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Note

On Hypergraphs without Two Edges Intersecting in a Given Number of Vertices

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Let X be a finite set of n-elements and suppose $t \geqslant 0$ is an integer. In 1975, P. Erdös asked for the determination of the maximum number of sets in a family $\mathscr{F} = \{F_1, ..., F_m\}$, $F_i \subset X$, such that $|F_i \cap F_j| \neq t$ for $1 \leqslant i \neq j \leqslant m$. This problem is solved for $n \geqslant n_0(t)$. Let us mention that the case $t = 0$ is trivial, the answer being 2^{n-1}. For $t = 1$ the problem was solved in [3]. For the proof a result of independent interest (Theorem 1.5) is used, which exhibits connections between linear algebra and extremal set theory.

1. INTRODUCTION

For an n-element set X we denote by 2^X the set of all the subsets of X. Thus a family $\mathscr{F}$ of subsets of X is just a subset of 2^X. For every integer t, $n \geqslant t \geqslant 0$, let us define

$$\mathscr{F}(n, t) = \begin{cases} n + t \text{ odd}, \{A \subseteq X : |A| \geqslant (n + t + 1)/2\} \\ n + t \text{ even}, \{A \subset X : |A \cap (X - x_0)| \geqslant (n + t)/2\}, x_0 \in X \text{ is fixed.} \end{cases}$$

It is easy to check that for $F, F' \in \mathscr{F}(n, t)$, $|F \cap F'| > t$ holds.

Following a conjecture of Erdös, Ko, Rado [2], Katona proved

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THEOREM 1.1. (Katona [5]). *Suppose* $\mathscr{F} \subset 2^X$, *an for every* $F, F' \in \mathscr{F}$ $|F \cap F'| > t$ *holds, then*

$$|\mathscr{F}| \leqslant |\mathscr{F}(n,t)|$$

Moreover, if $t \geqslant 1$, $|\mathscr{F}| = |\mathscr{F}(n,t)|$, *then* $\mathscr{F} = \mathscr{F}(n,t)$.

The main tool in Katona's proof was the next theorem which is interesting in its own right. To state it we need a definition. Suppose $g \geqslant 0$ is an integer, $\mathscr{A} \subset 2^X$. Define

$$\mathscr{A}^g = \{B : |B| = g, \exists A \in \mathscr{A}, B \subset A\}$$

THEOREM 1.2 (Katona [5]). *If* $0 \leqslant g < h$ *and* $g + t + 1 \geqslant h$ (g, h, t *are integers*), *and* $\mathscr{A}$ *is a family of h-subsets of X such that any two members of* $\mathscr{A}$ *intersects in at least* $t + 1$ *points. Then*

$$|\mathscr{A}^g| \geqslant |\mathscr{A}| \left[\binom{2h-t-1}{g} \Big/ \binom{2h-t-1}{h} \right].$$

Note that in the above theorem one can have equality by taking all the h-subsets of a $(2h-t-1)$-set.

In 1975, Erdös [1] proposed the following problem: What happens if in Theorem 1.1 we replace the condition $|F \cap F'| > t$ by the apparently weaker $|F \cap F'| \neq t$? Let us define

$$\mathscr{F}^*(n,t) = \mathscr{F}(n,t) \cup \{A \subset X, |A| < t\}.$$

Then obviously for F, $F' \in \mathscr{F}^*(n,t)$ we have $|F \cap F'| \neq t$. In [3] it was conjectured that this construction is best possible (for $n \geqslant n_0(t)$), and it was proved for the case $t = 1$. The main tool for the proof was an appropriate generalization of Theorem 1.2.

In this paper we prove this conjecture.

THEOREM 1.3. *Suppose* $\mathscr{F} \subset 2^X$, $|F \cap F'| \neq t$ *for* F, $F' \in \mathscr{F}$, $n > n_0(t)$. *Then* $|\mathscr{F}| \leqslant |\mathscr{F}^*(n,t)|$, *moreover equality holds only if* $\mathscr{F} = \mathscr{F}^*(n,t)$.

For the proof we need, again, a generalization of Theorem 1.2. It will be put together from two theorems.

Let $0 \leqslant l \leqslant n$ and $A_1, \ldots, A_{\binom{n}{l}}$ be all the different l-subsets of X. For $\mathscr{F} \subset 2^X$ we define the lth containment matrix $M(\mathscr{F}, l)$ in the following way. Let $\mathscr{F} = \{F_1, \ldots, F_m\}$, then M is m by $\binom{n}{l}$ and it has general entry

$$m_{i,j} = 1 \quad \text{if} \quad A_j \subset F_i$$
$$= 0 \quad \text{if} \quad A_j \not\subset F_i.$$

THEOREM 1.4 (Frankl and Singhi [4]). *Suppose $\mathscr{F}$ is a family of h-subsets of X, $n \geqslant h > t \geqslant 0$, and for every F, $F' \in \mathscr{F}$ we have $|F \cap F'| \neq t$. If $h - t$ has a prime power divisor which is greater than t, then the rows of $M(\mathscr{F}, h - t - 1)$ are independent over the rationals.*

Note that the conditions of Theorem 1.4 are satisfied if $h - t > \prod_{p^\alpha \leqslant t < p^{\alpha+1}} p^\alpha$. Set $q(t) = 1 + t + \prod_{p^\alpha \leqslant t < p^{\alpha+1}} p^\alpha$.

THEOREM 1.5. *Suppose $\mathscr{F}$ is a family of h-subsets of X such that the rows of $M(\mathscr{F}, h - t - 1)$ are independent over the rationals, and let g be an integer $0 \leqslant g < h$, $g + t + 1 \geqslant h \geqslant t + 1$. Then*

$$|\mathscr{F}^g| \geqslant |\mathscr{F}| \left[\binom{2h-t-1}{g} \Big/ \binom{2h-t-1}{h} \right].$$

Theorems 1.4 and 1.5 have the following:

COROLLARY 1.6. *If $h \geqslant q(t)$ then in Theorem 1.2 one can replace the condition $|A \cap A'| > t$ by $|A \cap A'| \neq t$, and still have the same conclusion.*

Let us remark that in [4] it is conjectured that the conclusion of Theorem 1.4 holds whenever $h \geqslant 2t + 1$. This would imply

CONJECTURE 1.7. *The statement of Corollary 1.6 holds whenever $h \geqslant 2t + 1$.*

2. THE PROOF OF THEOREM 1.5

First we consider the case $g = h - t - 1$. If $G \subset X$, $|G| = g$, $G \notin \mathscr{F}^g$ then in $M(\mathscr{F}, g)$ the column corresponding to G consists of zeros only. Thus we can omit all such columns without diminishing the row-rank of the matrix. Thus we obtain an $|\mathscr{F}|$ by $|\mathscr{F}^g|$ matrix of full row rank, yielding $|\mathscr{F}| \leqslant |\mathscr{F}^g|$, as desired.

Now we prove the theorem by induction on h. By the preceding case we may assume $g + t + 1 \geqslant h + 1$, and consequently $g \geqslant 1$.

For an $x \in X$ let $M(x)$ denote the submatrix of $M(\mathscr{F}, h - t - 1)$ spanned by all the $F \in \mathscr{F}$ satisfying $x \in X$ and all the $G \subset X$ satisfying $|G| = h - t - 1$, $x \notin G$. Also, set $\mathscr{F}(x) = \{F - \{x\} : x \in F \in \mathscr{F}\}$ Now $M(x)$ is just $M(\mathscr{F}(x), (h - 1) - t)$.

PROPOSITION 2.1. *$M(\mathscr{F}(x), (h - 1) - (t - 1) - 1)$ has full row-rank.*

Proof. Suppose the contrary and let $\alpha(B)$ be rational numbers for

$B \in \mathscr{F}(x)$ such that the linear combination, with coefficients $\alpha(B)$ of the rows of $M(\mathscr{F}(x), h-1-t)$ is zero. It means that

$$\forall\, G \subset (X - \{x\}), \qquad |G| = h - t - 1, \sum_{G \subset B \in \mathscr{F}(x)} \alpha(B) = 0. \tag{1}$$

We want to show that the linear combination of the corresponding rows of $M(\mathscr{F}, h-t-1)$, with the same coefficients $\beta(F) = \alpha(B)$ for $F = B \cup \{x\}$, is also zero.

In view of (1), for $G \subset X - \{x\}$, $|G| = h - 1 - t$ we have

$$\sum_{G \subset F \in \mathscr{F}} \beta(F) = \sum_{G \subset B \in \mathscr{F}(x)} \alpha(B) = 0.$$

If $G \subset X$, $|G| = h - t - 1$, $x \in G$, then, again, applying (1):

$$\sum_{G \subset F \in \mathscr{F}} \beta(F) = |F - G|^{-1} \sum_{y \in (X-G)} \sum_{(G \cup \{y\}) \subset F \in \mathscr{F}} \beta(F)$$

$$= |F - G|^{-1} \sum_{y \in (X-G)} \sum_{(G \cup \{y\} - \{x\}) \subset (F - \{x\}) \in \mathscr{F}(x)} \alpha(F - \{x\}) = 0.$$

Since $M(\mathscr{F}, h-t-1)$ is of full row rank, this is a contradiction, proving the proposition.

Now we want to apply the induction hypothesis to $\mathscr{F}(x)$ with $h' = h - 1$, $g' = g - 1$, $t' = t - 1$. We still have $(g-1) + (t-1) + 1 = g + t - 1 \geqslant h - 1$ (since $g + t + 1 \geqslant h + 1$), i.e., $g' + t' + 1 \geqslant h'$. As $h \geqslant t + 1$, $h' \geqslant t' + 1$ and $g \geqslant 1$ implies $0 \leqslant g' \leqslant h'$. Thus we have

$$|\mathscr{F}(x)^{g-1}| \geqslant |\mathscr{F}(x)| \left[\binom{2h-t-2}{g-1} \Big/ \binom{2h-t-2}{h-1} \right]$$

$$= |\mathscr{F}(x)| \frac{g}{h} \left[\binom{2h-t-1}{g} \Big/ \binom{2h-t-1}{h} \right]. \tag{2}$$

Since, obviously

$$g\, |\mathscr{F}^g| = \sum_{x \in X} |\mathscr{F}(x)^{g-1}|; \qquad \sum_{x \in X} |\mathscr{F}(x)| = h\, |\mathscr{F}|$$

using (2) we deduce

$$|\mathscr{F}^g| \geqslant \frac{1}{g} \sum_{x \in X} |\mathscr{F}(x)| \frac{g}{h} \left[\binom{2h-t-1}{g} \Big/ \binom{2h-t-1}{h} \right]$$

$$= |\mathscr{F}| \left[\binom{2h-t-1}{g} \Big/ \binom{2h-t-1}{h} \right]. \quad \blacksquare$$

3. The Proof of Theorem 1.3

Let us define for $0 \leqslant i \leqslant n$

$$\mathscr{F}_i = \{F \in \mathscr{F} : |F| = i\}, \qquad f_i = |\mathscr{F}_i|, \qquad \bar{\mathscr{F}}_i = \{X - F : F \in \mathscr{F}_i\}.$$

Proposition 3.1. *For* $t + 1 \leqslant i \leqslant (n+t)/2$

$$\mathscr{F}_i^{i-t} \cap \bar{\mathscr{F}}_{n+t-i} = \varnothing.$$

Proof. Suppose the contrary, i.e., there exist G, F such that $G \subset F \in \mathscr{F}$, $|F - G| = t$, $(X - G) \in \mathscr{F}$. But $(X - G) \cap F = F - G$ contradicting $|F' \cap F| \neq t$ for $F, F' \in \mathscr{F}$.

Consequently $|\mathscr{F}_i^{i-t}| + |\bar{\mathscr{F}}_{n+t-i}| \leqslant \binom{n}{i-t}$.

In view of Theorems 1.4 and 1.5 this inequality yields

$$\frac{i}{i-t} f_i + f_{n+t-i} \leqslant \binom{n}{i-t}, \qquad q(t) \leqslant i < \frac{n+t}{2} \tag{3}$$

$$f_{(n+t)/2} \leqslant (n-t)/2n \binom{n}{(n+t)/2} = \binom{n-1}{(n+t)/2} \quad \text{if} \quad n+t \text{ is even.} \tag{4}$$

Obviously we have also

$$f_j \leqslant \binom{n}{j}, \qquad 0 \leqslant j < q(t), \qquad n + t - q(t) \leqslant j \leqslant n. \tag{5}$$

If $f_j = 0$ for $t \leqslant j < q(t)$ then summing up the inequalities (3), (5) and for $n + t$ even also (4) we obtain

$$|\mathscr{F}| \leqslant |\mathscr{F}^*(n, t)| - \sum_{q(t) \leqslant i < (n+t)/2} \frac{t}{i-t} f_i, \tag{6}$$

yielding the desired bound, for $t > 0$, $|\mathscr{F}| = |\mathscr{F}^*(n, t)|$ is possible only if $f_i = 0$ for $q(t) \leqslant i < (n+t)/2$ and consequently $\mathscr{F} = \mathscr{F}^*(n, t)$, here in the case $n + t$ even we use the fact that equality holds in (4) iff $\mathscr{F}_{(n+t)/2} = \mathscr{F}^*(n, t)_{(n+t)/2}$ (cf. [2]).

Thus, we may assume now that there exists $F_0 \in \mathscr{F}$, $t \leqslant |F_0| \leqslant q(t)$. Let us set $a = |F|$, $b = [(n+t+2)/2]$. Then there are $\binom{a}{t}\binom{n-a}{b-t}$ b-subsets B of X with $|B \cap F_0| = t$. Of course, none of these sets is in $\mathscr{F}$. Thus

$$f_b \leqslant \binom{n}{b} - \binom{a}{t}\binom{n-a}{b-t}. \tag{7}$$

Setting $f_b = \binom{n}{b} - m$, from (3) we obtain $f_{n+t-b} \leqslant [(n-b)/(n+t-b)]\, m$; thus, in view of (7),

$$f_{n+t-b} + f_b \leqslant \binom{n}{b} - \frac{t}{n+t-b}\, m \leqslant \binom{n}{b} - \frac{t}{n+t-b} \binom{a}{t}\binom{n-a}{b-t}. \tag{8}$$

Summing up the inequalities (3) for $q(t) \leqslant i < [(n+t+2)/2]$, (4), (5) and (8) we obtain

$$|\mathscr{F}| \leqslant |\mathscr{F}^*(n,t)| - \left(\frac{t}{n+t-b}\binom{a}{t}\binom{n-a}{b-t} - \sum_{t \leqslant i < q(t)} \binom{n}{i}\right). \tag{9}$$

In (9) for t fixed the first term in the bracket is growing exponentially in $n (b = [(n+t+2)/2])$ while the second is bounded by $n^{q(t)}$. Thus for $n > n_0(t)$, $|\mathscr{F}| < |\mathscr{F}^*(n,t)|$. ∎

Let us note that more careful calculation shows that if Theorem 1.4 holds for $h \geqslant h_0(t)$, then Theorem 1.5 holds also for $n > 3h_0(t)$. Thus Conjecture 1.7 would imply Theorem 1.5 for $n \geqslant 6t$.

Remark 3.2. The same proof yields that for given t', t, $0 \leqslant t' \leqslant t$ and $n \geqslant n_0(t)$, any $\mathscr{F} \subset 2^X$ satisfying $|F \cap F'| < t'$ or $|F \cap F'| > t$ for every F, $F' \in \mathscr{F}$ has $|\mathscr{F}| \leqslant |\mathscr{F}(n,t)| + \sum_{0 \leqslant i < t'} \binom{n}{i}$. This was conjectured in [3].

4. Appendix

Here—for completeness' sake—we sketch the proof of Theorem 1.4. Let $q = p^s$ the prime power dividing $h-t$ and satisfying $q > t$. Let us suppose that some linear combination of the rows of $M(\mathscr{F}, h-t-1)$ is zero, let c_i denote the coefficient of the row of F_i, the c_i's can be supposed to be integers and such that not all of them are divisible by p. By symmetry assume $p \nmid c_1$.

This linear dependence is equivalent to

$$\sum_{T \subset F_i} c_i = 0 \qquad \text{for every } T \in \binom{X}{h-t-1}. \tag{10}$$

If $S \in \binom{X}{s}$, $s \leqslant h-t-1$, then (10) implies

$$\sum_{S \subset F_i} c_i = \left[1 \Big/ \binom{h-s}{h-t-1-s}\right] \sum_{S \subseteq T \subset F_i, |T| = h-t-1} c_i$$
$$= \left[1 \Big/ \binom{h-s}{h-t-1-s}\right] \sum_{S \subseteq T} \sum_{T \subset F_i} c_i = 0. \tag{11}$$

Summing up (11) for $S \in \binom{F_1}{s}$ we obtain

$$0 = \sum_{S \in \binom{F_1}{s}} \sum_{S \subset F_i} c_i = \sum_{1 \leqslant i \leqslant m} c_i \binom{|F_1 \cap F_i|}{s}. \tag{12}$$

Let the rational numbers a_s, $0 \leqslant i \leqslant h-t-1$, be defined by

$$\sum_{0 \leqslant s \leqslant h-t-1} a_s \binom{x}{s} = \frac{1}{(h-t-1)!} \prod_{t<i<h} (i-x) \overset{\text{def}}{=} p(x).$$

Now $p(x)=0$ if $t<i<h$ and $p(j)=\binom{h-j-1}{t-j}$ for $j=0,..., t-1$. All these numbers are divisible by p. However, $p(h)=(-1)^{h-t-1}$. Summing up (12) for $0 \leqslant s \leqslant h-t-1$ with coefficients a_s we infer $0 \equiv (-1)^{h-t-1} c_1 \pmod{p}$, a contradiction. ∎

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