

COMMUNICATION

AN EXACT RESULT FOR 3-GRAPHS

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The aim of this paper is to prove Theorem 1 which gives a full description of families of 3-subsets in which any 4 points contain 0 or 2 members of the family.

1. Introduction

A hypergraph $H = (V, \mathcal{E})$ consists of a vertex set V and an edge set \mathcal{E} , which is a family of subsets of V satisfying $\bigcup \mathcal{E} = V$. It is called an r -graph if all edges have r elements. For $W \subset V$ we set $\mathcal{E}_W = \{E \in \mathcal{E} : E \subset W\}$, these are the edges *spanned* by W .

Given integers $n > k > r$, $0 < s \leq \binom{k}{r}$ let us denote by $m(n, r, k, s)$ the maximum number of edges in an r -graph on n vertices in which any k vertices span less than s edges. The determination of $m(n, r, k, s)$ is a hopelessly difficult problem in general even for $r = 2$. The well-known Turán's theorem is the case $s = \binom{k}{2}$.

For $r \geq 3$ and $s = \binom{k}{r}$ we come to Turán's problem: What is the maximum number of edges in an r -graph without a complete subgraph on k vertices. This is a challenging open problem for all $k > r$, Erdős offers a monotone increasing sum for its solution (at present 3000 dollars).

Turán (cf. [3, 7]) conjectured that $m(n, 3, 4, 3)$ is asymptotic to $n^3/24$.

Let us consider the following 3-graph on 6 points: $S(6) = \{(123), (124), (345), (346), (561), (562), (135), (146), (236), (245)\}$ (cf. Fig. 1). One can check that any 4 points span 2 edges in $S(6)$.

Example 1. Suppose $|V| = n$ and V is partitioned into $V = V_1 \cup \dots \cup V_6$. Define a 3-graph $H_S = (V, \mathcal{E})$, where

$$\mathcal{E} = \{(v_{i_1} v_{i_2} v_{i_3}) : 1 \leq i_1 < i_2 < i_3 \leq 6, v_{i_j} \in V_{i_j}, (i_1 i_2 i_3) \in S(6)\}.$$

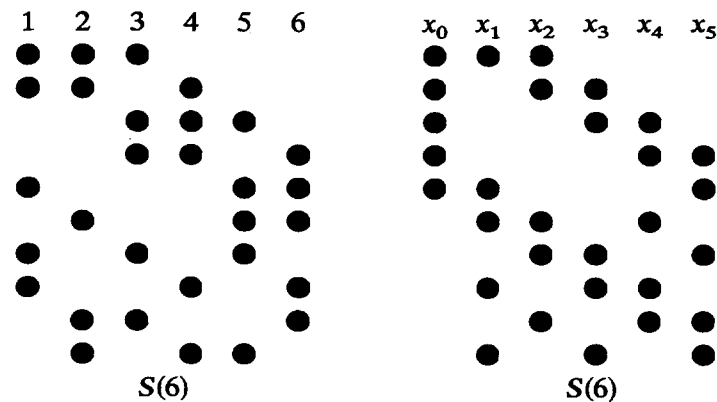


Fig. 1.

If we choose a partition satisfying $|V_i| \geq \lfloor n/6 \rfloor$, then H_S has more than $10 \lfloor n/6 \rfloor^3$ edges which is more than $n^3/24$, disproving Turán's conjecture – we will give a counter-example with more edges in Section 2.

Let us note that in H_S any four points span either zero or two edges.

We shall show that a 3-graph having this property cannot have more edges than H_S (Theorem 2). This result is deduced from Theorem 1, which gives a complete description of 3-graphs with the above properties: apart the 3-graphs given by Example 1 they are of the following form:

Example 2. Let V consist of n points on the unit circle, and let the edges be those 3-tuples that the origin is contained in the triangle formed by them (tacitly we assumed that the origin is contained in the convex hull of the points but it is on none of the lines joining two points).

The fact that in this 3-graph any 4 points span 0 or 2 edges can be verified easily.

Now we state our main results formally.

Theorem 1. Suppose $H = (V, \mathcal{E})$ is a 3-graph in which any 4 points span 0 or 2 edges. Then H is isomorphic to one of the 3-graphs in Examples 1 or 2.

Theorem 2. Suppose that $H = (V, \mathcal{E})$, $|V| = n \geq 5$ and any 4 points of V span 0 or 2 edges. Then $\max |\mathcal{E}|$ is attained exactly for H of the form H_S for some equipartition, i.e., $\lfloor n/6 \rfloor \leq |V_i| \leq \lceil n/6 \rceil$.

2. A remark on $m(n, 3, 4, 3)$

If one only wants to satisfy the condition: no four points span more than 2 edges, then one can add edges to H_S iteratively: first partition each V_i into 6 sets,

say W_{j1}, \dots, W_{j6} and add to H_S all 3-tuples $(w_{ji_1} w_{ji_2} w_{ji_3})$ with $(i_1 i_2 i_3) \in S(6)$. Then repeat this with the W_{ji} , etc.

Making the partitions always as equal as possible, finally one obtains $n^3(1+o(1))/21$ edges. This gives the first part of the following:

Theorem 3.

$$\frac{2+o(1)}{7} \binom{n}{3} \leq m(n, 3, 4, 3) \leq \frac{1}{3} \binom{n}{3} \frac{n}{n-2}.$$

The upper bound was proved by Caen [2].

Let us mention that the lower bound of the theorem was proved independently by Giraud [5] also.

We shall discuss other related extremal problems at the end of this paper.

3. The proof of Theorem 1

Suppose that $H = (V, \mathcal{E})$ is a 3-graph satisfying the assumptions, i.e., whenever $(xyz) \in \mathcal{E}$ and v is another element of V , then there is exactly one more edge spanned by $\{x, y, z, v\}$.

For $v \in V$ let us define the *neighbourhood* of v : $N(v) = \{(xy) : (xyv) \in \mathcal{E}\}$.

We distinguish two cases.

(a) For some $v \in V$, $N(v)$ contains an odd cycle: $(x_1 x_2), (x_2 x_3), \dots, (x_{2t+1} x_1)$.

Proposition 1. $N(v)$ contains a cycle of length 5.

Proof. Suppose x_1, \dots, x_{2t+1} form an odd cycle of minimal length. Since $t=1$ would yield ≥ 3 edges on 4 points, $t \geq 2$. Suppose for contradiction, $t \geq 3$.

By the minimal choice of t , $(v, x_i x_j) \notin \mathcal{E}$ holds unless $i=j+1$ or $j=i+1$. Looking at 4 vertices v, x_i, x_{i+1}, x_j with $j \neq i-1, i+1, i+2$, we conclude $(x_i x_{i+1} x_j) \in \mathcal{E}$. Consequently the 4 points x_1, x_2, x_4, x_5 span 4 edges, a contradiction. \square

Actually our argument gave for the five-cycle x_1, \dots, x_5 that $(x_i x_{i+1} x_{i+3}) \in \mathcal{E}$ holds for $i=1, \dots, 5$ (the subscripts are understood mod 5). That is the 3-graph spanned by v, x_1, \dots, x_5 is isomorphic to $S(6)$. Set $x_0 = v$.

First we show that w, x_0, x_1, \dots, x_5 span a 3-graph of the form H_S , given by Example 1.

Looking at the 4-tuple w, x_0, x_1, x_2 and using that the automorphism group of $S(6)$ (which is A_5) is doubly transitive, we may assume $(wx_1 x_2) \in \mathcal{E}$.

Similarly, with w, x_1, x_2, x_3 we infer $(wx_2 x_3) \in \mathcal{E}$.

In particular $(wx_0 x_i) \notin \mathcal{E}$ holds for $i=1, 2, 3$.

We claim that the same holds for $i=4, 5$. Consider $wx_1 x_2 x_4$, we infer

$(wx_2x_4) \notin \mathcal{E}$. Now look at $wx_0x_2x_4$. They span no edge, especially $(wx_0x_4) \notin \mathcal{E}$. Similarly $(wx_0x_5) \notin \mathcal{E}$ holds. Looking at the sets $wx_0x_ix_{i+1}$ we conclude $(wx_ix_{i+1}) \in \mathcal{E}$, i.e., w, x_1, x_2, \dots, x_5 span $S(6)$.

These show that if we have $S(6)$ plus one point, then they span a 3-graph of the form H_S . Now the general case follows easily, e.g. by induction on n .

(b) $N(v)$ is bipartite for all $v \in V$.

Let x be a fixed vertex and (A, B) a fixed bipartition of $N(x)$.

Proposition 2. *If $(ab) \in N(x)$, $(ab') \notin N(x)$, then $(abb') \in \mathcal{E}$.*

Proof. It is sufficient to consider the 4-tuple $(xabb')$. \square

Denote by $\deg(c)$ the degree of the vertex c in $N(x)$. Order the elements of A and B according to their degrees: $A = \{a_1, \dots, a_k\}$; $B = \{b_1, \dots, b_l\}$; $\deg(a_1) \geq \dots \geq \deg(a_k)$, $\deg(b_1) \geq \dots \geq \deg(b_l)$.

Proposition 3. *Suppose $(a_ib_j) \in N(x)$, $i' \leq i, j' \leq j$. Then $(a_ib_{j'}) \in N(x)$ holds.*

Proof. Clearly, it is sufficient to prove the statement in the case $i = i'$. Set $a = a_i$ and suppose for contradiction $(ab_{j'}) \notin N(x)$. Now $\deg(b_{j'}) \geq \deg(b_j)$ implies the existence of $a' \in A$ with $(a'b_j) \in N(x)$, $(a', b_j) \notin N(x)$.

Applying Proposition 2 four times we infer that a, a', b'_j, b_j span 4 edges, a contradiction. \square

Proposition 4. *Neither A nor B contains an edge of H .*

Proof. Suppose by symmetry $(a_1a_2a_3)$ is an edge of H contained in A . Considering the four points x, a_1, a_2, a_3 we arrive at a contradiction with (A, B) being a bipartition of $N(x)$. \square

Proposition 5. *Suppose $a \in A, b, b' \in B$. Then $(abb') \in \mathcal{E}$ if and only if exactly one out of (ab) and (ab') belongs to $N(x)$.*

Proof. Since $N(x)$ is bipartite, $(xabb') \notin \mathcal{E}$. Now the statement follows by considering the four point x, a, b, b' . \square

Let us call $v, w \in V$ *equivalent* if $N(v) = N(w)$. It is easy to see that adding or removing equivalent points does not change the four points property.

Proposition 6. *Two points v, w are equivalent if and only if there is no edge in H containing both of them.*

Proof. If for some z , $(vwz) \in \mathcal{E}$, then $(vz) \notin N(v)$ but $(vz) \in N(w)$, that is v and w are not equivalent. Suppose now that $\{v, w\}$ is not contained in any edge. Choose $y, z \in V - \{v, w\}$. Considering v, w, y, z it follows that either both (vyz) and (wyz) are edges or none of them, proving the proposition. \square

Clearly it is sufficient to show that H is isomorphic to Example 2 in the case: there are no two equivalent vertices. In view of Proposition 6 we assume that any pair of vertices is contained in at least one edge. Consequently $N(x)$ has no isolated vertices.

If $a, a' \in A$ have the same degree, then by Proposition 3, they have the same neighbourhood in $N(x)$. Thus, by Propositions 4 and 5, $\{a, a'\}$ is contained in no edge, a contradiction. We infer

$$l \geq \deg(a_1) > \dots > \deg(a_k) \geq 1; \quad k \geq \deg(b_1) > \dots > \deg(b_l) \geq 1.$$

This is only possible if $k = l$ and $\deg(a_i) = \deg(b_i) = k - i + 1$. More exactly – using Proposition 3 – $(a_i, b_j) \in N(x)$ if and only if $i + j \leq k + 1$.

Now imagine that we have placed these $2k + 1$ points in the vertices of a regular $(2k + 1)$ -gon on the unit circle in the order $x, a_k, \dots, a_1, b_1, \dots, b_k$. Then $(a_i, b_j) \in N(x)$ if and only if the triangle $xa_i b_j$ contains the origin. Now Proposition 5 yields that 3 points form an edge in H if and only if the corresponding triangle contains the origin, concluding the proof of Theorem 1. \square

Remark 1. Our proof showed that in Example 2 one can always move the points to the vertices of some regular $(2k + 1)$ -gon without altering the 3-graph. In particular, if any two vertices are covered by an edge, then n is odd.

To prove Theorem 2 just note that if H can be obtained by putting d_1, \dots, d_{2k+1} points into the vertices of a regular $(2k + 1)$ -gon, then the number of edges is maximized if the d_i 's are as equal as possible. Thus it is upperbounded by

$$(n/(2k + 1))^3 (2k + 1) \binom{k + 1}{2} / 3 \quad (< n^3/24),$$

which is always less than or equal to the maximal size of a 3-graph coming from Example 1, with equality holding only for $n \leq 5$ – then the two examples coincide.

4. Concluding remarks and open problems

The value of our Theorem 1 is given partially by the fact that there are very few exact results concerning 3-graphs.

Following a conjecture of Katona [6], Bollobás [1] proved that if in a 3-graph on n vertices no edge is containing the symmetric difference of two other edges,

then it has at most $\lfloor n/3 \rfloor \lfloor (n+1)/3 \rfloor \lfloor (n+2)/3 \rfloor$ edges. In [4] for $n > 800$ we gave a more exact form of this result by showing that if the 3-graph has at least as many as above edges then either it contains 3 edges of the form (123), (124), (345) or it is the complete 3-partite graph, that is $V = V_1 \cup V_2 \cup V_3$, the V_i 's are disjoint and the edges are the triples meeting all the V_i (Bollobás excluded the configuration (123), (124), (134) also, however, his result holds for all n).

Let us mention the following:

Conjecture 1 (Erdős and Sós [3]). *Suppose $H = (V, \mathcal{E})$ is a 3-graph in which $N(x)$ is bipartite for all $x \in V$. Then $|\mathcal{E}| < n^3/24$.*

Example 2 shows that one can have as much as $n^3(1+o(1))/24$ edges. Another, more general example is provided by taking a random tournament on n points and the 3-cycles of it as edges.

Problem 1. Suppose $H = (V, \mathcal{E})$ is an r -graph on n points in which any $r+1$ points span zero or two edges. Determine $\max |\mathcal{E}|$.

An interesting example of such r -graphs is given by

Example 3. Put n points on the surface of the unit sphere in $r-1$ dimension. Let r points form an edge if the corresponding simplex contains the origin.

It is easy to see, that choosing the points at random gives $\binom{n}{r}(1+o(1))/2^{r-1}$ edges and one cannot have more edges in this example.

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