of $\mathscr{P}$, i.e. the intersection of any three such half-planes is not empty. Therefore by the Helly theorem (see [5]) the intersection of the whole family is not empty.

Similarly, it is easy to prove that

PROPOSITION 2. *If* $p>(n-1)/2$, *then* $\mathrm{Conv}_p(\mathscr{P})=\varnothing$.

Moreover, if $\mathrm{Conv}_p(\mathscr{P})\neq\varnothing$ holds for $p=(n-1)/2$, then it contains a single point only.

The Caratheodory theorem says (see [2], [5]) that if $X\in\mathrm{Conv}(\mathscr{P})$, then there exists a closed triangle $P_iP_jP_k$ which covers X. In [3] Birch proved that there are at least $n-2$ such triangles. In other words, if $\mathscr{P}\cup\{X\}$ is independent, $X\in\mathrm{Conv}(\mathscr{P})$, then $f(\mathscr{P},X)\geqslant n-2$. Here we improve this result.

THEOREM 2. *Let* $\mathscr{P}\cup\{X\}$ *be an independent family of points in the plane* $n=|\mathscr{P}|$. *If* $X\in\mathrm{Conv}_p(\mathscr{P})$, *then*

$$(5)\qquad f(\mathscr{P},X)\geqslant\binom{p+2}{2}n-\frac{1}{2}\binom{2p+4}{3}.$$

Moreover, if $X\notin\mathrm{Conv}_{p+1}(\mathscr{P})$, *then*

$$(6)\ f(\mathscr{P},X) \leqslant \begin{cases} \frac{1}{4}\left(\binom{n}{3}-\binom{n+2p-2}{3}\right)+\frac{(p+1)(n-p-2)}{2} & \text{if } n \text{ is even,} \\ \frac{1}{4}\left(\binom{n+1}{3}-\binom{n-2p-1}{3}\right) & \text{if } n \text{ is odd.} \end{cases}$$

These bounds are best possible.

Proof. Let n be fixed. If $0 \leqslant p \leqslant (n-1)/2$, then the lower bound in (5) increase and the upper bounds in (6) decrease. Hence, we may suppose that $X \in \operatorname{Conv}_p(\mathscr{P}) \setminus \operatorname{Conv}_{p+1}(\mathscr{P})$. X is an inner point of this set, since $\mathscr{P} \cup \{X\}$ is independent.

Let η be a closed half-plane with $X \in b(\eta)$, containing $\operatorname{Conv}_{p+1}(\mathscr{P})$ such that $|\mathscr{P} \cap \eta| = n-p-1$ and $b(\eta) \cap \mathscr{P} = \varnothing$. By the definition of the pth core such a half-plane exists. Then applying Lemma 1 for this half-plane and for $k = p+1$ we have

$$(7) \qquad f(\mathscr{P},X) = \sum_{s=1}^{p+1} a_s(2s+n-3-2p-a_s).$$

As $X \in \operatorname{Conv}_p(\mathscr{P})$ is an inner point, every half-plane passing through X with its boundary line contains at least $p+1$ and at most $n-p-1$ points of $\mathscr{P}$. Hence $n-p-1 \geqslant |\mathscr{P} \cap \sigma(X,P_s)| \geqslant p+1$ for every $P_s \in \mathscr{P} \setminus \eta$ and therefore $n-2p-3+s \geqslant a_s \geqslant s$ for $s = 1,2,\dots,p+1$. In this case the terms of the sum in (7) are minimal if $a_s = s$, and are maximal if their factors are close, i.e. if $a_s = s-p-1+(n-1)/2$ if n is odd and if $a_s = s-p-1+(n-2)/2$ if n is even. Thus (5) and (6) follow from (7) by simple calculation.

The sharpness of the bounds can be proved by constructions (see Figs. 1 and 2). Let us consider a regular $(2p+3)$-gon $P_0, P_1, \dots, P_{2p+2}$ with center X. Suppose that $n \geqslant 2p+3$. Let the point set $\mathscr{P}_1$ consist of $P_0, P_1, \dots, P_{2p+2}$ and a $(n-2p-3)$-element point set around P_0. Let the point set $\mathscr{P}_2$ consist of $P_0, P_1, \dots, P_{2p+2}$ and a $[(n-2p-3)/2]$-element point set around P_{p+2} and a $[(n-2p-2)/2]$-element point set around P_0.

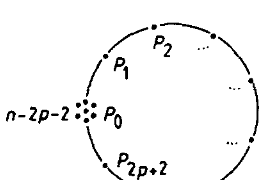

Fig. 1

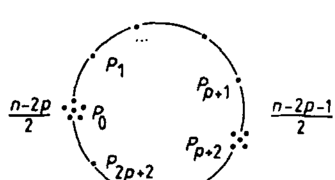

Fig. 2

It is easy to see that $\mathscr{P}_1$ and $\mathscr{P}_2$ satisfy the conditions and equality holds in (5) for $\mathscr{P}_1$ and in (6) for $\mathscr{P}_2$. (1) follows from Theorem 2 immediately with $p=(n-1)/2$ if n is odd and with $p=(n-2)/2$ if n is even.

4. THE CENTER OF AN n-SET

We have to find an appropriate point X_0 for the given $\mathscr{P}$ such that $f(\mathscr{P},X_0)\geqslant n^3/27$. By Proposition 1, $\mathrm{Conv}_{[n-1/3]}(\mathscr{P})\neq 0$. Hence, if $X_0\in \mathrm{Conv}_{[(n-1)/3]}(\mathscr{P})$, then Theorem 2 gives that $f(\mathscr{P},X_0)\geqslant (n^3/27)\cdot(20/24)$ and the construction given by Figure 1 shows that this result is the best possible. Nevertheless, for the proof of Theorem 1, we shall choose X_0 from $\mathrm{Conv}_{[(n-1)/3]}(\mathscr{P})$. We need an additional lemma.

Suppose that q is an integer such that $\mathrm{Conv}_q(\mathscr{P})\neq\varnothing$ and either $\mathrm{Conv}_{q+1}(\mathscr{P})=\varnothing$ or it contains only one point. By Propositions 1 and 2 we have $(n-1)/3\leqslant q\leqslant (n-1)/2$. Suppose $q<(n-2)/2$.

LEMMA 3. *There exists an inner point $X\in\mathrm{Conv}_q(\mathscr{P})$ and three closed half-planes η_1,η_2,η_3 such that X lies on their boundaries, η_1,η_2 and η_3 cover the plane and $|\eta_i\cap\mathscr{P}|=n-q-1$ for $i=1,2,3$.*
Call such a point X the center of $\mathscr{P}$.

Proof. For the proof we are going to introduce a function on the set of closed half-planes.

Let α be an arbitrary closed half-plane with $e=b(\alpha)$. Now let $\alpha_0\supsetneqq\alpha_1\supsetneqq\cdots\supsetneqq\alpha_r$ be the set of closed half-planes with $e_i=b(\alpha_i)$, such that the line e_i is parallel to e and passes through at least one of the points of $\mathscr{P}$. Then define $\mu(\alpha)$ as follows:

$$\mu(\alpha):=\begin{cases}0 & \text{if } \alpha\supseteq\alpha_0\\ (n-1-|\alpha_i\cap\mathscr{P}|)+|e_i\cap\mathscr{P}|+\dfrac{d(e,e_i)}{d(e_i,e_{i+1})}|e_{i+1}\cap\mathscr{P}| & \text{if } \alpha_i\supsetneqq\alpha\supseteq\alpha_{i+1}\\ n-1 & \text{if } \alpha_r\supsetneqq\alpha\end{cases}$$

where $d(e,f)$ denotes the Euclidean distance between lines e and f.

It is clear that if α is moved over the plane parallel to a fixed position, then the function $\mu(\alpha)$ changes continuously; and if the boundary line of α contains exactly one point of $\mathscr{P}$, then $\mu(\alpha)=n-|\alpha\cap\mathscr{P}|$. As the point set $\mathscr{P}$ is finite the distances $d(e_i,e_{i+1})$ in the definition of μ are bounded by a certain real D from above. Hence, if $\alpha_0 \supset \alpha \supset \beta \supset \alpha_r$ and $d(b(\alpha), b(\beta)) = \varepsilon$, then

(8) $\qquad \mu(\beta) \geqslant \mu(\alpha) + \varepsilon/D.$

In this proof we shall consider only those half-planes with boundary lines parallel to the lines formed by the point of $\mathscr{P}$. Actually this restriction does not change our statements, but the proof becomes clearer.

For every real $x, 0 \leqslant x \leqslant n-1$ let $\operatorname{Conv}_x(\mathscr{P}) := \bigcap \{\alpha \mid \alpha$ is a closed half-plane, $b(\alpha)$ is parallel to some (P_iP_j) and $\mu(\alpha) = x\}$.

It is easy to prove that $\operatorname{Conv}_x(\mathscr{P})$ is a convex, closed polygon in the plane and for an integer $x = k \operatorname{Conv}_x(\mathscr{P})$ is just the kth core of $\mathscr{P}$; moreover, if $\operatorname{Conv}_y(\mathscr{P}) \neq \emptyset$, then $\operatorname{Conv}_x(\mathscr{P}) \supset \operatorname{Conv}_y(\mathscr{P})$ for $0 < x < y$.

From these facts it follows that there is a greatest real x_0 for which $\operatorname{Conv}_{x_0}(\mathscr{P}) \neq \emptyset$. It is clear that $q = [x_0]$. Using (8) it can be proved that $\operatorname{Conv}_{x_0}(\mathscr{P})$ has no inner point. We state that it contains only one point, say X, otherwise $q \geqslant (n-2)/2$ would follow, contradicting our assumption.

Let us consider the finitely many opened half-planes $\mathring{\alpha}$ for which $\mu(\mathring{\alpha} \cup b(\mathring{\alpha})) = x_0$. Then the intersection of these half-planes is empty by the definition of x_0. Hence there are three such half-planes $\mathring{\eta}_1, \mathring{\eta}_2, \mathring{\eta}_3$ which have empty intersection by the Helly theorem. Let η_1, η_2, η_3 be the closure of these half-planes. Then $X \in b(\eta_1) \cap b(\eta_2) \cap b(\eta_3)$ by its definition; η_1, η_2 and η_3 cover the plane and $|\eta_i \cap \mathscr{P}| = n - q - 1$.

We note that an analogous statement also holds in higher dimensions.

5. The proof of the lower bound in Theorem 1

Let $X = X_0$ be the center of $\mathscr{P}$ given by Lemma 3. Suppose that $g(S_1) = n - q - 1$. By Lemma 3 we have that there exist indices i, j $(1 < i < j < n)$ such that $g(S_i) = q + 1$, $g(S_j) = n - q - 1$, $g(S_{n+1}) = q + 1$ holds by (2). According to (2) and (3) we get that $\sum_{i=1}^{2n} (g(S_i) - n/2)^2$ is maximal with respect to these constraints, e.g. for the function g given in Figure 3. Hence

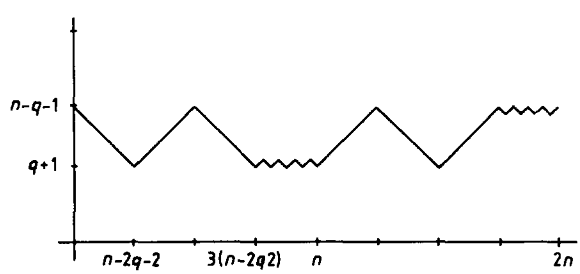

Fig. 3

we have, using (2),

$$\sum_{i=1}^{2n}\left(g(S_i)-\frac{n}{2}\right)^2=\sum_{i=1}^{2n} g^2(S_i)-\frac{n^3}{2}$$

$$\leqslant 3\sum_{i=q+2}^{n-q-1} i^2+3\sum_{i=q+1}^{n-q-2} i^2+(3q+3-n)$$

$$\times((q+1)^2+(q+2)^2+(n-q-1)^2+(n-q-2)^2)-\frac{n^3}{2}$$

$$=\tfrac{1}{2}(n-2q-2)(n-2q-4)(4q+6-n)+n.$$

By Propositions 1 and 2 we have $(n/3)-1\leqslant q\leqslant (n/2)-1$. The last expression increases in this interval. Hence, we get

$$\sum_{i=1}^{2n}\left(g(S_i)-\frac{n}{2}\right)^2\leqslant\frac{n^3}{54}+\frac{n}{3}.$$

Then $f(\mathscr{P},X)\geqslant n^3/27$ follows by Lemma 2.

6. A construction for the proof of the upper bound in Theorem 1

We now define $\mathscr{P}_n$. Let C be the unit circle, with center O, and let Q be a point on its circumference. Let $\mathscr{P}_n:=\mathscr{A}\cup\mathscr{B}\cup\mathscr{C}$, with

$$\mathscr{A}=\{A_i \mid \sphericalangle QOA_i=i/n^2,\quad 1\leqslant i\leqslant \lfloor n/3\rfloor\};$$

$$\mathscr{B}=\{B_j \mid \sphericalangle QOB_j=(2\pi)/3+n^{-j},\quad 1\leqslant j\leqslant \lfloor (n+1)/3\rfloor\}$$

and

$$\mathscr{C}=\{C_k \mid \sphericalangle QOC_k=(4\pi)/3-n^{-k},\quad 1\leqslant k\leqslant \lfloor (n+2)/3\rfloor\},$$

where the points of $\mathscr{P}_n$ also belong to the circumference of C.

PROPOSITION 3. *For all X we have $f(\mathscr{P}_n,X)<n^3/27+n^2$.*

Proof. If X is covered by every triangle $A_iB_jC_k$, then $f(\mathscr{P}_n,X)=[n/3][(n+1)/3][(n+2)/3]\leqslant n^3/27$.

If X belongs to the convex hull of two groups of $\mathscr{P}_n$, say that $X\in\mathrm{Conv}(\mathscr{A}\cup\mathscr{B})$, but it is not contained in any triangle $A_iA_jA_k$ or $B_iB_jB_k$, then Lemma 1 can be applied.

Consider the half-plane η which separates $\mathscr{A}$ from $\mathscr{B}\cup\mathscr{C}$ with X on its boundary line and apply Lemma 1 with this half-plane and with $k=[n/3]$. It is easy to prove that there is an index $t\leqslant k$ such that $a_s=a_1$

for all $1 \leqslant s \leqslant t$, and $a_s = k$ for $k \geqslant s \geqslant t+2$. Hence by Lemma 1 we obtain

$$\begin{aligned} f(\mathscr{P}_n, X) &= \sum_{s=1}^{t} a_1(2s - 2k + n - 1 - a_1) \\ &\quad + \sum_{s=t+3}^{k} k(2s - 2k + n - 1 - k) \\ &\quad + a_{t+1}(2t + 1 - 2k + n - a_{t+1}) \\ &\quad + a_{t+2}(2t + 3 - 2k + n - a_{t+2}) \\ &\leqslant t \cdot a_1(k + t - a_1) + k(k-t)(k+t) \\ &\quad + \left(t - k + \frac{n+1}{2}\right)^2 + \left(t - k + \frac{n+3}{2}\right)^2. \end{aligned}$$

This is maximal, if $t = a_1 = k (= [n/3])$, thus $f(\mathscr{P}_n, X) \leqslant n^3/27 + n^2/2 + 2n + \frac{5}{2}$ in this case, too.

Finally, if X belongs to the convex hull of one group of $\mathscr{P}_n$, say to Conv$(\mathscr{A})$, then there is a nearest line (A_iA_j) which separates it from the points of $\mathscr{B}$ and $\mathscr{C}$. Then moving X through this line, $f(\mathscr{P}_n, X)$ increases at least by $[n/3]$; hence X does not maximize the function f.

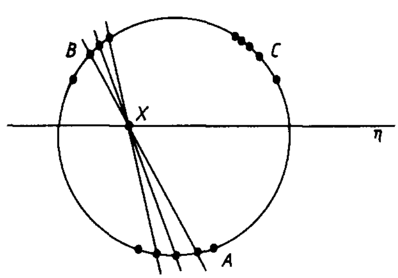

Fig. 4

7. A REMARK ON THE HIGHER DIMENSIONAL CASE

Let $\mathscr{P} \subset \mathbb{R}^d$ be an n-element set, and $X \in \mathbb{R}^d$ a point. We can define $f^d(\mathscr{P}, X)$ as the number of (open) d-simplices covering X with vertices from $\mathscr{P}$. It is easy to see that

$$f^d(\mathscr{P}, X) \leqslant \frac{1}{2^d} \binom{n}{d+1}.$$

(Bárány [2] determined exactly the value of $\max f^d(\mathscr{P}, X)$.) Similarly, using Tverberg's theorem [8] and a generalization of Caratheodory's theorem, Bárány proved the following in [2]:

For each independent $\mathscr{P} \subset \mathbb{R}^d$, *there exists an* $X \in \mathbb{R}^d$ *such that*

$$(9) \qquad f^d(\mathscr{P}, X) \geqslant n^{d+1}/(d+1)!\,(d+1)^{d+1} - O(n^d).$$

His proof is suitable to obtain a d-dimensional version of Theorem 2, proving that

$$(10) \qquad f^d(\mathscr{P}, X) \geqslant n \cdot k^d/d!\, d^d$$

holds for $X \in \mathrm{Conv}_k(\mathscr{P})$.

This generalizes a result of Baker [1]: $f^d(\mathscr{P}, X) \geqslant n - d$ holds for all $X \in \mathrm{Conv}(\mathscr{P})$. Formulas (9) and (10) give the best possible bounds, apart from a constant factor, but the determination of the exact values is an open problem.

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