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Geometrical Solution of an Intersection Problem for Two Hypergraphs

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Let $A_1, A_2, \ldots, A_m$ be at most a and let $B_1, \ldots, B_m$ be at most b-element sets and let t be a non-negative integer with the following property $|A_i \cap B_i| \leq t$ and $|A_i \cap B_j| > t$ for $i \neq j$. Then $m \leq \binom{a+b-2t}{a-t}$. The proof uses Lovász's geometrical method and leads to several open problems.

1. Introduction

The following theorem plays an important role in the theory of τ-critical hypergraphs (see Berge [1], Lovász [14]):

(1) *Let $A_1, \ldots, A_m$ be a-element and let $B_1, \ldots, B_m$ be b-element sets with the following property. $A_i \cap B_j = \emptyset$ iff $i = j$. Then $m \leq \binom{a+b}{a}$.*

The case $a = 2$ was proved by Erdős, Hajnal and Moon [6], and the general case by Bollobás [3]. Later other proofs were given by Jaeger and Payan [10], Katona [11] and Lovász [12, 13]. However, only Bollobás's original proof yields that in (1) equality holds iff the sets A_i and B_j are all a and b-element subsets of a given $(a+b)$-set.

Lovász [12, 13] proved the following two geometrical generalizations of (1).

(2) *Let $A_1, \ldots, A_m$ be a-dimensional and let $B_1, \ldots, B_m$ be b-dimensional subspaces of a linear space with the following property.* $\dim(A_i \cap B_j) = 0$ *iff $i = j$. Then $m \leq \binom{a+b}{a}$.*

(3) *Let $A_1, \ldots, A_m$ be a-dimensional subspaces of a linear space and let $B_1, \ldots, B_m$ be b-element point-sets with the following property $A_i \cap B_j = \emptyset$ iff $i = j$. Then $m \leq \binom{a+b}{a}$.*

2. Results

Most of the above-mentioned authors conjectured the following generalization.

THEOREM 1. *Let $A_1, \ldots, A_m$ be a-element and let $B_1, \ldots, B_m$ be b-element sets and let t be a nonnegative integer, $a, b \geq t$. Suppose further that $|A_i \cap B_j| \leq t$ iff $i = j$. Then $m \leq \binom{a+b-2t}{a-t}$.*

Let X be an $(a+b-2t)$-element and let T be a t-element set and $X \cap T = \emptyset$. Define, $\mathscr{A} = \{A: |A| = a,\ T \subset A \subset X \cup T\}$, $\mathscr{B} = \{B: |B| = b,\ T \subset B \subset X \cup T\}$. Pairing the members of $\mathscr{A}$ and $\mathscr{B}$ in the obvious way shows that the upper bound in Theorem 1 is exact. But I cannot prove the uniqueness of the extremal families.

THEOREM 2. *Let $A_1, \ldots, A_m$ be a-dimensional and let $B_1, \ldots, B_m$ be b-dimensional subspaces of the real Euclidean space, and let t be a non-negative integer, $a, b \geq t$. Suppose further that* $\dim(A_i \cap B_j) \leq t$ *iff $i = j$. Then $m \leq \binom{a+b-2t}{a-t}$.*

The investigation of (3) leads to new problems. The statement (3) could not be generalized in the same way as (1) and (2). Define $m_t(a, b)$ as the greatest number m such

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that there exist subspaces $A_1, \dots, A_m$ of *rank a* (i.e. dimension $a-1$) of the real *projective* space and pointsets $B_1, \dots, B_m$ of b elements with the following property. $|A_i \cap B_j| \leq t$ *iff* $i=j$. Clearly,

$$\binom{a+b-2t}{a-t} \leq m_t(a,b) \leq \binom{a+b-t}{b-t}. \tag{4}$$

The upper bound is obtained from (3) by replacing each B_i by the $(b-t)$-set $B_i - A_i$. There is no equality in (4), e.g.

PROPOSITION 3. *For* $a=2$, $t=1$, $b \geq 3$ *we have*

$$1 + \lfloor b(b+3)/6 \rfloor \leq m_1(2,b) \leq \binom{b}{2} + 1.$$

Here

$$\binom{a+b-2t}{b-t} = b < 1 + \lfloor b(b+3)/6 \rfloor \qquad \text{and} \qquad \binom{b}{2} + 1 < \binom{b+1}{2} = \binom{a+b-t}{b-t}.$$

The simplest counterexample for the evident (but wrong) conjecture $m_t(a,b) = \binom{a+b-2t}{b-t}$ is the following. Set $a=2$, $t=1$, $b=3$ and let A_1, A_2, A_3, A_4 be four lines in general position on the projective plane. Let us denote by A_{ij} the intersection point of A_i and A_j, and let $B_1 = (A_{23}, A_{34}, A_{42})$, $B_2 = (A_{13}, A_{34}, A_{41})$ and so on.

3. PROBLEMS AND REMARKS

3.1. Each statement stays true if we replace the assumptions $|A_i| = a$, $|B_j| = b$, $\dim|A_i| = a \dots$ with $|A_i| \leq a$, $\dim|A_i| \leq a$ and so on.

3.2. Bollobás [4, 5] and Pin [15] conjectured and Frankl [7] proved that the assumptions of (1)–(3)

$$A_i \cap B_j = \varnothing \qquad \text{iff} \qquad i=j$$

can be replaced with the following weaker assumption.

$$A_i \cap B_i = \varnothing \qquad \text{and} \qquad A_i \cap B_j \neq \varnothing \qquad \text{for } 1 \leq i < j \leq m.$$

These stronger theorems have several applications in graph theory (Bollobás [4, 5]) and in extremal hypergraph theory (Füredi and Tuza [9]).

Theorems 1 and 2 are valid if we suppose our assumptions only for $1 \leq i \leq j \leq m$.

3.3. Theorems (1)–(3) have Helly-type reformulations (see Lovász [12, 13]). E.g.

(2)′ *Let a collection* $\mathscr{A}$ *of a-dimensional subspaces of a linear space have the property that for every* $\binom{a+b}{a}$ *of them there exists a b-dimensional subspace meeting each of them in a* nonzero *subspace. Then there exists a b-dimensional subspace meeting each member of* $\mathscr{A}$ *in a nonzero subspace.*

We can reformulate (1), (3) and Theorem 1 and 2 in the same way.

3.4. The theorems (2), (3), (2)′ hold for flats of matroids if this matroid can be coordinated over a commutative field (Lovász [12, 13]). (Rank a stands instead of dimension a.) Similarly, Theorem 2 holds for subspaces of a linear space over a 'great enough' commutative field (See the next section).

3.5. Tarján [16] generalized (1) proving that

$$\sum 1 \Big/ \binom{|A_i|+|B_i|}{|A_i|} \leq 1.$$

In the case of Theorem 1 a similar inequality seems to be true,

$$\sum 1 \Big/ \binom{|A_i|+|B_i|-2t}{|A_i|-t} \leq 1,$$

but I cannot prove it.

3.6. We get a new problem in all three versions (1), (2) and (3) if we modify the assumptions in the following way: $|A_i \cap B_j| > t$ and $|A_i \cap B_i| \leq l$ $(l \leq t)$. These problems seem to be much more difficult, I have no established conjecture.

4. Proofs

4.1. Proof of Theorem 1. It follows from Theorem 2 in the same way as (2) implies (1). I.e. let $X = (\cup A_i) \cup (\cup B_j)$, $|X| = N$. Let us assign a vector $\mathbf{v}(x) \in \mathbb{R}^N$ to each $x \in X$ so that $\{\mathbf{v}(x) \colon x \in X\}$ forms a basis of $\mathbb{R}^N$. Finally let $\overline{A_i}$ (and $\overline{B_j}$) be the subspaces generated by $\{\mathbf{v}(a) \colon a \in A_i\}$. Now, Theorem 2 can be applied.

4.2. Proof of Theorem 2. Suppose that A_i, $B_j \subset \mathbb{R}^N$. We can suppose that N is finite. For a subspace C let us define $C^{\perp} =: \{y \in \mathbb{R}^N \colon (c, y) = 0 \text{ for each } c \in C\}$, i.e. the orthogonal subspace to C. Two subspaces D and C of dimensions d and c are in *general position* if $\dim(D \cap C) = \max\{0, d + c - N\}$.

There exists a subspace C of dimension $(N - a - b + t)$ which is in general position with respect to each A_i, B_i and $\{A_i \cup B_j\}$ where $\{A_i \cup B_j\}$ denotes the subspace generated by $A_i \cup B_j$. Projecting A_i and B_j to $C^{\perp}$, we get A'_i and B'_j. Now $\dim(A'_i) = \dim(A_i)\text{-}\dim(A_i \cap C) = a$ holds and similarly $\dim B'_i = b$, $\dim\{A'_i \cup B'_i\} = a + b - t$ and $\dim\{A'_i \cup B'_j\} \leq a + b - t - 1$ hold for $i \neq j$. I.e. $\dim(A'_i \cup B'_j) \leq t$ iff $i = j$.

Now find a subspace $C' \subset C^{\perp}$ of dimension $a + b - 2t$ which is in general position with respect to each $A'_i \cap B'_i$. $(\dim(A'_i \cap B'_i) = t)$. Let $A''_i = A'_i \cap C'$ and $B''_i = B'_i \cap C'$. Then $\dim A''_i = a - t$, $\dim B''_i = b - t$, $\dim(A''_i \cap B''_i) = 0$ and for $i \neq j$ we have $\dim(A''_i \cap B''_j) = \dim((A'_i \cap B'_j) \cap C') \geq 1$. Hence (2) can be applied to $\{A''_i, B''_i\}$.

4.3. Proof of Proposition 3. The fact $m_1(2, b) \leq \binom{b}{2} + 1$ is trivial, because the lines $A_2, A_3, \ldots, A_m$ contain at least two points from B_1 but A_i and A_j contain different pairs.

The lower bound is a construction. Burr, Günbaum and Sloane [2] gave $b + 3$ points $P_1, \ldots, P_{b+3}$ on the plane and $1 + \lfloor b(b+3)/6 \rfloor$ lines $L_1, \ldots, L_{1+\lfloor b(b+3)/6 \rfloor}$ such that each L_i contains exactly three $P_j - s$. A much simplier construction can be found in Füredi and Palásti [9]. Let $A_i = L_i$ and $B_i = \{P_\alpha \colon P_\alpha \notin L_i\}$.

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