COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
37. FINITE AND INFINITE SETS, EGER (HUNGARY), 1981.

FAMILIES OF FINITE SETS WITH MISSING INTERSECTIONS

P. FRANKL – Z. FÜREDI

ABSTRACT

Let X be an n-element set, l, s positive integers satisfying $s \geqslant 3l + 2$. Suppose that $\mathscr{F}$ is a family of subsets of X having the property that for any two different $F, F' \in \mathscr{F}$ we have $|F \cap F'| \leqslant s$ and $|F \cap F'| \neq l$. We prove that $|\mathscr{F}| \leqslant (1 + o(1))\binom{n-l-1}{s-l}$. Taking all the $(s+1)$-sets containing a given $(l+1)$-set shows that this is asymptotically best possible.

1. PRELIMINARIES

Let X be an n-element set. For an integer k we use the notation $\binom{X}{k} = \{A \subset X\colon |A| = k\}$ and $\binom{X}{\leqslant k} = \{A \subset X\colon |A| \leqslant k\}$.

If $L \subset \{0, 1, \ldots, n-1\}$ and $\mathscr{F}$ is a family of subsets of X then $\mathscr{F}$ is called an (n, L)-system ((n, k, L)-system) if for any two different $F, F' \in \mathscr{F}$ we have $|F \cap F'| \in L$ (and in addition $|F| = k$), respectively. We denote by $m(n, L)$ $(m(n, k, L))$ the maximum cardinality of an (n, L)-system ((n, k, L)-system), respectively. In this terminology two of the basic theorems in extremal set theory, can be stated as follows:

Theorem (Erdős, Ko, Rado [5]). *Suppose* $n > n_0(k, t)$, *where* $k > t > 0$, *are integers. Then*

$$m(n, k, \{t, t+1, \dots, k-1\}) = \binom{n-t}{k-t}. \tag{1}$$

Theorem (Katona [16]). *Suppose* $n > t > 0$. *Then*

$$m(n, \{t, t+1, \dots, n-1\}) = \begin{cases} \sum_{i \geq \frac{n+t}{2}} \binom{n}{i} & \text{if } (n+t) \text{ is even,} \\ 2 \sum_{i \geq \frac{n+t-1}{2}} \binom{n-1}{i} & \text{if } (n+t) \text{ is odd.} \end{cases} \tag{2}$$

Erdős [4] asked for the values of $m(n, k, \{0, 1, \dots, k-1\} - \{l\})$ and $m(n, \{0, 1, \dots, n-1\} - \{l\})$. He conjectured in particular that for $k \geq 2l+1$, $k \geq 3$, $n > n_0(k)$ the value of the first function is $\binom{n-l-1}{k-l-1}$. If true, this is a strengthening of (1). This conjecture was proved in [9] for $l = 1$. For the general case it is known

Theorem (Frankl [7]). *If* $k \geq 3l+2$, *then*

$$m(n, k, \{0, 1, \dots, k-1\} - \{l\}) = (1 + o(1))\binom{n-l-1}{k-l-1}. \tag{3}$$

If $k \geq 3l$, *then*

$$m(n, k, \{0, 1, \dots, k-1\} - \{l\}) = O\left(\binom{n-l-1}{k-l-1}\right). \tag{4}$$

If $3l > k > l$, *then*

$$m(n, k, \{0, 1, \dots, k-1\} - \{l\}) = O(n^{\frac{k+l-1}{2}}). \tag{5}$$

For the non-uniform case it is conjectured in [4]:

Conjecture 1. *If* $n > n_0(l)$, *then*

$$m(n, \{0, 1, \dots, n-1\} - \{l\}) = \tag{6}$$

$$= m(n, \{l+1, \ldots, n-1\}) + \sum_{i \leq l-1} \binom{n}{i}.$$

If true this conjecture would generalize (2) – to an $(n, \{l+1, \ldots, n-1\})$-system we can always adjoin $\binom{X}{< l}$, and still have an $(n, \{0, 1, \ldots \ldots, n-1\} - \{l\})$-system. In [8] the conjecture was proved for $l = 1$. In [11] the validity of the conjecture is showed apart from a polynomial remainder term. Recently the authors [10] have proved (6) for $n > 3^l$. However, (6) seems to be true for all n and l.

A general result is the following

Theorem (Frankl, Wilson [12], Ray-Chaudhuri, Wilson [18])

$$m(n, L) \leq \sum_{i \leq |L|} \binom{n}{i}, \tag{7}$$

$$m(n, k, L) \leq \binom{n}{|L|}. \tag{8}$$

There are several more results on $m(n, k, L)$ for $n > n_0(k)$ but very little is known about $m(n, L)$. In [13] it is shown that for $n > n_0(r)$ we have $m(n, \{0, r\}) = \binom{\lfloor \frac{n}{r} \rfloor}{2} + \lfloor \frac{n}{r} \rfloor + 1 + \left(n - r\lfloor \frac{n}{r} \rfloor\right)$, and it was proved that $m(n, \{0, 2, 3\}) = O(n^2)$. Other results can be found in [14].

2. RESULTS

The main result of this paper is the following

Theorem 1. *Suppose* $s \geq 3l + 2$, n, s, l *are positive integers. Then*

$$m(n, \{0, 1, \ldots, s\} - \{l\}) = (1 + o(1))\binom{n-l-1}{s-l}.$$

By taking $\binom{X}{< l}$ and all the members of $\binom{X}{\leq s+1}$ which contain a fixed $(l+1)$-element set, we see

$$m(n, \{0, 1, \ldots, s\} - \{l\}) \geq \sum_{i \leq l-1} \binom{n}{i} + \sum_{i=l+1}^{s+1} \binom{n-l-1}{i-l-1}. \tag{9}$$

Conjecture 2. *For* $n > n_0(s)$ *and* $s > 2l + 1$ *equality holds in* (9).

For $l = 0$ we can prove our conjecture.

Theorem 2. *If* $n \leq 2s + 2$ *or* $n > 100 \dfrac{s^2}{\log (s+1)}$ *then*

$$m(n, \{1, 2, \ldots, s\}) = \sum_{i \leq s} \binom{n-1}{i}. \tag{10}$$

For $s \geq 2$ *the extremal families are* $\mathcal{G}_x = \{A \subset X\colon |A| \leq s+1,\ x \in A\}$ *and* $\mathcal{G}'_x = \{A \subset X\colon |A| \leq s+1,\ x \in A,\ |A| \geq 2\} \cup \{X - \{x\}\}$.

The case $L = \{1\}$ was investigated by de Bruijn and Erdős [1]. They proved $m(n, \{1\}) = n$ and the extremal families are: $\mathcal{G}_x, \mathcal{G}'_x$ and the finite projective plane on n points (if there exists). L. Pyber [17] (a student of G.O.H. Katona) proved (10) for $6s < n < \dfrac{s^2}{6}$.

3. STAR-SYSTEMS

We need some lemmas. We say that the sets $A_1, \ldots, A_k$ form a *star-system* (or Δ*-system*) of cardinality k with kernel B if for $1 \leq i < j \leq k$ $A_i \cap A_j = B$ holds. The use of star-systems is based on the following

Lemma 1 (Deza, Erdős, Frankl [3]). *Suppose the members* $F_1, \ldots, F_k$ *and* $G_1, \ldots, G_k$ *of the* (n, L)*-system* $\mathcal{F}$ *form a star-system with kernels* B *and* B', *respectively. Moreover* $|F_i| < k$, $|G_i| < k$ *for* $1 \leq i \leq k$. *Then*

(11) $|B \cap B'| \in L.$

(12) *For each* $F \in \mathcal{F}$, *satisfying* $|F| < k$ *we have* $|F \cap B| \in L$.

This lemma follows from the definitions.

Lemma 2. *Suppose* $\mathcal{A} \subset \binom{X}{\leq r}$ $|A \cap A'| < a$ *for every* $A, A' \in \mathcal{A}$ $(r \geq a,\ k \geq 2$ *integers) moreover* $|\mathcal{A}| > k^a \prod_{i=0}^{a-2} (r - i)$. *Then* $\mathcal{A}$ *contains a star-system of cardinality greater than* k.

Proof. We apply induction on a. If $a = 1$, then $\mathscr{A}$ consists of pairwise disjoint sets, i.e. it is a star-system itself. Thus the statement is true. Suppose now a is the smallest value for which we have not shown the lemma yet. Let $A_1, \ldots, A_s$ be a maximal collection of pairwise disjoint sets in $\mathscr{A}$. If $s > k$, we are done. Thus we may assume $s \leqslant k$. Let us set $Y = A_1 \cup \ldots \cup A_s$. Then $|Y| \leqslant kr$ and for every $A \in \mathscr{A}$ we have $A \cap Y \neq \phi$. Hence we may find an $y \in Y$ satisfying $|\mathscr{A}(y)| \geqslant \frac{|\mathscr{A}|}{kr}$, where $\mathscr{A}(y) = \{A \in \mathscr{A} : y \in \mathscr{A}\}$. Define now $\mathscr{B} = \{A - \{y\} : A \in \mathscr{A}(y)\}$. Then for $B, B' \in \mathscr{B}$ we have $|B| \leqslant r - 1$, $|B \cap B'| < a - 1$. Moreover $|\mathscr{B}| \geqslant \frac{|\mathscr{A}|}{kr} > k^{a-1}(r-1)\ldots(r-a+2)$. By the induction hypothesis $\mathscr{B}$ contains a star-system of cardinality $k+1$, say $B_1, \ldots, B_{k+1}$. Setting $A_i = B_i \cup \{y\}$, we obtain the desired star-system in $\mathscr{A}$. ∎

Corollary 1. *Suppose* $B \subset \binom{X}{\leqslant r}$ *and* $1 \leqslant |B \cap B'| < a$ *for every* $B, B' \in \mathscr{B}$, *moreover* $|\mathscr{B}| > k^{a-1} \prod_{i=0}^{a-2} (r-i)$. *Then* $\mathscr{B}$ *contains a star-system of cardinality greater than* k.

Proof. Just let $\mathscr{A}$ consist of k pairwise disjoint copies of $\mathscr{B}$, then $|\mathscr{A}| = k|\mathscr{B}| > k^a \prod_{i=0}^{a-2} (r-i)$. We may apply Lemma 2 and find a star-system $A_1, \ldots, A_{k+1}$. By the construction of $\mathscr{A}$, it contains no $k+1$ pairwise disjoint sets, whence the kernel of the star-system is non-empty. Thus all the A_i's are from one copy of $\mathscr{B}$, yielding the result. ∎

Remark 1. Lemma 2 is a generalization of a theorem due to Erdős and Rado [6]. He proved the case $a = r$.

Remark 2. It would be very interesting to know what happens if we replace the condition $|A \cap A'| < a$ by $|A \cap A'|$ takes at most a different values. For the case $a = 1$, this more general problem was solved by Deza [2], who showed that if $|\mathscr{A}| > r^2 - r + 1$ then $\mathscr{A}$ is a star-system.

4. THE PROOF OF THEOREM 1

Suppose now that $\mathscr{F}$ is an $(n, \{0, 1, \ldots, l-1, l+1, \ldots, s\})$-system. By Theorem 2 we may suppose $l \geq 1$. We break up $\mathscr{F}$ into 4 parts: $\mathscr{F} = \mathscr{F}_0 \cup \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, where

$$\mathscr{F}_0 = \{F \in \mathscr{F} : |F| \leq l\},$$

$$\mathscr{F}_1 = \{F \in \mathscr{F} : l+1 \leq |F| < n^{\frac{\epsilon}{l}}\},$$

$$\mathscr{F}_2 = \{F \in \mathscr{F} : n^{\frac{\epsilon}{l}} \leq |F| < n^{\epsilon} n^{\frac{l+1}{s+1}}\},$$

$$\mathscr{F}_3 = \{F \in \mathscr{F} : n^{\epsilon} n^{\frac{l+1}{s+1}} \leq |F|\}.$$

For convenience we set $k(n) = n^{\epsilon} n^{\frac{l+1}{s+1}}$. The value of ϵ is $\frac{1}{10s}$. We shall estimate the cardinalities of $\mathscr{F}_i$'s separately.

Of course we have

$$|\mathscr{F}_0| \leq \sum_{i=0}^{l} \binom{n}{i} = o(n^{s-l-\epsilon}). \tag{13}$$

Moreover we have

$$|\mathscr{F}_3| = o(n^{s-l-\epsilon}). \tag{14}$$

Proof. For $F, F' \in \mathscr{F}_3$ we have $\binom{F}{s+1} \cap \binom{F'}{s+1} = \phi$. Moreover $|F| \geq k(n)$ implies $\binom{|F|}{s+1} > \frac{(k(n)-s)^{s+1}}{(s+1)!} > n^{l+1+2\epsilon}$ for $n > n_0(s)$.

Thus $|\mathscr{F}_3| \leq \frac{\binom{n}{s+1}}{n^{l+1+2\epsilon}} = o(n^{s-l-\epsilon})$.

Now we deal with $\mathscr{F}_2$. From now on we use simply $k(n) \leq n^{\frac{1}{3}+\epsilon}$. As $\mathscr{F}_2$ is an $(n, < k(n), \{0, 1, \ldots, s\} - \{l\})$-system, it can be embedded into a non-extendable $(n, < k(n), \{0, 1, \ldots, s\} - \{l\})$-system, $\mathscr{F}_2^*$.

Proposition 1. *If* $F_1, F_2, \ldots, F_t \in \mathscr{F}_2^*$ *form a star-system with kernel* K, *and* $t \geq k(n)$, *then* $K \in \mathscr{F}_2^*$.

Proof. Let $F \in \mathscr{F}_2^*$. Then $|F| < k(n)$, thus F cannot intersect all of the pairwise disjoint sets $F_1 - K, F_2 - K, \ldots, F_t - K$. By symmetry, assume $F \cap (F_1 - K) = \phi$. This means, however, $|F \cap K| = |F \cap F_1| \in$ $\in (\{0, 1, \ldots, s\} - \{l\})$. Hence $K \in \mathscr{F}_2^*$ follows. ∎

Let us set $\mathscr{B} = \mathscr{B}(\mathscr{F}_2^*) =: \{B \in \mathscr{F}_2^*: l + 1 \leqslant |B| \leqslant s$ and $\nexists B' \in \mathscr{F}_2^*$ such that $B \subset B'$, $|B'| \leqslant s\}$. For every set $F \in \mathscr{F}_2$ define $\mathscr{B}(F) =$ $= \{B \in \mathscr{B}: B \subset F\}$, $\mathscr{C}_l(F) = \left\{C \in \binom{F}{l}: \nexists B \in \mathscr{B}(F) \text{ such that } C \subset B\right\}$. Moreover we set for $l < i \leqslant s$:

$$\mathscr{C}_i(F) = \left\{C \in \binom{F}{l}: \exists B \in \mathscr{B},\ |B| = i,\ C \subset B \subset F\right\}.$$

We have $\binom{F}{l} = \bigcup_{i=l}^{s} \mathscr{C}_i(F)$ by definition. Hence we may fix for every $F \in \mathscr{F}_2$ the least integer $i(F)$ $(l \leqslant i \leqslant s)$ such that

$$(15) \qquad |\mathscr{C}_{i(F)}(F)| > \frac{1}{n^\epsilon} \binom{|F|}{l}.$$

(Of course $n^\epsilon > s - l + 1$). Let $\mathscr{B}_i(F) = \{B \in \mathscr{B}(F): |B| = i\}$. Then we obviously have

$$(16a) \qquad |\mathscr{B}_{i(F)}(F)| > \frac{\binom{|F|}{l}}{\binom{i(F)}{l}} \frac{1}{n^\epsilon} \quad \text{for } l \leqslant i(F) < s$$

and

$$(16b) \qquad |\mathscr{B}_{i(F)}(F)| > \frac{\binom{|F|}{l}}{\binom{i(F)}{l}} \left(1 - \frac{s}{n^\epsilon}\right) \quad \text{for } i(F) = s.$$

Let us set $\mathscr{F}^i = \{F \in \mathscr{F}_2: i(F) = i\}$. We want to give upper bounds on $|\mathscr{F}^i|$ in function of i.

(a) $i = s$. We only use that for every $F \in \mathscr{F}^s$ contains at least $\frac{n^\epsilon}{s^l}$ $B \in \mathscr{B}(F)$ such that $|B| = s$ by (16b). In view of (3) the number of choices for B is $\leqslant (1 + o(1))\binom{n - l - 1}{s - l - 1}$, and each particular B is contained in at most $(n-s+1)$ members of $\mathscr{F}^s$. Thus

$$(17) \qquad |\mathscr{F}^s| \leqslant (1+o(1))\binom{n-l-1}{s-l-1}(n-s+1)O(n^{-\epsilon}) = O(n^{s-l-\epsilon}).$$

(b) $3l \leqslant i < s$. Let us set $\mathscr{B}(i) = \{B \in \mathscr{B}: |B| = i\}$. Then in view of (4) $|\mathscr{B}(i)| \leqslant O(n^{i-l-1})$. Let $\mathscr{F}^i(r) = \{F \in \mathscr{F}^i: |F| = r\}$.

Proposition 2. *Every* $B \in \mathscr{B}(i)$ *is contained in at most* $2(k(n))^{s-i} \cdot r^{s-i-2}n$ *members of* $\mathscr{F}^i(r)$.

Proof. The definition of $\mathscr{B}$ and Proposition 1 imply that for $B \in \mathscr{B}(i)$, $x \notin B$ the set $B \cup \{x\}$ is not contained in the kernel of the star-system $F_1, F_2, \ldots, F_{[k(n)]} \in \mathscr{F}^i(r)$. So applying Lemma 2 we have

$$|\{F \in \mathscr{F}^i(r): B \cup \{x\} \subset F\}| \leqslant (k(n))^{s-i} \prod_{t=0}^{s-i-2} (r-t).$$

Hence

$$|\{F \in \mathscr{F}^i(r): B \subset F\}| \leqslant \frac{n}{r-i}(k(n))^{s-i} \prod_{t=0}^{s-i-2} (r-t). \ \blacksquare$$

Now (16a) shows that every $F \in \mathscr{F}^i(r)$ contains at least $\dfrac{\binom{r}{l}}{\binom{i}{l}n^{\epsilon}}$ members of $\mathscr{B}(i)$. Thus we deduce

$$|\mathscr{F}^i(r)| \leqslant \frac{|\mathscr{B}(i)|2(k(n))^{s-i}r^{s-i-2}n \cdot n^{\epsilon}\binom{i}{l}}{\binom{r}{l}} =$$

$$= O(n^{i-l-1+(\frac{1}{3}+\epsilon)(s-i)+1+\epsilon})r^{s-i-2-l}.$$

If $s-i-l-2 \leqslant 0$ then we deduce simply

$$(18) \qquad |\mathscr{F}^i| \leqslant k(n)O(n^{i-l+(\frac{1}{3}+\epsilon)(s-i)+\epsilon}) = o(n^{s-l-\epsilon}).$$

If $s-i-l-2 > 0$ then using $\sum_{r \leqslant k(n)} r^t < (k(n))^{t+1}$ we conclude

$$(19) \qquad |\mathscr{F}^i| < O(n^{i-l+(\frac{1}{3}+\epsilon)(s-i)+\epsilon+(\frac{1}{3}+\epsilon)(s-i-1-l)}) = o(n^{s-l-\epsilon}).$$

(c) $l < i < 3l$. We know $|\mathscr{B}(i)| \leqslant O(n^{\frac{i+l-1}{2}})$ from (5), and using Proposition 2 and distinguishing the same two cases as in (b) we deduce

$$(20) \qquad |\mathscr{F}^i| = o(n^{s-l-\epsilon}).$$

(d) $i = l$. Let $D \in \binom{X}{l}$. Then in view of Corollary 1 and Proposition 1 for fixed r, there are at most $(k(n)r)^{s-l}$ members of $\mathscr{F}^l(r)$ containing D. Using (15), we deduce

$$(21) \qquad |\mathscr{F}^l| \leqslant \sum_{r \leqslant k(n)} n^\epsilon \frac{\binom{n}{l}}{\binom{r}{l}} (k(n)r)^{s-l} = o(n^{s-l-\epsilon}).$$

Now summing up the bounds for $\mathscr{F}^i$ $(s \geqslant i \geqslant l)$ we obtain

$$(22) \qquad |\mathscr{F}_2| \leqslant O(n^{s-l-\epsilon}).$$

Now let us consider the set system $\mathscr{F}_1$. We can investigate it in the same way as we have done by set-sysem $\mathscr{F}_2$. $\mathscr{F}_1$ can be embedded into a non-extendable $(n, < n^{\frac{\epsilon}{l}}, \{0, 1, \ldots, s\} - \{l\})$-system, $\mathscr{F}_1^*$.

Proposition 1′. *If* $F_1, F_2, \ldots, F_t \in \mathscr{F}_1^*$ *form a star-system with kernel* K, *and* $t \geqslant n^{\frac{\epsilon}{l}}$, *then* $K \in \mathscr{F}_1^*$.

Let us set $\mathscr{B}' = \mathscr{B}'(\mathscr{F}_1^*) =: \{B \in \mathscr{F}_1^*: l+1 \leqslant |B| \leqslant s$ and $\nexists B' \in \mathscr{F}_1^*$ such that $B \subset B'$ $|B'| \leqslant s\}$. For every set $F \in \mathscr{F}_1$ define $\mathscr{B}'(F) = \{B \in \mathscr{B}: B \subset F\}$, $\mathscr{C}_l'(F) = \left\{C \in \binom{F}{l}: \nexists B \in \mathscr{B}'(F), C \subset B\right\}$. Moreover we set for $l < i \leqslant s$:

$$\mathscr{C}_i'(F) = \left\{C \in \binom{F}{l}: \exists B \in \mathscr{B}', |B| = i, C \subset B \subset F\right\}.$$

Let $i(F)$ the minimal i for which $|\mathscr{C}'_{i(F)}(F)| > \frac{\binom{|F|}{l}}{n^\epsilon}$. Then (16a) and (16b) holds for $\mathscr{B}_i'(F)$, too. Let us set $\mathscr{G}^i = \{F \in \mathscr{F}_1: i(F) = i\}$. Similarly to the cases (b), (c) and (d) we obtain $|\mathscr{G}^i| = o(n^{s-l-\epsilon})$ for $l \leqslant i < s$. Now we deal with $\mathscr{G}^s$. Let us set

$$\mathscr{G}_1^s = \{F \in \mathscr{G}^s \colon |\mathscr{B}_s'(F)| < s - l\}$$

and

$$\mathscr{G}_2^s = \{F \in \mathscr{G}^s \colon |\mathscr{B}_s'(F)| \geqslant s - l\}.$$

If $F \in \mathscr{G}_1^s$ then there exists a subset $A = A(F) \subset D$ $|A| = s - l - 1$ which is not contained in any $B \in \mathscr{B}_s'(F)$. By (15) and (16a) $\mathscr{B}_s'(F) = \mathscr{B}'(F)$, that is A is not contained in any $B \in \mathscr{B}'(F)$. So, applying Lemma 2, we have $|\{F \in \mathscr{G}_1^s \colon A(F) = A\}| < (n^{\frac{\epsilon}{l}})^{l+2}(n^{\frac{\epsilon}{l}})^{l+1} = o(n^{1-\epsilon})$. Thus we deduce

$$|\mathscr{G}_1^s| \leqslant \binom{n}{s-l-1} o(n^{1-\epsilon}) = o(n^{s-l-\epsilon}). \tag{23}$$

Finally, similarly to (17) we have

$$\begin{aligned} |\mathscr{G}_2^s| &\leqslant \frac{1}{s-l} \sum_{F \in \mathscr{G}_2^s} |\mathscr{B}_s'(F)| \leqslant \frac{n-s+1}{s-l} |\mathscr{B}'(s)| \leqslant \\ &\leqslant \frac{n-s+1}{s-l} (1 + o(1)) \binom{n-l-1}{s-l-1} = (1 + o(1)) \binom{n-l-1}{s-l}. \end{aligned} \tag{24}$$

So we have

$$|\mathscr{F}_1| \leqslant (1 + o(1)) \binom{n-l-1}{s-l}. \tag{25}$$

Adding up (13), (14), (22) and (25) the statement of Theorem 1 follows. ∎

5. THE PROOF OF THEOREM 2

Let $\mathscr{F}$ be a $\{1, 2, \ldots, s\}$-system on n points. Pyber [17] has found a very simple proof for the case $n \leqslant 2s + 3$ so our complicated proof can be omitted.

Let us suppose that $n > \dfrac{100s^2}{\log (s+1)}$ and $\mathscr{F} = \mathscr{F}(1) \cup \mathscr{F}(2) \cup \ldots$ where $\mathscr{F}(i) = \{F \in \mathscr{F} \colon |F| = i\}$. Moreover we can suppose that $|\mathscr{F}| \geqslant \sum_{i \leqslant s} \binom{n-1}{i}$. If $i \leqslant s$, then applying (1) we have $|\mathscr{F}(i)| \leqslant \binom{n-1}{i-1}$. Hence

(26) $\quad |\mathcal{F}(\geqslant s+1)| \geqslant \binom{n-1}{s}.$

Now we give an upper bound for $|\mathcal{F}(\geqslant s+3)|$. Let $F_0 \in \mathcal{F}$ be an arbitrary edge. Then every $F \in \mathcal{F}(\geqslant k)$ contains at least $\binom{k-1}{s}$ members of $\binom{X}{s+1} - \binom{X-F_0}{s+1}$. Hence

$$(27) \qquad |\mathcal{F}(\geqslant k)| \leqslant \frac{\binom{n}{s+1} - \binom{n-|F_0|}{s+1}}{\binom{k-1}{s}} \leqslant \frac{|F_0|}{\binom{k-1}{s}} \binom{n-1}{s}.$$

Here the coefficient of $\binom{n-1}{s}$ is less than 1 if $|F_0| = k \geqslant s+3$ so, by (26), we get that $|\mathcal{F}(s+1) \cup \mathcal{F}(s+2)| > 0$. Thus F_0 can be chosen from $\mathcal{F}(\leqslant s+2)$, and we get by (27) that

$$(28) \qquad |\mathcal{F}(\geqslant k)| \leqslant \frac{s+2}{\binom{k-1}{s}} \binom{n-1}{s}.$$

In particular, for $k = s+3$ we have

$$(29) \qquad |\mathcal{F}(\geqslant s+3)| \leqslant \frac{2}{s+1} \binom{n-1}{s}.$$

Now we deal with $\mathcal{F}(s+1) \cup \mathcal{F}(s+2) = \mathcal{G}$.

(30) $\quad$ If $\bigcap \mathcal{G} = \phi$ then $|\mathcal{G}| \leqslant \binom{3s+3}{2}\binom{n-2}{s-1}.$

(30) can be proved by considering a set T of cardinality $\leqslant 3s+3$ which meets every $G \in \mathcal{G}$ in at least 2 points. Such a T always exists. (Either there is an edge of $\mathcal{G}$ meeting the requirements, or we can find $E_1, E_2 \in \mathcal{G}$, $E_1 \cap E_2 = \{p\}$. Then there exists an edge $E_3 \in \mathcal{G}$ not containing p, and in this case $E_1 \cup E_2 \cup E_3$ is suitable for T.)

The sum of the right-hand sides of (29) and (30) is less than $\binom{n-1}{s}$, if $n > 27s^3$. This contradicts (26). Thus, we may assume $\bigcap \mathcal{G} \neq \phi$, say $p \in \bigcap \mathcal{G}$. Let $F_1 \in \mathcal{F}(\geqslant s+3)$ be a minimal edge not containing p $|F_1| = k$. Then

$$|\{F \in \mathcal{F}(\geqslant s+1):\ p \in F\}| \leqslant$$

$$\leqslant \left| \binom{X - \{p\}}{s} - \binom{X - F_1 - \{p\}}{s} \right|.$$

So by (28) we have

$$\binom{n-1-k}{s} \leqslant |\{F \in \mathcal{F}(\geqslant s+1):\ p \notin F\}| \leqslant \frac{s+2}{\binom{k-1}{s}} \binom{n-1}{s}.$$

This yields that $k > n - s - 3 > \frac{n+s}{2}$, thus we get $|\{F \in \mathcal{F}(\geqslant s+3):\ p \notin F\}| \leqslant 1$. This implies that $\mathcal{F}(s+2) = \phi$ and $\mathcal{F}(\geqslant s+3)$ is either $\{X - \{p\}\}$ or ϕ. This completes the proof for $n > 27s^3$.

For $s \geqslant 10$ we can improve (30) using the following theorem due to Hilton and Milner [15].

(31) If $\bigcap \mathcal{F}(s+1) = \phi$ then

$$|\mathcal{F}(s+1)| \leqslant \binom{n-1}{s} - \binom{n-s-1}{s} + 1.$$

((31) holds iff $n \geqslant 2s+2$.) Moreover we need

(32) $$|\mathcal{F}(s+2)| < \frac{3s+2}{\binom{s+2}{2}} \binom{n-1}{s}.$$

To prove (32) consider the members of $\binom{X}{s}$, and let the weight of $S \in \binom{X}{s}$ be $w(S) = |\{F \in \mathcal{F}(s+2):\ S \subset F\}|$. It is easy to see that if $S_1, \ldots, S_7 \in \binom{X}{s}$ are pairwise disjoint edges then one of them has weight $\leqslant 2$. Thus, the number of edges of $\binom{X}{s}$ with weight larger than 2 is less than $6\binom{n-1}{s-1}$. Hence

$$\mathcal{F}(s+2)\binom{s+2}{s} \leqslant$$

$$\leqslant 6\binom{n-1}{s-1}\frac{n-s}{6} + \left(\binom{n}{s} - 6\binom{n-1}{s-1}\right)2 < (3s+2)\binom{n-1}{s}.$$

Finally, taking the sum of (29) and (32), we obtain

$$\mathscr{F}(\geqslant s+2) < \frac{8}{s+2}\binom{n-1}{s} < \binom{n-s-1}{s}$$

for $n > \dfrac{100s^2}{\log s}$. So (26) implies

(30′) If $n > \dfrac{100s^2}{\log s}$ $(s \geqslant 10)$ then $\bigcap \mathscr{F}(s+1) \neq \phi$.

From this point the proof goes in exactly the same way as above. ∎

REFERENCES

[1] N.G. de Bruijn – P. Erdős, On a combinatorial problem, *Indag. Math.*, Akademia Amsterdam, 10 (1948), 421–423.

[2] M. Deza, Solution d'un probleme de Erdős–Lovász, *J. Combinatorial Th. B*, 16 (1974), 166–167.

[3] M. Deza – P. Erdős – P. Frankl, Intersection properties of the systems of finite sets, *Proc. London Math. Soc.*, (3) 36 (1978), 369–384.

[4] P. Erdős, Problems and results in graph theory and combinatorial analysis, *Proc. Fifth British Comb. Conf.* (Aberdeen, 1975) (C.St.J.A. Nash-Williams and J. Sheenan eds.) 169–192. *Congressus Numerantium,* 15, Utilitas Math. Publ., Winnipeg, 1976.

[5] P. Erdős – C. Ko – R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford,* (2) 12 (1961), 313–320.

[6] P. Erdős – P. Rado, Intersection theorems for systems of sets, *J. London Math. Soc.*, 35 (1960), 85–90.

[7] P. Frankl, Extremal problems and coverings of the space, *European J. Comb.,* 1 (1980), 101–106.

[8] P. Frankl, An intersection problem for finite sets, *Acta Math. Acad. Sci. Hungar.*, 30 (1977), 371–373.

[9] P. Frankl, On families of finite sets no two of which intersect in a singleton, *Bull. Austral. Math. Soc.*, 17 (1977), 125–134.

[10] P. Frankl – Z. Füredi, On hypergraphs without two edges intersecting in a given number of vertices, *J. Combinatorial Th. A*, submitted.

[11] P. Frankl – N.M. Singhi, Linear dependencies among subsets of a finite set, to appear.

[12] P. Frankl – R.M. Wilson, Intersection theorems with geometric consequences, *Combinatorica,* 1 (1981), 357–368.

[13] Z. Füredi, Set-systems with prescribed cardinalities for pairwise intersections, *Discrete Math.*, 40 (1982), 53–67.

[14] Z. Füredi, An intersection problem whose extremum is the finite projective space, *J. Combinatorial Th. A,* 32 (1982), 66–72.

[15] A.J.W. Hilton – E.C. Milner, Some intersection theorems for systems of finite sets, *Quart. J. Math. Oxford,* 2 (1967), 369–384.

[16] G.O.H. Katona, Intersection theorems for systems of finite sets, *Acta Math. Acad. Sci. Hungar.*, 15 (1964), 329–337.

[17] L. Pyber, personal communication.

[18] D.K. Ray-Chaudhuri – R.M. Wilson, On *t*-designs, *Osaka J. Math.,* 12 (1975), 737–744.

P. Frankl

CNRS 54 Bd. Raspail, 75006 Paris, France.

Z. Füredi

Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Reáltanoda u. 13–15, H-1053, Hungary.