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HYPERGRAPHS IN WHICH ALL DISJOINT PAIRS HAVE DISTINCT UNIONS

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Let $\mathcal{F}$ be a set-system of r-element subsets on an n-element set, $r \geqq 3$. It is proved that if $|\mathcal{F}| > 3.5 \binom{n}{r-1}$ then $\mathcal{F}$ contains four distinct members A, B, C, D such that $A \cup B = C \cup D$ and $A \cap B = C \cap D = \emptyset$.

1. Introduction and the Theorem

Let n and r $(n \geqq r)$ be positive integers and let X be an n-element set. We denote by $\binom{X}{r}$ the family of all r-element subsets of X. If $\mathcal{G} \subset \binom{X}{2}$ then we call $\mathcal{G}$ a *graph* on the vertex-set X. Furthermore, if $\mathcal{G} \subset \binom{A \cup B}{2} - \binom{A}{2} - \binom{B}{2}$, then we say $\mathcal{G}$ is a bipartite graph with parts A and B. The *degree* of the point $p \in X$ in the set-system $\mathcal{F}$ is denoted by $d_{\mathcal{F}}(p)$ or briefly $d(p) = |\{F\colon p \in F \in \mathcal{F}\}|$. As usual, $\Gamma_{\mathcal{G}}(p)$ denotes the *neighbourhood* of the point p in the graph $\mathcal{G}$, i.e. $\Gamma_{\mathcal{G}}(p) = \{q\colon \{p, q\} \in \mathcal{G}\}$. $|\Gamma_{\mathcal{G}}(p)| = d_{\mathcal{G}}(p)$. We call the family $\mathcal{F}$ *disjoint-union-free* if for every $A, B, C, D \in \mathcal{F}$, $A \cap B = \emptyset$, $C \cap D = \emptyset$ and $A \cup B = C \cup D$ imply $\{A, B\} = \{C, D\}$. That is, all disjoint pairs have distinct unions.

Erdős [4] asked to determine the maximum cardinality of $\mathcal{F} \subset \binom{X}{r}$, $\mathcal{F}$ is disjoint-union-free. In the case $r=2$ the question becomes: what is the maximum number of edges in a graph which contains no C_4 (C_4 is the cycle of length 4) as a subgraph (not necessarily induced subgraph). This problem goes back to 1938 (see Erdős [3]).

Definition 1.1. *Let $f_r(n)$ be the maximum number of edges in a disjoint-union-free family $\mathcal{F}$, $\mathcal{F} \subset \binom{X}{r}$, $|X| = n$.*

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Erdős, Rényi and T. Sós [6] and Brown [2] proved (see also Blanchard [1]) $f_2(n)=\left(\frac{1}{2}+o(1)\right)n^{3/2}$. Recently, the author [10] determined the exact value of $f_2(n)$ for an infinitely many values, namely, for $q=2^\alpha$. We have $f_2(q^2+q+1)=\frac{1}{2}q(q+1)^2$.

As Erdős and Frankl pointed out in 1975, one can prove for all $r\geqq 2$ that $f_r(n)<O(n^{r-0.5})$ (unpublished). In [4] Erdős mentions that he and Bollobás proved that $c_1n^2<f_3(n)<c_2n^2$ for some positive $c_i>0$. However, they have not published this result since that time (1977) and failed to reconstruct the proof.

Theorem 1.2. *If* $r\geqq 3$ *then* $\binom{n-1}{r-1}+\left\lfloor\frac{n-1}{r}\right\rfloor\leqq f_r(n)<3.5\binom{n}{r-1}$.

We obtain the lower bound considering the set-system of all r-element subsets of X which contain a fixed element plus an arbitrary system of $\left\lfloor\frac{n-1}{r}\right\rfloor$ pairwise disjoint r-element subsets not containing that element. In the case $r=3$ one can construct a little bit larger example:

Example 1.3. Let $\mathcal{S}$ be an $S_1(n, 5, 2)$ Steiner-system, i.e. $\mathcal{S}\subset\binom{X}{5}$ and for all $x, y\in X$ there exists exactly one member $S\in\mathcal{S}$ containing them. This Steiner-system exists iff $n\equiv 1$ or 5 (mod 20) (see Hanani [11]). Replace each $S\in\mathcal{S}$ by $\binom{S}{3}$. So we get a disjoint-union-free set-system $\mathcal{F}\subset\binom{X}{3}$ such that $|\mathcal{F}|=\binom{n}{2}$.

Conjecture 1.4. *If* $r\geqq 3$ *and* $n>n_0(r)$ *then* $f_r(n)\leqq\binom{n}{r-1}$. *Moreover, for* $r\geqq 4$, $f_r(n)=\binom{n-1}{r-1}+\left\lfloor\frac{n-1}{r}\right\rfloor$ *holds.*

The proof of Theorem 1.2 is based on a rather technical lemma (Lemma 3.3), which we state in §3 and prove in §4, along with other lemmas, needed for its proof. For those who don't have the patience to go through its proof, we recommend just to read its statement and then go directly to §5 and see how it implies Theorem 1.2.

2. Other Union-free Properties

In [7] and [8] Frankl and the author have introduced the following notions and functions: Call the family $\mathcal{F}$ *union-free* if, for every $A, B, C, D\in\mathcal{F}$, $A\cup B=C\cup D$ implies $\{A, B\}=\{C, D\}$. Let $u_r(n)$ denote the maximum cardinality of a union-free family $\mathcal{F}\subset\binom{X}{r}$. A family $\mathcal{F}$ is *weakly union-free* if for every *distinct* A, B, C, D we have $A\cup B\neq C\cup D$. Let $w_r(n)$ denote $\max\left\{|\mathcal{F}|: \mathcal{F}\subset\binom{X}{r}, \text{ weakly union-free}\right\}$. A family $\mathcal{F}$ is called *intersection-union-free* if for every four distinct $A, B, C, D\in\mathcal{F}$

either $A\cup B\neq C\cup D$ or $A\cap B\neq C\cap D$ holds. Denote by $i_r(n)$ its maximal possible cardinality, for $\mathcal{F}\subset\binom{X}{r}$.

Clearly, $u_r(n)\leqq w_r(n)\leqq i_r(n)\leqq f_r(n)$. It is easy to see that $i_r(n)\leqq c_r n^{\alpha_r}$ for some $c_r>0$, where $\alpha_r=[4r/3]/2\sim(2/3)r+O(1)$. In [8] it is proved

Theorem 2.1. *There exists a positive constant c_r' such that $c_r' n^{\alpha_r}<u_r(n)$.*

3. The Main Idea of the Proof: Lemmas

3.1. It is a usual technique to investigate r-partite hypergraphs only. In this way we can lose only a constant factor. (See Lemma 5.1.) Then we reduce the general problem to the three-uniform case (see Lemma 5.2). Finally, this is the most difficult step, in the case of $r=3$, when we want to prove $f_3(n)=O(n^2)$, we try to get a one-to-one correspondence between the disjoint-union-free set-system $\mathcal{F}$ and the pairs of the underlying-set X. In general, we will not succeed, but we will correspond a pair of X to a few members of $\mathcal{F}$ only.

3.2. Lemmas. From now on we denote by $\mathcal{G}$, $\mathcal{G}_i$, $\mathcal{G}'$, etc., only bipartite graphs with parts A and B. Let us denote by $\mathcal{D}(\mathcal{G})$ the set of diagonals of quadrilaterals in $\mathcal{G}$, i.e. $\mathcal{D}(\mathcal{G})=:\left\{\{u,v\}\in\binom{A\cup B}{2}:\ \exists x\neq y\in A\cup B \text{ such that } \{u,x\},\ \{x,v\},\ \{v,y\},\ \{y,u\}\in\mathcal{G}\right\}$. Note that we have $\mathcal{D}(\mathcal{G})\cap\mathcal{G}=\emptyset$.

Lemma 3.1. *Let $\mathcal{G}$ be a bipartite graph and let $\mathcal{D}(\mathcal{G})$ be the set of diagonals of its quadrilaterals. Then we can get a quadrilateral-free subgraph $\mathcal{A}(\mathcal{G})\subset\mathcal{G}$ deleting at most $|\mathcal{D}(\mathcal{G})|$ edges from $\mathcal{G}$.*

For the proof of this lemma we will use the following statement:

Lemma 3.2. *Let $\mathcal{P}=\{P_1, P_2, \dots, P_m\}$ be a collection of finite sets and suppose that $d_{\mathcal{P}}(p)\geqq\left|\bigcup_{p\in P\in\mathcal{P}} P\right|$ holds for every element p. Then $|\mathcal{P}|\geqq|\bigcup\mathcal{P}|$.*

Our main tool in establishing the validity of Theorem 1.2 is the next lemma. The bounds of this lemma are not exact and improving this lemma could eventually lead to decreasing the coefficient of $\binom{n}{r-1}$, the upper bound of the theorem.

Lemma 3.3. *Let $\mathcal{G}_i$ be a bipartite graph with parts A and B, $i=1,2,\dots,t$. Suppose that $\mathcal{G}_i\cap\mathcal{G}_j$ does not contain two disjoint edges and $\mathcal{G}_i\cup\mathcal{G}_j$ does not contain a C_4 with two-two neighbouring edges from $\mathcal{G}_i$ and $\mathcal{G}_j$, $1\leqq i\leqq j\leqq t$. (I.e., $\mathcal{G}_i\cap\mathcal{G}_j$ is a star and there is no quadrilateral (a,b,c,d) such that $\{a,b\}$, $\{b,c\}\in\mathcal{G}_i$, $\{c,d\}$, $\{d,a\}\in\mathcal{G}_j$). Then $\sum_{1\leqq i\leqq t}|\mathcal{G}_i|\leqq 2\binom{|A|}{2}+2\binom{|B|}{2}+(|A|+|B|)t/2+\binom{t}{2}$.*

4. Proof of the Lemmas

4.1. **Proof of Lemma 3.2.** We use induction on $|\bigcup\mathscr{P}|$. Now, it is permitted that a member belongs to $\mathscr{P}$ with multiplicity. Let $p\in\bigcup\mathscr{P}$ be an arbitrary point and let $\bigcup_{p\in P\in\mathscr{P}} P=N$, $\mathscr{N}=\{P-N\colon P\in\mathscr{P},\ P-N\neq\emptyset\}$. We can apply the induction hypothesis for $\mathscr{N}$, hence we get $|\mathscr{P}|\geqq d_{\mathscr{P}}(p)+|\mathscr{N}|\geqq|N|+|\bigcup\mathscr{N}|=|\bigcup\mathscr{P}|$. ▮

4.2. **Proof of Lemma 3.1.** We use induction on the number of edges of $\mathscr{G}$. If there exists an edge $E\in\mathscr{G}$ which is not contained in a quadrilateral of $\mathscr{G}$ then we can use the induction hypothesis for $\mathscr{G}-\{E\}$ and set $\mathscr{A}(\mathscr{G})=\mathscr{A}(\mathscr{G}-\{E\})\cup\{E\}$.

From now on we suppose that the edges of quadrilaterals in $\mathscr{G}$ cover the edges of $\mathscr{G}$. We shall define a set of edges, $\mathscr{B}(\mathscr{G})$, $\mathscr{B}(\mathscr{G})\subset\mathscr{G}$, $|\mathscr{B}(\mathscr{G})|\leqq|\mathscr{D}(\mathscr{G})|$, $\mathscr{G}-\mathscr{B}(\mathscr{G})=\mathscr{A}(\mathscr{G})$ is quadrilateral-free. Let p be an arbitrary vertex of $\mathscr{G}$ (with $d_{\mathscr{G}}(p)>0$). Define $P=\{p_1,p_2,\dots,p_m\}=\Gamma_{\mathscr{D}}(p)$, $Q=\Gamma_{\mathscr{G}}(p)$ and $P_i=Q\cap\Gamma_{\mathscr{G}}(p_i)$. (See Fig. 1.) Each edge is contained in a C_4 so we have $\bigcup P_i=Q$.

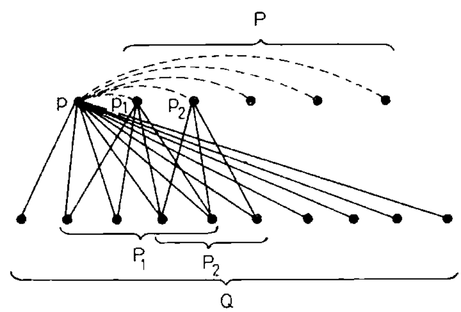

Fig. 1

If $|P|\geqq|Q|$ then delete from $\mathscr{G}$ the edges adjacent to p. Using the induction hypothesis for the obtained smaller $\mathscr{G}'$ we get a $\mathscr{B}(\mathscr{G}')$. Setting $\mathscr{B}(\mathscr{G})=\mathscr{B}(\mathscr{G}')\cup\{\{p,q\}\colon q\in\Gamma_{\mathscr{G}}(p)\}$ we obtain:

$$|\mathscr{B}(\mathscr{G})| = |\mathscr{B}(\mathscr{G}')|+|Q| \leqq |\mathscr{D}(\mathscr{G}')|+|P| \leqq |\mathscr{D}(\mathscr{G})|,$$

and $\mathscr{G}-\mathscr{B}(\mathscr{G})=\mathscr{G}'-\mathscr{B}(\mathscr{G}')$ is quadrilateral free.

If $|P|<|Q|$ then by Lemma 3.2 there exists a $q\in Q$ such that $d_{\mathscr{P}}(q)<|\bigcup_{q\in P_i} P_i|=|Q\cap\Gamma_{\mathscr{D}}(q)|+1$, i.e., $|P\cap\Gamma_{\mathscr{G}}(q)|\leqq|Q\cap\Gamma_{\mathscr{D}}(q)|$. (Note that $d_{\mathscr{P}}(q)>0$ for all $q\in Q$.) If the pair $\{x,y\}$ is a diagonal of a quadrilateral of $\mathscr{G}$ and $\{x,y\}\in\binom{Q}{2}\cap\mathscr{D}(\mathscr{G})$ then the remaining two vertices of this quadrilateral belong to $P\cup\{p\}$. According to this, delete the edges joining q to $P\cap\Gamma_{\mathscr{G}}(q)$. We get the graph $\mathscr{G}''$. In the graph $\mathscr{D}(\mathscr{G}'')$ there is no edge from q to Q, hence we get by the induction hypothesis that $|\mathscr{B}(\mathscr{G})|=|\mathscr{B}(\mathscr{G}'')|+|P\cap\Gamma_{\mathscr{G}}(q)|\leqq|\mathscr{D}(\mathscr{G}'')|+|Q\cap\Gamma_{\mathscr{D}}(q)|\leqq|\mathscr{D}(\mathscr{G})|$. ▮

4.3. **Proof of Lemma 3.3.** (First step.) Let us denote by $\mathscr{E}(\mathscr{G})$ the set of pairs $\{u,v\}$ which possess a common neighbour w. I.e., $\mathscr{E}(\mathscr{G})=\{\{u,v\}\colon \exists w$ such that $\{u,w\},\{w,v\}\in\mathscr{G}\}$.

It is easy to see that, for $i \neq j$, $\mathscr{D}(\mathscr{G}_i) \cap \mathscr{D}(\mathscr{G}_j) = \emptyset$. Now, apply Lemma 3.1 to $\mathscr{G}_i$. We obtain $\mathscr{A}(\mathscr{G}_i) = \mathscr{G}'_i$ which is quadrilateral-free. Of course, $\sum |\mathscr{D}(\mathscr{G}_i)| \leqq \binom{|A|}{2} + \binom{|B|}{2}$ and $|\bigcup \mathscr{E}(\mathscr{G}'_i)| \leqq \binom{|A|}{2} + \binom{|B|}{2}$. Consequently, we are done if we prove the following statement.

Lemma 4.4. *Let $\mathscr{G}'_i$ be a bipartite, quadrilateral-free graph with parts A and B, $i = 1, 2, \dots, t$. Suppose that $\mathscr{G}'_i \cap \mathscr{G}'_j$ does not contain two disjoint edges and $\mathscr{G}'_i \cup \mathscr{G}'_j$ does not contain a C_4 with two-two neighbouring edges from $\mathscr{G}'_i$ and $\mathscr{G}'_j$ for all $1 \leqq i < j \leqq t$. Then*

$$\sum_{1 \leqq i \leqq t} |\mathscr{G}'_i| \leqq (|A| + |B|)t/2 + \binom{t}{2} + |\bigcup \mathscr{E}(\mathscr{G}'_i)|.$$

Proof. We apply induction on $\sum |\mathscr{G}'_i|$. We need several definitions and a lemma.

Define a graph $\mathscr{H}(p)$ over the points $\{1, 2, \dots, t\}$ for each point $p \in A \cup B$. $\mathscr{H}(p) = \{\{i, j\}: \exists u, v \in A \cup B$ such that $\{u, p\}, \{p, v\} \in \mathscr{G}'_i \cap \mathscr{G}'_j\}$, i.e. the pair $\{i, j\} \in \mathscr{H}(p)$ iff $\mathscr{G}'_i$ and $\mathscr{G}'_j$ intersect each other in a star with center p. Hence $\mathscr{H}(p) \cap \mathscr{H}(q) = \emptyset$ for $p \neq q$, and $\bigcup \mathscr{H}(p) \subset \binom{T}{2}$. $(T = \{1, 2, \dots, t\}.)$

Define a set $M(E) \subset T$ for each pair $E \in \binom{A}{2} \cup \binom{B}{2}$. Let $M(E) = \{i: E \in \mathscr{E}(\mathscr{G}'_i)\}$.

Finally, define a subset of T for each edge F from $\bigcup \mathscr{G}'_i$. Let $N(F) = \{i: F \in \mathscr{G}'_i\}$. Of course, in some cases $\mathscr{H}(p)$ or $M(E)$ may be empty.

Lemma 4.5. *If there exists an edge $\{p, q\} = F \in \bigcup \mathscr{G}'_i$ such that $i \in N(F)$ and $2 + d_{\mathscr{H}(p)}(i) \leqq d_{\mathscr{G}'_i}(p)$ for some $i \in T$ then we can choose subgraphs $\mathscr{G}''_j \subset \mathscr{G}'_j$ such that*

$$\sum |\mathscr{G}'_j| - \sum |\mathscr{G}''_j| \leqq |\bigcup \mathscr{E}(\mathscr{G}'_j)| - |\bigcup \mathscr{E}(\mathscr{G}''_j)|.$$

Proof of Lemma 4.5. Set $\mathscr{G}''_i = \mathscr{G}'_i - \{F\}$ and $\mathscr{G}''_j = \mathscr{G}'_j - \{F\}$ if $j \in N(F) \cap \Gamma_{\mathscr{H}(p)}(i)$ and $\mathscr{G}''_j = \mathscr{G}'_j$ otherwise. We deleted the edge F at most $(d_{\mathscr{H}(p)}(i) + 1)$ times, hence the left-hand side is at most $(d_{\mathscr{H}(p)}(i) + 1)$. However, we deleted every edge $\{q, r\}$, $r \in \Gamma_{\mathscr{G}'_i}(p)$ from $\bigcup \mathscr{E}(\mathscr{G}')$. Hence the right-hand side is at least $d_{\mathscr{G}'_i}(p) - 1$. ∎

4.6. Now we can conclude the proof of Lemma 4.4 and thus that of Lemma 3.3.

We apply induction on $\sum |\mathscr{G}'_i|$. If there are F and i fulfilling the assumptions of Lemma 4.4 then we are ready. Thus, we can suppose that for each $p \in A \cup B$ and each $i \in T$ we have $1 + d_{\mathscr{H}(p)}(i) \geqq d_{\mathscr{G}'_i}(p)$. Summing up these inequalities for all $p \in (A \cup B)$ we get

$$\sum_p d_{\mathscr{H}(p)}(i) + |A| + |B| \geqq \sum_p d_{\mathscr{G}'_i}(p) = 2|\mathscr{G}'_i|.$$

That is, $|\mathscr{G}'_i| \leqq (|A| + |B| + t - 1)/2$. Hence, in this case, $\sum |\mathscr{G}'_i| \leqq t(|A| + |B|)/2 + \binom{t}{2}$. ∎

5. Proof of the Theorem

5.1. Lemmas. We use the following theorem of Erdős and Kleitman [5]:

Lemma 5.1. *Let $\mathscr{F}$ be an r-graph over the n-element set X, $n=n_1+n_2+\ldots+n_r$ ($n_i\geqq 1$, integer). Then there exists a partition $\{X_1, X_2, \ldots, X_r\}$ of X with $|X_i|=n_i$ such that the subcollection $\mathscr{G}$ of $\mathscr{F}$ defined by $\mathscr{G}=\{F\in\mathscr{F}\colon |F\cap X_i|=1$ for all $1\leqq i\leqq r\}$ has at least $|\mathscr{F}|n_1n_2\ldots n_r/\binom{n}{r}$ sets.* ∎

We need the following lemma.

Lemma 5.2. *Let $\mathscr{G}$ be an r-partite r-graph with parts $X_1, \ldots, X_r$ where $|X_i|=n_i$, $n_1\leqq n_2\leqq\ldots\leqq n_r$ and $r\geqq 3$. For a subset $E\subset(\bigcup X_i)=X$ we denote by $\mathscr{G}[E]$ the set-system $\{G\in\mathscr{G}\colon E\subset G\}$. Then there exist pairwise disjoint subsets $E_1, E_2, \ldots, E_{n_1}$ such that $|E_i|=r-2$, $|E_i\cap X_j|=1$ for all $1\leqq i\leqq n_1$ and $1\leqq j\leqq r-2$ and*

$$(*)\qquad \sum_{1\leqq i\leqq n_1}|\mathscr{G}[E_i]|\geqq\frac{n_1}{n_1n_2\ldots n_{r-2}}|\mathscr{G}|.$$

Proof of Lemma 5.2. Let us consider all systems $E_1, \ldots, E_{n_1}$ satisfying the conditions of the Lemma except $(*)$. We can choose such an $E_1, \ldots, E_{n_1}$ in $n_1!\binom{n_1}{n_1}\times n_1!\binom{n_2}{n_1}\times\ldots\times n_1!\binom{n_{r-2}}{n_1}$ way. Let us note that each $F\in\mathscr{G}$ belongs to $\bigcup\mathscr{G}[E_i]$ exactly $n_1\times(n_1-1)!\binom{n_1-1}{n_1-1}\times(n_1-1)!\binom{n_2-1}{n_1-1}\times\ldots\times(n_1-1)!\binom{n_{r-2}-1}{n_1-1}$-times. Thus the mean value of $|\bigcup\mathscr{G}[E_i]|$ is $(n_1/n_1n_2\ldots n_{r-2})|\mathscr{G}|$. ∎

5.2. **Proof of Theorem 1.2.** Let $\mathscr{F}\subset\binom{X}{r}$ be a disjoint-union-free set-system, $|X|=n$. Let $n_i=\lfloor(n+i-1)/r\rfloor$ $1\leqq i\leqq r$. Use Lemma 5.1. We obtain $X_1, \ldots, X_r$ and the r-partite $\mathscr{G}$ with $|\mathscr{G}|\geqq n_1n_2\ldots n_r|\mathscr{F}|/\binom{n}{r}$. Now using Lemma 5.2 we get $\mathscr{G}[E_1], \ldots, \mathscr{G}[E_{n_1}]$ with

$$\sum|\mathscr{G}[E_i]|\geqq\frac{n_1}{n_1\ldots n_{r-2}}|\mathscr{G}|\geqq\frac{n_1}{n_1\ldots n_{r-2}}\times\frac{n_1\ldots n_r}{\binom{n}{r}}|\mathscr{F}|=$$

$$=\frac{n_1n_{r-1}n_r}{(n-r+1)/r}\frac{|\mathscr{F}|}{\binom{n}{r-1}}\geqq n_{r-1}n_r\frac{|\mathscr{F}|}{\binom{n}{r-1}}.$$

Let us define the bipartite graph $\mathscr{G}_i=\{E-E_i\colon E\in\mathscr{G}[E_i]\}$. It is easy to check that the system $\mathscr{G}_1, \ldots, \mathscr{G}_{n_1}$ satisfies the assumptions of Lemma 3.3 (with $A=X_{r-1}$, $B=X_r$, $t=n_1$). So applying it we infer

$$|\mathscr{F}|\leqq\binom{n}{r-1}\frac{1}{n_{r-1}n_r}\sum|\mathscr{G}_i|\leqq$$

$$\leqq\binom{n}{r-1}\frac{1}{n_{r-1}n_r}\left[2\binom{n_r}{2}+2\binom{n_{r-1}}{2}+\frac{n_r+n_{r-1}}{2}n_1+\binom{n_1}{2}\right]<\frac{7}{2}\binom{n}{r-1}.\ ∎$$

6. Remarks

Proposition 6.1. *The limit* $\lim_{n\to\infty} f_3(n)/\binom{n}{2}$ *exists and it satisfies* $1 \leqq \lim_{n\to\infty} f_3(n)/\binom{n}{2} \leqq 3.5$.

Proof. Let $\mathscr{F}_k$ be a disjoint-union-free set-system on k points with $|\mathscr{F}_k| = f_3(k)$. Set $f_3(k)/\binom{k}{2} = c_k$. If $n > n_0(k)$ and $n \equiv 1 \pmod{k(k-1)}$ then there exists an $S_1(n, k, 2)$ Steiner-system $\mathscr{S}$. Replace each member of $\mathscr{S}$ by a copy of $\mathscr{F}_k$. We get a disjoint-union-free system on n points. Hence for $n > n_0(k, \varepsilon)$ we have $c_n \geqq c_k - \varepsilon$ for all $\varepsilon > 0$, proving the existence of the limit. The inequality follows from Theorem 1.2. ▮

I cannot prove the corresponding statement for $f_r(n)$.

6.2. Lemma 3.3 belongs to the *structure intersection* problems posed by V. T. Sós (cf. [12]).

6.3. The determination of $f_r(n)$ belongs to the so-called Turán-type problems, i.e. what is the maximum number of r-subsets of an n-set if it contains no sub-system isomorphic to one member of a set of r-graphs $\mathscr{H}$. (Generally this $\mathscr{H}$ is finite.) This maximum is usually denoted by $\mathrm{ex}_r(n, \mathscr{H})$. Let us define U as the class of set-systems having four distinct members A, B, C, D such that $A \cup B = C \cup D$ and $A \cap B = C \cap D = \phi$. In this terminology we proved in Theorem 1.2 that

$$\binom{n-1}{r-1} < \mathrm{ex}_r(n, U) < \frac{7}{2}\binom{n}{r-1}.$$

Let us denote by W those class of the set-systems consisting of 3 r-sets, A, B, C such that $A \cap B = \emptyset$, $C \subset A \cup B$ and $|C \cap A| = 1$. With this notation, Frankl and the author [9] have proved that $\mathrm{ex}_r(n, W) = \binom{n-1}{r-1}$ holds for $n > n_0(r)$. Thus, Conjecture 1.4, if true, shows that $\mathrm{ex}_r(n, U) - \mathrm{ex}_r(n, W) = O(n)$.

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