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A NEW EXTREMAL PROPERTY OF STEINER TRIPLE-SYSTEMS

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Suppose $\mathcal{S}$ is a Steiner triple-system on the n-element set X, i.e., for every pair of distinct vertices of X there is exactly one triple in $\mathcal{S}$ containing them. Necessarily, $|\mathcal{S}| = n(n-1)/6$ holds. It is easy to see that, for $S, T, S', T' \in \mathcal{S}$, $S \cup T = S' \cup T'$ implies $\{S, T\} = \{S', T'\}$.

We show that, conversely, this condition, for any family $\mathcal{S}'$ of 3-subsets of X, implies $|\mathcal{S}'| \leq n(n-1)/6$. A similar type of result is obtained for a weaker union condition. The corresponding problems for graphs are still open.

1. Introduction

Let n, k $(n > k)$ be positive integers and let X be an n-element set. We denote by 2^X $(\binom{X}{k})$ the family of all subsets (all k-element subsets) of X, respectively. A subset of $\binom{X}{2}$ $(\binom{X}{3})$ is called a graph (a triple-system), respectively. We call the family $\mathcal{F}$ *union-free* if, for every $F, G, F', G' \in \mathcal{F}$, $F \cup G = F' \cup G'$ implies $\{F, G\} = \{F', G'\}$. We call $\mathcal{F}$ *weakly union-free* if the following weaker condition holds: for any four *distinct* members F_1, F_2, F_3, F_4 of $\mathcal{F}$ we have $F_1 \cup F_2 \neq F_3 \cup F_4$.

Erdös [5] asked to determine the maximum cardinality of $\mathcal{F} \subset \binom{X}{k}$, $\mathcal{F}$ is union-free. In the case $k = 2$ the question is what the maximum number of edges is in a graph which contains no C_3 or C_4 (C_r is the cycle of length r) as a subgraph (not necessarily induced subgraph). This problem goes back to 1938 [3]. In that paper Erdös also asked to determine the maximum number of edges in a graph without C_4, i.e., if it is weakly union-free.

Let us introduce two sets of functions.

Definition 1.1. $f_k(n)$ $(f(n))$ is the maximum number of edges in a union-free family $\mathcal{F}$, $\mathcal{F} \subset \binom{X}{k}$ $(\mathcal{F} \subset 2^X)$, respectively.

Definition 1.2. $F_k(n)$ $(F(n))$ is the maximum number of edges in a weakly union-free family $\mathcal{F}$, $\mathcal{F} \subset \binom{X}{k}$ $(\mathcal{F} \subset 2^X)$, respectively.

Reiman [12] (see also [1]) proved $(1/2\sqrt{2})\ n^{\frac{3}{2}} < f_2(n) < \frac{1}{2}\ n^{\frac{3}{2}}$ and it is conjectured that $f_2(n) = ((1+o(1))/2\sqrt{2}))n^{\frac{3}{2}}$ holds [4]. Let us mention for curiosity that Erdös and Simonovits [6] proved the exactitude of this bound if the graph contains no C_4 or C_5.

It is known (see [2, 7]) that $F_2(n) = (\frac{1}{2} + o(1))n^{\frac{3}{2}}$.

Quite recently, Füredi [8] determined the exact value of $F_2(n)$ for an infinity of values. More exactly, he proved that, for $q = 2^\alpha$, $F_2(q^2+q+1) = \frac{1}{2}q(q+1)^2$ holds.

Surprisingly, the determination of $f_3(n)$ and $F_3(n)$ is easier.

Definition 1.3. An $\mathcal{S} \subset \binom{X}{k}$ is called an $S_\lambda(n, k, t)$ if, for every $T \in \binom{X}{k}$, there exist exactly λ sets $S_1, \ldots, S_\lambda \in \mathcal{S}$ such that $T \subset S_i$ holds, $1 \leqslant i \leqslant \lambda$. An $S_1(n, 3, 2)$ is also called a Steiner triple-system.

It is easy to see that an $S_1(n, 3, 2)$ is always union-free (already, $A \cup B \supset C$ implies $A = C$ or $B = C$ for A, B, $C \in S$, $\mathcal{S}$ an $S_1(n, 3, 2)$). For infinitely many values of n we shall construct $S_2(n, 3, 2)$'s, which are weakly union-free.

Theorem 1.4. *We have*

$$f_3(n) = [n(n-1)/6]. \tag{1}$$

Remark 1.5. If $n \equiv 1$ or $3 \pmod 6$, $n \geqslant 7$, then Steiner triple-systems provide equality in Theorem 1.4. However, they are not characterized by the union-free property; many other examples exist, too.

Theorem 1.6. $F_3(n) \leqslant n(n-1)/3$, *and if equality holds for the weakly union-free family* $\mathcal{F}$, *then* $\mathcal{F}$ *is an* $S_2(n, 3, 2)$. *Moreover, if* $n \equiv 1 \pmod 6$, *then equality holds for* $n > n_0$.

Corollary 1.7. *If* $n > n_0$, *then we have*

$$n(n-1)/3 - \tfrac{10}{3}n < F_3(n) \leqslant [n(n-1)/3].$$

We review the known bounds on $f_k(n)$, $f(n)$, $F_k(n)$ and $F(n)$ in Section 4.

2. The proof of the upper bounds

Let $\mathcal{F}$ be any triple system, i.e., $\mathcal{F} \subset \binom{X}{3}$. Let us define, for every i, $0 \leqslant i \leqslant n-2$,

$$\mathcal{G}_i = \left\{\{x, y\} \in \binom{X}{2} : |\{z \in X, \{x, y, z\} \in \mathcal{F}\}| = i\right\}.$$

With words, $A \in \binom{X}{2}$ is in $\mathcal{G}_i$ if there are exactly i sets in $\mathcal{F}$ which contain A. Set

$$g_i = |\mathcal{G}_i|.$$

Of course, $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{n-2}$ form a partition of $\binom{X}{2}$. Thus we have

$$\sum_{0 \leqslant i \leqslant n-2} g_i = \binom{n}{2}. \tag{3}$$

Counting the number of pairs (A, F), $A \subset F \in \mathcal{F}$, $|A| = 2$, in two ways, we obtain

$$\sum_{0 \le i \le n-2} i g_i = 3\,|\mathcal{F}|\,. \tag{4}$$

For $A \in \binom{X}{2}$, define $T(A) = \{z \in X : (A \cup \{z\}) \in \mathcal{F}\}$.

Claim 1. *If $\mathcal{F}$ is (weakly) union-free, then for $A, A' \in \binom{X}{2}$,*

$$\binom{T(A)}{2} \cap \binom{T(A')}{2} = \emptyset$$

holds.

Proof. Suppose the contrary and let $\{z, z'\}$ belong to the intersection. Then $A \cup \{z\}$, $A \cup \{z'\}$, $A' \cup \{z\}$, $A' \cup \{z'\}$ are four different members of $\mathcal{F}$ and $(A \cup \{z\}) \cup (A' \cup \{z'\}) = (A \cup \{z'\}) \cup (A' \cup \{z\})$, a contradiction. □

Thus, for a weakly union-free family $\mathcal{F}$, we have

$$\sum_{2 \le i \le n-2} \binom{i}{2} g_i \le \binom{n}{2}. \tag{5a}$$

Adding (3) and (5a), we obtain

$$\sum_{0 \le i \le n-2} \left(1 + \binom{i}{2}\right) g_i = \sum_{0 \le i \le n-2} i g_i + \sum_{0 \le i \le n-2} \left(1 + \binom{i}{2} - i\right) g_i \le n(n-1). \tag{6}$$

In the middle part of (6) the first term is, by (4), just $3\,|\mathcal{F}|$, while the second is non-negative. Thus $|\mathcal{F}| \le n(n-1)/3$ follows, giving the upper bound of Theorem 1.6. To have equality, equality must hold in (5a) and also

$$\sum_{0 \le i \le n-2} \left(1 + \binom{i}{2} - i\right) g_i = 0.$$

This latter condition implies $g_0 = g_3 = g_4 = \cdots = g_{n-2} = 0$. Putting this back into the first one, we obtain $g_2 = \binom{n}{2}$, i.e., $\mathcal{F}$ is an $S_2(n, 3, 2)$.

Claim 2. *If $\mathcal{F}$ is union-free, then, for every $A \in \binom{X}{2}$, $\binom{T(A)}{2} \subseteq \mathcal{G}_0$* holds.

Proof. Suppose the contrary and take some $\{z, z'\} \in \binom{T(A)}{2}$ such that $\{z, z'\} \notin \mathcal{G}_0$ holds. Then, for some $i > 0$, $\{z, z'\} \in \mathcal{G}_i$ and consequently, for some $z'' \in X$, $\{z, z', z''\} \in \mathcal{F}$ holds. However, $(A \cup \{z\}) \cup \{z, z', z''\} = (A \cup \{z'\}) \cup \{z, z', z''\}$, a contradiction. □

In view of Claim 1 and Claim 2 the sets $\binom{T(A)}{2}$ are pairwise disjoint in $\mathcal{G}_0$ for $A \in (\mathcal{G}_2 \cup \mathcal{G}_3 \cup \cdots \cup \mathcal{G}_{n-2})$. Thus we have

$$\sum_{2 \le i \le n-2} \binom{i}{2} g_i \le g_0. \tag{5b}$$

Putting back (5b) into (3), we obtain

$$\binom{n}{2}=\sum_{0\le i\le n-2} g_i \ge \sum_{2\le i\le n-2} g_i\binom{i}{2}+\sum_{1\le i\le n-2} g_i$$
$$=\sum_{1\le i\le n-2} ig_i+\sum_{1\le i\le n-2}\left(1+\binom{i}{2}-i\right)g_i. \tag{7}$$

Again, the first term on the right-hand side of (7) is just $3|\mathscr{F}|$ while the second is non-negative. Thus $|\mathscr{F}|\le\frac{1}{3}\binom{n}{2}$ follows. Since $|\mathscr{F}|$ is an integer, we obtain $|\mathscr{F}|\le\lfloor n(n-1)/6\rfloor$, proving the upper bound of Theorem 1.4. Note that in case of equality the second term in the right-hand side of (7) must be zero and thus $g_3=g_4=\cdots=g_{n-2}=0$. Also, equality must occur in (5b), yielding $g_0=g_2$.

3. The constructions

We say that $\mathscr{S}\subset(\binom{X}{3}\cup\binom{X}{4})$ is a quasi-design, $QS_1(n,\{3,4\},2)$, if $|S\cap S'|\le 1$, for every $S,S'\in\mathscr{S}$, and there exists at most one set $A\in\binom{X}{2}$ which is not contained in any member of $\mathscr{S}$.

Proposition 3.1. *Suppose $\mathscr{F}_1\subset\binom{X}{3}$, $\mathscr{F}_2\subset\binom{X}{4}$ and $\mathscr{F}_1\cup\mathscr{F}_2$ is a $QS_1(n,\{4,3\},2)$. For $F\in\mathscr{F}_2$, let $A(F)$ and $B(F)$ be two distinct 3-subsets of F. Then $\mathscr{F}=\mathscr{F}_1\cup\{A(F):F\in\mathscr{F}_2\}\cup\{B(F):F\in\mathscr{F}_2\}$ is union-free and $|\mathscr{F}|=\lfloor n(n-1)/6\rfloor$ holds.*

Proof. As $\mathscr{F}_1\cup\mathscr{F}_2$ is a quasi-design $QS_1(n,\{4,3\},2)$, we have $\binom{n}{2}-1\le 3|\mathscr{F}_1|+6|\mathscr{F}_2|\le\binom{n}{2}$. Hence, $|\mathscr{F}|=|\mathscr{F}_1|+2|\mathscr{F}_2|=\lfloor n(n-1)/6\rfloor$ holds, proving the second part of the proposition.

Suppose $F,G,F',G'\in\mathscr{F}$ and $F\cup G=F'\cup G'$ holds, but $\{F,G\}\ne\{F',G'\}$. By symmetry we may assume $F'\notin\{F,G\}$ holds. As $F'\subset F\cup G$, $|F\cap F'|$ or $|G\cap F'|$ is at least 2. By symmetry assume $|F\cap F'|\ge 2$. But $\mathscr{F}_1\cup\mathscr{F}_2$ is a $QS_1(n,\{3,4\},2)$, thus the only possibility is $F,F'\subset H$, for some $H\in\mathscr{F}_2$. Then $G'\notin H$, consequently $|G'\cap F|\le 1$. We deduce $|G\cap G'|=2$ and consequently, for some $K\in\mathscr{F}_2$, G, $G'\subset K$ holds. $F\cup G=F'\cup G'$ implies $(F-F')\subset G'$, $(F'-F)\subset G$. Thus $H\cap K$ contains $F-F'$ and $F'-F$. Since $\mathscr{F}_1\cup\mathscr{F}_2$ is a $QS_1(n,\{3,4\},2)$, $H=K$ must hold, yielding $|\{F,F',G,G'\}|\le 2$, a contradiction. □

Corollary 3.2. *If a $QS_1(n,\{3,4\},2)$ exists, then $f_3(n)\ge\lfloor n(n-1)/6\rfloor$ holds.*

Next, we want to show that a $QS_1(n,\{3,4\},2)$ exists for almost all values of n. For this we shall use an important theorem of Ray-Chaudhuri and Wilson [11].

Definition 3.3. Suppose $\mathscr{S}$ is an $S_1(6t+3,3,2)$, $t\ge 1$, and $\mathscr{S}=\mathscr{S}_1\cup\cdots\cup\mathscr{S}_{3t+1}$ with each $\mathscr{S}_i$ being a partition of X, i.e., $|\mathscr{S}_i|=2t+1$ and $\bigcup_{\mathscr{S}\in S_i}S=X$ hold for $1\le i\le 3t+1$. Then $\mathscr{S}$ is called a *Kirkman design* and the $\mathscr{S}_i$ its parallel classes.

Theorem 3.4 ([11]). *Kirkman designs exist for every $n=6t+3$, $t\geq 1$.*

Proposition 3.5. *A $QS_1(n, \{3,4\}, 2)$ exists for every n except $n=5, 6, 8$ and eventually $n=20, 32$.*

Proof. If $n=1,2$, take $\mathcal{F}=\emptyset$. If $f=3,4$, take $\mathcal{F}=\{X\}$. If $n=7$, take the unique $S_1(7,3,2)$, the lines of the projective plane of order 2. In the remaining cases, $n\geq 9$. Suppose $n\neq 14$, $n=6t+3+i$ with $0\leq i\leq 5$, $t\geq 1$.

Let $X=\{1,2,\ldots,n\}$ and let $\mathcal{S}$ be a Kirkman design on $\{i+1,\ldots,n\}$ with parallel classes $\mathcal{S}_1,\ldots,\mathcal{S}_{3t+1}$.

Define $\mathcal{S}'_j=\{S\cup\{j\}: S\in\mathcal{S}_j\}$ if $0\leq j\leq i$. Then $\mathcal{S}'=\mathcal{S}'_1\cup\cdots\cup\mathcal{S}'_i\cup\mathcal{S}_{i+1}\cup\cdots\cup\mathcal{S}_{3t+1}$ is a $QS_1(6t+3+i,3,2)$ if $i=0$, 1 or 2 while for $i=3$ or 4 we can take $\mathcal{S}'\cup\{\{1,2,\ldots,i\}\}$.

If $n=6t+8$, we write n as $6(t-1)+3+11$. Suppose first $t\geq 5$. Let $\mathcal{S}$ be a Kirkman design on $\{12,13,\ldots,n\}$ with parallel classes $\mathcal{S}_1,\ldots,\mathcal{S}_{3(t-1)+1}$. Define $\mathcal{S}'_j=\{S\cup\{j\}: S\in\mathcal{S}_j\}$ for $j=1,\ldots,11$. Let $\mathcal{T}$ be a $QS_1(11,\{3,4\},2)$ on $\{1,\ldots,11\}$. Then $\mathcal{S}'=\mathcal{S}'_1\cup\cdots\cup\mathcal{S}'_{11}\cup\mathcal{S}_{12}\cup\cdots\cup\mathcal{S}_{3(t-1)+1}\cup\mathcal{T}$ is a $QS_1(n,\{3,4\},2)$.

Four cases remain, $n=14, 20, 26, 32$. If $m=12r+4$, then, by a theorem of Hanani [9], there exists $\mathcal{S}$, an $S_1(m,4,2)$ on $\{1,2,\ldots,m\}$. Let S_0 be the unique set in $\mathcal{S}$ containing $\{m-1,m\}$. Then $\mathcal{S}'=\{S\cap\{1,2,\ldots,m-2\}: S\in\mathcal{S},\ S\neq S_0\}$ is a $QS_1(m-2,\{3,4\},2)$. Setting $r=1$ or 2 we obtain a $QS_1(n,\{3,4\},2)$ for $n=14$ or 26.

For the cases $n=20, 32$ we could not decide whether a $QS_1(n,\{3,4\},2)$ exists or not. □

Now Proposition 3.5 implies, in view of Corollary 3.2, $f_3(n)\geq\lfloor n(n-1)/6\rfloor$, unless $n=5$, 6, 8, 20 or 32.

For these cases we give a direct construction.

(i) $n=5$. Take $\mathcal{F}=\{\{1,2,i\}: i=3,4,5\}$.

(ii) $n=6$. Take $\mathcal{F}=\{\{1,i,i+1\}: i=2,3,4,5\}\cup\{\{1,2,6\}\}$.

(iii) $n=8$. Let $\mathcal{F}$ be the family given by the rows of the following incidency matrix,

$$\begin{bmatrix}1&1&1&0&0&0&0&0\\1&1&0&1&0&0&0&0\\1&1&0&0&1&0&0&0\\0&0&1&0&0&0&1&1\\0&0&0&1&0&1&0&1\\0&0&0&0&1&1&1&0\\1&0&1&0&0&1&0&0\\1&0&0&1&0&0&1&0\\1&0&0&0&1&0&0&1\end{bmatrix}$$

(iv) $n = 20$ or 32. Let $\mathcal{S}$ be a Kirkman design on $\{6, \ldots, n\}$ with parallel classes $\mathcal{S}_1, \ldots, \mathcal{S}_{(n-6)/2}$. Define again $\mathcal{S}'_i = \{S \cup \{i\} : S \in \mathcal{S}_i\}$ and let $\mathcal{F}_i$ denote the triple-system which we obtain from $\mathcal{S}'_i$ by replacing each of its members by two of its 3-subsets. Take $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_5 \cup \mathcal{S}_6 \cup \cdots \cup \mathcal{S}_{(n-6)/2} \cup \{\{1, 2, j\} : j = 3, 4, 5\}$.

Weakly union-free systems.

Let p be an odd prime power, $p > 7$, $p \equiv 1 \pmod 3$. Let further $X = GF(p)$, and $1, g, g^2$ be the solutions of $x^3 = 1$. Let us define

$$\mathcal{F} = \{\{a, b, c\} \in \tbinom{X}{3} : a + bg + cg^2 = 0\}.$$

Proposition 3.6. *$\mathcal{F}$ is an $\mathcal{S}_2(p, 3, 2)$ and $\mathcal{F}$ is weakly union-free.*

Proof. Suppose $\{x, y\} \in \binom{X}{2}$. Then $\{x, y, z\} \in \mathcal{F}$ if and only if $z = -gx - g^2y$ or $z = -g^2x - gy$, and $-gx - g^2y = -g^2x - gy$ would imply $(x-y)(g^2 - g) = 0$, i.e., $x = y$. Thus $\mathcal{F}$ is an $S_2(p, 3, 2)$, in particular, $|\mathcal{F}| = p(p-1)/3$.

Now we suppose indirectly that F_1, F_2, F_3, F_4 are four different sets in $\mathcal{F}$ and $F_1 \cup F_2 = F_3 \cup F_4$ holds. We want to derive a contradiction. As $F_3 \subset (F_1 \cup F_2)$, we may assume $|F_1 \cap F_3| = 2$. Let $\{x, y\}$ be this intersection. Again, by symmetry, we may assume

$$F_1 = \{x, y, -xg - yg^2\}, \qquad F_3 = \{x, y, -xg^2 - yg\},$$

and, consequently, $(-xg - yg^2) \in F_4$, $(-xg^2 - yg) \in F_2$. Suppose $F_2 = \{v, w, -xg^2 - yg\}$. We distinguish 3 cases:

(i) $F_4 = \{v, w, -xg - yg^2\}$. Eventually exchanging v, w, we may assume

$$-vg - wg^2 = -xg - yg^2, \qquad -vg^2 - wg = -xg^2 - yg,$$

and thus $v = x$, $w = y$, i.e., $F_1 = F_4$, $F_2 = F_3$, a contradiction.

(ii) $|F_1 \cup F_2| = 4$. By symmetry we may assume

$$F_2 = \{x, -xg - yg^2, -xg^2 - yg\}, \qquad F_4 = \{y, -xg - yg^2, -xg^2 - yg\},$$

and

$$x + g(-xg - yg^2) + g^2(-xg^2 - yg) = 0.$$

Consequently, using $g^3 = 1 = -g - g^2$, we have $2(x - y) = 0$, i.e., $x = y$, a contradiction.

(iii) Neither (i) nor (ii) holds. Then $|F_1 \cup F_2| = 5$. By symmetry we may assume $v = x$, $w \neq y$. Since (i) does not hold we must have $F_4 = \{y, w, -xg - yg^2\}$. Using $F_4 \in \mathcal{F}$, $F_4 \neq F_1$, we obtain $-yg - wg^2 = -xg - yg^2$. Using $F_2 \in \mathcal{F}$, $F_3 \neq F_2$, we obtain $-xg - wg^2 = -xg^2 - yg$. Taking the difference of the two equations we infer $(x - y)(2g + g^2) = 0$, i.e., $x = y$ $(2g + g^2 = g - 1 \neq 0)$, the final contradiction. □

Proposition 3.7. *Suppose $n \equiv 1 \pmod 6$ and $n > n_0$. Then there exists a weakly union-free $\mathcal{F} \subset \binom{X}{3}$ with $|\mathcal{F}| = n(n-1)/3$.*

Proof. By Wilson's existence theorem [13] there exists an $S_1(n, \{13, 19\}, 2)$, $\mathcal{S}$ on X (this means that $\mathcal{S} \subset \binom{X}{13} \cup \binom{X}{19}$ and for every $T \in \binom{X}{2}$ there exists exactly one set $S \in \mathcal{S}$ such that $T \subset S$ holds). By Proposition 3.6 on 13 (on 19) points there exists a weakly union-free family of size $(13 \cdot 12)/2$ $((19 \cdot 18)/2)$, respectively. Replace every block of $\mathcal{S}$ by some such family. The new family is easily seen to be weakly union-free and has size $n(n-1)/3$. □

Now, to prove the lower bound of Corollary 1.7 for any $n > n_0$, let n' be the greatest integer satisfying $n-5 \leq n' \leq n$ and $n' \equiv 1 \pmod 6$. Take a weakly union-free family of size $n'(n'-1)/3$ on $\{1, \ldots, n'\}$; such a family exists in view of Proposition 3.7 and

$$n'(n'-1)/3 \geq (n-5)(n-6)/3 > (n^2-n)/3 - \tfrac{10}{3}n.$$

Remark 3.8. It would be very interesting to know for which values of n a weakly union-free $S_2(n, 3, 2)$ exists. We believe that, for $n > n_0$, the condition $3 \mid n(n-1)$ is sufficient—as for the existence of $S_2(n, 3, 2)$ (see [10]).

4. The case $k \geq 4$ and the non-uniform case

We shall return to these problems in a later paper. Here we only list the existing results.

The next proposition shows that $f_k(n)$ and $F_k(n)$ are of the same order of magnitude.

Proposition 4.1. $f_k(n) \leq F_k(n) \leq (k^k/k!) f_k(n)$.

Theorem 4.2. *We have*

$$(\tfrac{1}{24} - o(1))n^3 < f_4(n) < \tfrac{1}{24}n^3.$$

In general we have:

Theorem 4.3.

$$c_k n^{\lceil 4k/3 \rceil/2} \leq f_k(n) \leq c_{k'} n^{\lceil 4k/3 \rceil/2}$$

where $\lceil\ \rceil$ *denotes upper integer part.*

Proposition 4.4. *For* $n > 1000$ *we have*

$$1.19^n < \tfrac{1}{2}(27/19)^{n/2} < f(n) < 2\sqrt{2}^n.$$

Proposition 4.5. *For* $n > 30$ *we have*

$$1.25^n < 2^{(n-1)/3} < F(n) < 2 \cdot 8^{n/4}.$$

Conjecture 4.6. There exists a positive ε such that, for $n > n_0$,

$$F(n)/f(n) > (1+\varepsilon)^n$$

holds.

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