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AN INTERSECTION PROBLEM WITH 6 EXTREMES

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I. Introduction

Let X be a finite set of n elements. A family $\mathcal{F}$ of the subsets of X is *intersecting* if any two members of $\mathcal{F}$ intersect.

Erdős, Ko and Rado [3] proved that if $\mathcal{F}$ is an intersecting set-system of r-tuples of X and $n \geqq 2r$ then $|\mathcal{F}| \leqq \binom{n-1}{r-1}$. Equality holds in the case $n > 2r$ if and only if the members of $\mathcal{F}$ have a common element.

Let c be a real number, $0 < c \leqq 1$. The *degree* of the point (that is an element) x in the set-system $\mathcal{F}$ is denoted by $d_{\mathcal{F}}(x)$ or simply $d(x) =: |\{F : x \in F \in \mathcal{F}\}|$.

Erdős, Rotschild and Szemerédi [5] raised the following question: How large can be the intersecting set-system $\mathcal{F}$ of r-tuples of X if each point has degree at most $c|\mathcal{F}|$? For the case $c = 2/3$, $n > n_0(r)$ they proved that

$$|\mathcal{F}| \leqq |\mathcal{F}_{3,2}| \tag{1}$$

where $\mathcal{F}_{3,2} = \{F \subset X : |F| = r,\ |F \cap D| \geqq 2\}$, $|D| = 3$.

Frankl [6] proved that (1) holds for any $2/3 \leqq c < 1$ if n is large enough $(n > n_0(r, c))$, and he solved the cases $3/7 < c \leqq 3/5$ as well, proving the conjectures of Erdős—Rotschild—Szemerédi. The aim of this paper is to settle the missing case $3/5 < c < 2/3$.

II. Results

For a finite set-system $\mathcal{H}$ the underlying set of which is a subset of X (i.e. $\bigcup \mathcal{H} \subset X$), we write $\mathcal{F}(\overline{\mathcal{H}}) = \{F \subset X : |F| = r$ and there exists an $H \in \mathcal{H}$ such that $H \subset F\}$, $\mathcal{F}(\mathcal{H}) = \{F \subset X : |F| = r$ and $(F \cap (\bigcup \mathcal{H})) \in \mathcal{H}\}$. Evidently, $\mathcal{F}(\mathcal{H})$ and $\mathcal{F}(\overline{\mathcal{H}})$ are intersecting set-systems if $\mathcal{H}$ is intersecting, and $\mathcal{F}(\mathcal{H}) \subseteqq \mathcal{F}(\overline{\mathcal{H}})$.

P. Frankl [6] proved that if $1/2 < c \leqq 3/5$ and $n > n_0(r, c)$ then

$$|\mathcal{F}| \leqq 10\binom{n-6}{r-3} + 15\binom{n-6}{r-4} + 6\binom{n-6}{r-5} + \binom{n-6}{r-6} =$$

$$= 10\binom{n-5}{r-3} + 5\binom{n-5}{r-4} + \binom{n-5}{r-5}.$$

If equality holds here then there exists a 3-uniform, 5-regular, intersecting set-system $\mathcal{H}_1$ on a 6-element set such that $\mathcal{F} = \mathcal{F}(\overline{\mathcal{H}_1})$. There exists exactly one such $\mathcal{H}_1$ (see Figure 1).

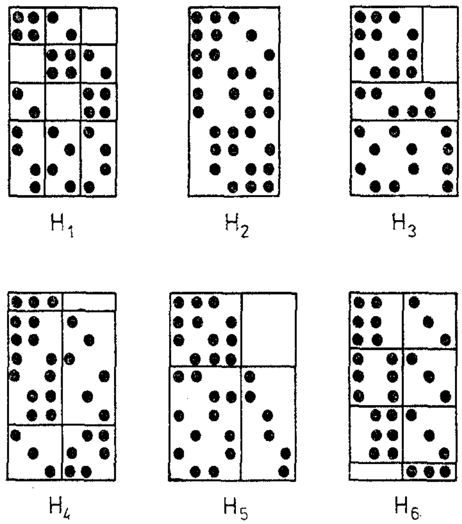

Fig. 1

We describe 6 hypergraphs. (The elements of the underlying set are denoted by positive integers, see also Figure 1.)

$\mathcal{H}_1 = \{123, 124, 345, 346, 156, 256, 135, 146, 236, 245\}$,

$\mathcal{H}_2 = \begin{bmatrix}5\\3\end{bmatrix} = \{123, 124, 125, 134, 135, 145, 234, 235, 245, 345\}$,

$\mathcal{H}_3 = \{123, 124, 134, 234, 125, 345, 136, 246, 146, 236\}$,

$\mathcal{H}_4 = \{123, 124, 125, 134, 136, 235, 236, 156, 246, 345\}$,

$\mathcal{H}_5 = \{123, 124, 134, 234, 125, 345, 136, 246, 147, 237\}$,

$\mathcal{H}_6 = \{124, 125, 126, 134, 135, 136, 234, 235, 236, 456\}$.

THEOREM 1. *Let $\mathcal{F}$ be an intersecting family consisting of r-element subsets of X, $|X| = n$. Suppose that for some $3/5 < c < 2/3$, for $n > n_0(r, c)$ and for every $x \in X$, $d_{\mathcal{F}}(x) \leqq c|\mathcal{F}|$ holds. Then*

$$|\mathcal{F}| \leqq 10\binom{n-5}{r-3} + 5\binom{n-5}{r-4} + \binom{n-5}{r-5}. \tag{2}$$

Furthermore equality holds in (2) *iff $\mathcal{F} \equiv \mathcal{F}(\overline{\mathcal{H}_i})$ for some $1 \leqq i \leqq 6$.*

Furthermore, if $c = 1/2$ and $n > n_0(r)$ then

$$|\mathcal{F}| \leqq 10\binom{n-6}{r-3}, \tag{3}$$

and equality holds here if and only if $\mathcal{F} = \mathcal{F}(\mathcal{H}_1)$.

So the cardinality of a maximum $\mathscr{F}$ is constant on the whole interval (1/2, 2/3). Our theorem differs from the theorem of Frankl because in case $3/5<c<2/3$ five more extremal systems are allowed. So we have non-isomorphic optimal families. This phenomenon is not rare in combinatorics even in the Erdős—Ko—Rado type theorems, cf. the theorem of Hilton—Milner for $r=3$ (see [6] or [8]).

The following is a consequence of Theorem 1.

THEOREM 2. *Let $\mathscr{F}$ be a family of intersecting r-subsets of X, $|X|=n$. Suppose that $|\mathscr{F}|>(10+\varepsilon)\binom{n-3}{r-2}$ where $\varepsilon>0$ is a positive constant. Then for $n>n_0(r,\varepsilon)$ there exists an $x\in X$ such that $d(x)>(2/3-\varepsilon)|\mathscr{F}|$.*

(This is also an improvement of a theorem of Frankl. He proved the lower bound $3/5+\min(0.01, 0.01\varepsilon)$ instead of $2/3-\varepsilon$.)

III. Definitions and lemmas

Define an *edge-contraction* as the following operation on a set-system $\mathscr{H}$: we substitute an edge $E\in\mathscr{H}$ by a smaller, nonempty $E'\subsetneqq E$, and thus we get the set-system $\mathscr{H}-\{E\}\cup\{E'\}$. An intersecting set-system is *v-critical* if it has no multiple edges and the hypergraph obtained by contracting any of its edges is non-intersecting. That is

(4) For all $E\in\mathscr{H}, x\in E$ there exists an $F\in\mathscr{H}$ such that $E\cap F=\{x\}$.

Every v-critical intersecting set-system is a Sperner-family, that is

(5) If $E\in\mathscr{H}$ and $F\subsetneqq E$ then $F\notin\mathscr{H}$.

Erdős and Lovász proved the following theorem [4]:

(6) If $\mathscr{H}$ is a v-critical intersecting set-system and $\max\{|E|: E\in\mathscr{H}\}=k$, then $|\mathscr{H}|\leqq k^k$.

We can get a v-critical intersecting set-system from any intersecting set-system $\mathscr{F}$ by contracting its edges as far as possible and deleting all but one copy of the appearing multiple edges. This $\mathscr{H}$ is called the *nucleus* of the set-system $\mathscr{F}$. Split $\mathscr{H}$ according to the cardinality of its members: $\mathscr{H}=\mathscr{H}^1\cup\mathscr{H}^2\cup\ldots\cup\mathscr{H}^r$ where $E\in\mathscr{H}^i$ implies $|E|=i$.

Denote by $\mathscr{B}$ the nucleus of $\mathscr{H}^1\cup\mathscr{H}^2\cup\mathscr{H}^3$. In what follows $\mathscr{B}$ is called the *nucleus of rank* 3 *of* $\mathscr{F}$. Of course, $\mathscr{B}$ is not unique, but this is not important.

(7) If $\mathscr{F}$ is an r-uniform, v-critical intersecting set-system with underlying set X, $|X|=n$, then there exists a set-system $\mathscr{B}$ such that

(a) $\mathscr{B}$ is v-critical, intersecting and for all $B\in\mathscr{B}$, $|B|\leqq 3$ (possibly $\mathscr{B}=\emptyset$);

(b) $|\mathscr{F}-\mathscr{F}(\overline{\mathscr{B}})|\leqq r^r\binom{n-4}{r-4}$.

Indeed, applying (6) we get

$$|\mathscr{F}-\mathscr{F}(\overline{\mathscr{B}})|\leqq\sum_{i=4}^{r}|\mathscr{F}\cap\mathscr{F}(\overline{\mathscr{H}^i})|\leqq\sum_{i=4}^{r}|\mathscr{F}(\overline{\mathscr{H}^i})|\leqq|\mathscr{H}|\binom{n-4}{r-4}\leqq r^r\binom{n-4}{r-4}.$$

Q.E.D.

IV. The first part of the proof of Theorem 1. The main lemma

We shall consider the whole interval [1/2, 2/3), thus we will prove the above mentioned theorem of Frankl at the same time. So let $1/2 \leqq c < 2/3$ be fixed and let $\mathcal{F}$ be an r-uniform intersecting set-system on X with $\max\{d_{\mathcal{F}}(x): x \in X\} \leqq c|\mathcal{F}|$. We are looking for $\mathcal{F}$ with maximal cardinality, hence we may suppose $|\mathcal{F}| \geqq |\mathcal{F}(\mathcal{H}_1)| = 10\binom{n-6}{r-3}$.

Write $\mathcal{B}$ for the nucleus of rank 3 of $\mathcal{F}$. For each $F \in \mathcal{F} \cap \mathcal{F}(\overline{\mathcal{B}})$ let us choose a $B \in \mathcal{B}$ with $B \subset F$. Let $\mathcal{F}_B$ denote the set of those members of $\mathcal{F}$ for which B is chosen. Thus $|\mathcal{F}| = \sum_{B \in \mathcal{B}} |\mathcal{F}_B| + |\mathcal{F} - \mathcal{F}(\overline{\mathcal{B}})|$. Define a weight $w(B)$ of B by $w(B) = |\mathcal{F}_B| \Big/ \binom{n-3}{r-3}$. Since by (7b) $|\mathcal{F} - \mathcal{F}(\overline{\mathcal{B}})| \leqq r^r \binom{n-4}{r-4}$ we get

$$\sum_{B \in \mathcal{B}} w(B) \geqq \frac{10\binom{n-6}{r-3}}{\binom{n-3}{r-3}} - \frac{r^r\binom{n-4}{r-4}}{\binom{n-3}{r-3}} > 9.9 \tag{8}$$

provided n is large enough ($n > 10r^{r+1}$). Moreover for any $x \in X$ we have

$$\sum_{B \ni x} w(B) < \frac{2}{3} \sum w(B). \tag{9}$$

Indeed

$$\frac{\sum_{B \ni x} w(B)}{\sum w(B)} = \frac{\sum_{B \ni x} w(B)\binom{n-3}{r-3}}{\sum w(B)\binom{n-3}{r-3}} = \frac{\sum_{B \ni x} |\mathcal{F}_B|}{\sum |\mathcal{F}_B|} \leqq \frac{d_{\mathcal{F}}(x)}{|\mathcal{F} \cap \mathcal{F}(\overline{\mathcal{B}})|} \leqq \frac{c|\mathcal{F}|}{|\mathcal{F}| - O\left(\binom{n-4}{r-4}\right)} < \frac{2}{3}$$

provided $n > n_0(r, c)$ $\left(n > \frac{1}{(2/3)-c} 10r^{r+1}\right)$.

The following lemma is the crucial point of the proof.

MAIN LEMMA. *Suppose that $\mathcal{B}$ is a ν-critical, intersecting set-system of rank* 3. *Suppose further that there exists a non-negative weight function $w: \mathcal{B} \to \mathbf{R}$ such that* (8) *and* (9) *hold and $w(B) \leqq 1$ if $|B| = 3$. Then $\mathcal{B} \equiv \mathcal{H}_i$ for some* $1 \leqq i \leqq 6$ (see Figure 1).

By (6) $|\mathcal{B}| \leqq 27$ thus the proof of this lemma is reduced to the investigation of finitely many "small" set-systems. After this lemma the proof of Theorem 1 is not hard. But we cannot hope for a simple proof of the lemma because its conclusion is somewhat complicated, and any proof must yield a description of the structures of the $\mathcal{H}_i$'s.

The following two parts of this paper (Chapters V and VI) contain only the proof of the Main Lemma. If the reader believies that the author has examined all (finitely many) cases of the ν-critical intersecting set-systems of rank 3, then he or she can continue reading Chapter VII.

V. The first part of the proof of the lemma. The nucleus of rank 3 of a maximal $\mathcal{F}$ is 3-uniform

The formula (9) yields that $\mathcal{B}$ has no member with 1 element, since then $|\mathcal{B}|=1$. We will show that

(10) *$\mathcal{B}$ is 3-uniform.*

We will prove this by way of contradiction. Denote by $\mathcal{B}^2$ the members of $\mathcal{B}$ with 2-elements. In what follows the points of the underlying set of $\mathcal{B}$ will be denoted by the positive integers.

If $|\mathcal{B}^2|\geqq 4$ then its edges have a common point since $\mathcal{B}^2$ is intersecting. E.g. $B_1=\{1,2\}$, $B_2=\{1,3\}$, $B_3=\{1,4\}$, $B_4=\{1,5\}$. (See Figure 2.) By (4) there exists an edge B_5 not containing the point 1. But B_5 intersects B_1, B_2, B_3 and B_4 thus $\{2,3,4,5\}\subset B_5$. This is a contradiction.

If $|\mathcal{B}^2|=3$ *and* the edges of $\mathcal{B}^2$ have a common point then let this be e.g. the point 1 and $B_1=\{1,2\}$, $B_2=\{1,3\}$, $B_3=\{1,4\}$. (See Figure 3.) Since there exists an edge B_4 not containing the point 1 we get that $B_4=\{2,3,4\}$. There are no other edges of $\mathcal{B}$ which do not contain the point 1. Moreover there is no other edge of $\mathcal{B}$ which contains 1 because it would contain some B_i $(1\leqq i\leqq 3)$ contradicting (5). Thus in this case $\mathcal{B}=\{B_1, B_2, B_3, B_4\}$. Considering (9) at point 1 we have $(w_1+w_2+w_3)<\frac{2}{3}(w_1+w_2+w_3+w_4)$. This and the inequality $w_4\leqq 1$ give that $\sum w_i<3$. This contradicts (8) $(w_i=w(B_i))$.

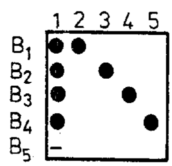

Fig. 2

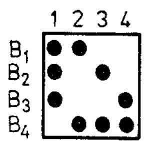

Fig. 3

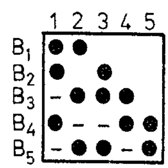

Fig. 4

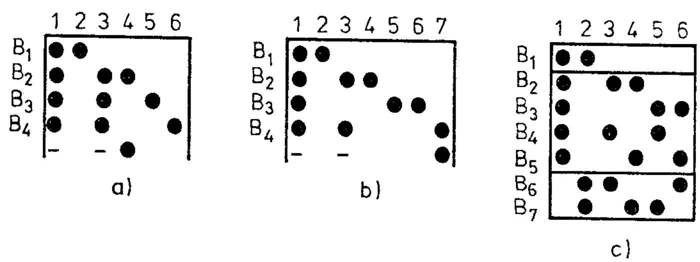

Fig. 5

If $|\mathcal{B}^2|=3$ *and* the edges of $\mathcal{B}^2$ have no a common point then they form a triangle, i.e. $B_1=\{1,2\}$, $B_2=\{1,3\}$, $B_3=\{2,3\}$. The set-system $\mathcal{B}=\{B_1, B_2, B_3\}$, similarly to the above mentioned cases, is a maximal ν-critical intersecting system. (I.e. if $\mathcal{B}'$ is a ν-critical intersecting set-system and $\mathcal{B}\subset\mathcal{B}'$ then $\mathcal{B}=\mathcal{B}'$.) However for the triangle (9) does not hold.

If $|\mathscr{B}^2|=2$ then let $B_1=\{1, 2\}$, $B_2=\{1, 3\}$ (see Figure 4). By (4) there exists an edge B_3 not containing the point 1, $B_3=\{2, 3, 4\}$. There exists an edge B_4 meeting B_3 only in the point 4, i.e. $2, 3\notin B_4$, $4\in B_4$, thus $B_4=\{1, 4, 5\}$. There exists an edge B_5 which meets B_4 only in the point 5, i.e. $1, 4\notin B_5$, $5\in B_5$, thus $B_5=\{2, 3, 5\}$. The set-system $\mathscr{B}$ has no further edges containing the point 1, and it has no further edges not containing the point 1. Hence the set-system obtained above is maximal ν-critical, i.e. $\mathscr{B}=\{B_1, \dots, B_5\}$. Applying (9) at the point 1 we have $(w_1+w_2+w_4)<\frac{2}{3}(w_1+w_2+w_3+w_4+w_5)$. Moreover $w_3, w_4, w_5\leqq 1$ hence $\sum w_i<6$, but that contradicts (8).

Finally if $|\mathscr{B}^2|=1$ then let $B_1=\{1, 2\}$. Applying (9) at the points 1 and 2 and summing we get $2w_1+(\sum_{i>1} w_i)<\frac{4}{3}(\sum w_i)$. From this $w_1<\frac{1}{2}(\sum_{i>1} w_i)$ and $(\sum w_i)<\frac{3}{2}(\sum_{i>1} w_i)$. By (8) $\sum w_i\geqq 9$ hence $(\sum_{i>1} w_i)>6$. Consequently at least 7 members of $\mathscr{B}$ with 3 elements intersect B_1. Thus at least 4 edges $(B_2\dots B_5)$ contain the point 1 (and by (5) they do not contain the point 2). There are no three sets from $B_2\setminus\{1\}, B_3\setminus\{1\}, B_4\{1\}, B_5\setminus\{1\}$ which have a common point because if we suppose on the contrary that (see Figure 5a) $B_2=\{1, 3, 4\}$ $B_3=\{1, 3, 5\}$, $B_4=\{1, 3, 6\}$ then we get a contradiction applying (4) to the edges B_2 at the point 4. Consequently among the sets $B_i-\{1\}$ $(2\leqq i\leqq 5)$ there are two disjoint, e.g. $B_2=\{1, 3, 4\}$, $B_3=\{1, 5, 6\}$ (see Figure 5b and 5c). If the edge B_4 (or B_5) would contain a further point (say 7) then we immediately get a contradiction applying (4) to the edge B_4 (or B_5) at the point 7 (see Figure 5b). Thus $B_4\setminus\{1\}, B_5\setminus\{1\}\subset(3, 4, 5, 6)$ and they are disjoint (see Figure 5c). Consequently there is no further edge containing the point 1. The edges not containing 1 contain the point 2 and intersect $(3, 4, 5, 6)$ in $(3, 6)$ or $(4, 5)$. Thus there are only at most six 3-element edges of $\mathscr{B}$ which contradicts the assumption $(\sum_{i>1} w_i)>6$.

Consequently $\mathscr{B}^2=\emptyset$ and this completes the proof of (10).

VI. Proof of the main lemma (last part). The nucleus of rank 3 of a maximal $\mathscr{F}$ is $\mathscr{H}_i$ $(1\leqq i\leqq 6)$

By (9) there is no point contained in all the edges of $\mathscr{B}$. Further there is no pair $\{x, y\}$ covering all the edges of $\mathscr{B}$ (i.e. $\forall B\in\mathscr{B}$, $\{x, y\}\cap B\neq\emptyset$). Indeed, if we suppose the contrary then joining the edge $\{x, y\}$ with weight 0 to $\mathscr{B}$ we get an intersecting set-system which satisfies the assumptions of the Lemma. But (10) says that this is impossible. Consequently

(11) *For all the points* x, y *there exists an edge* $B\in\mathscr{B}$ *such that* $B\cap\{x, y\}=\emptyset$.

Since $(\sum w_i)>9$ we get $|\mathscr{B}|\geqq 10$. Then we can apply a theorem of Deza [1] which in this case states: If at least 8 3-element sets are given so that any two of them intersect in exactly 1 element, then all the sets have a common point. Consequently $\mathscr{B}$ has two edges (B_1 and B_2) intersecting in 2 elements. E.g. $B_1=\{1, 2, 3\}$, $B_2=\{1, 2, 4\}$ (see Fig. 6).

Firstly, suppose that

(12) there do not exist $B', B'', B''' \in \mathcal{B}$ such that $|B' \cap B'' \cap B'''| = 2$.

We will prove that in this case $\mathcal{B} = \mathcal{H}_1$.

By (11) there exists an edge B_3 such that $1, 2 \notin B_3$, say $B_3 = \{3, 4, 5\}$. We will show that $d_{\mathcal{B}}(1) \leqq 5$ (and similarly $d_{\mathcal{B}}(2) \leqq 5$). Indeed, if $d_{\mathcal{B}}(1) \geqq 6$, then there exists at least 4 edges (B_4, B_5, B_6, B_7) which contain the point 1 and by (12) do not contain the point 2. Also by (12) there are no two of them which contain the point 3 (or the point 4) (see Figure 6a). Thus there are two edges (B_6, B_7) which do not contain the points 3, 4 (and 2) e.g. $B_6 = \{1, 5, 6\}$, $B_7 = \{1, 5, 7\}$, see Fig. 6b. Then there is no edge not containing the points $\{1, 5\}$; however it meets all the edges B_1, B_2, B_3, B_6, B_7. This contradicts (11).

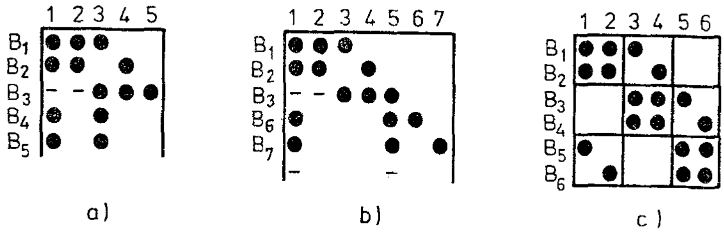

Fig. 6

Thus, by $|\mathcal{B}| \geqq 10$ and $d(1), d(2) \leqq 5$, there are at least (and by (12) at most) two edges which are disjoint from the points $\{1, 2\}$. They contain the points 3 and 4 (see Figure 6c) $B_3 = \{3, 4, 5\}$, $B_4 = \{3, 4, 6\}$. Since $|B_3 \cap B_4| = 2$ we again can say that $d(3) \leqq 5$, $d(4) \leqq 5$ and there exists exactly two further edges (B_5, B_6) which are disjoint from $\{3, 4\}$. Consequently $B_5 = \{5, 6, 1\}$, $B_6 = \{5, 6, 2\}$. Then the minimal number of points covering the system $\{B_1 \dots B_6\}$ is 3, hence the set $\{1, 2, \dots, 6\}$ contains all the edges of $\mathcal{B}$. Thus $d(1) = \dots = d(6) = 5$ and $|\mathcal{B}| = 10$. For each pair of points from $\{1, \dots, 6\}$ there is an edge B_i $(1 \leqq i \leqq 6)$ containing it, thus by (12) all the intersections $B_\alpha \cap B_\beta$ $(7 \leqq \alpha, \beta \leqq 10)$ have only 1 point in common. Moreover B_α $(7 \leqq \alpha \leqq 10)$ intersects each of the sets $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ in one point only. This last two properties (up to isomorphisms) uniquely define the edges $B_7, \dots, B_{10}$. (Since the edges $B_7, \dots, B_{10}$ and the sets $\{1, 2, x\}$, $\{3, 4, x\}$, $\{5, 6, x\}$ form the finite projective plane of order 2.) So we get $\mathcal{H}_1$.

Now we suppose that (12) does not hold, i.e. $|B_1 \cap B_2 \cap B_3| = 2$, $B_1 = \{1, 2, 3\}$, $B_2 = \{1, 2, 4\}$, $B_3 = \{1, 2, 5\}$. (See Figure 7.) By (11) there exists an edge B_4 which does not contain 1 and 2, thus $B_4 = \{3, 4, 5\}$. An arbitrary other edge B of $\mathcal{B}$

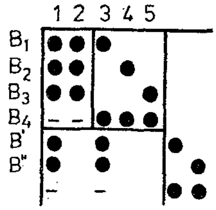

Fig. 7

is called *inner* or *outer* according to whether it is contained in $\{1, 2, 3, 4, 5\}$ or not. I.e. either $|B\cap\{1, 2\}|=1$ and $|B\cap\{3, 4, 5\}|=2$ or $|B\cap\{1, 2\}|=|B\cap\{3, 4, 5\}|= =|B\setminus\{1, \dots, 5\}|=1$, respectively.

First of all we show that there are no two outer edges B', B'' which intersect $\{1, \dots, 5\}$ at the same points. Suppose on the contrary that $B'=\{1, 3, x\}$, $B''= =\{1, 3, y\}$ $(x, y>5)$ then applying (11) to the points 1 and 3 we get a contradiction (see Fig. 7). Hence the number of outer edges (and the number of inner edges, too) is at most 6. These 6 and 6 sets form 6 complementary pairs (e.g. the complement of the outer edge $\{1, 4, x\}$ is the inner edge $\{2, 3, 5\}$). Naturally, $\mathscr{B}$ contains at most one member of each complementary pair, thus $|\mathscr{B}-\{B_1, B_2, B_3, B_4\}|\leqq 6$. It follows that $|\mathscr{B}|\leqq 10$, thus

(13) $$|\mathscr{B}| = 10.$$

This yields $d_{\mathscr{B}}(x)\leqq 6$ for all points x, because $d(x)\geqq 7$ would imply $\sum_{B\not\ni x} w(B)\leqq 3$ thus writing (9) at the point x we get

$$\Big(\sum_{B\ni x} w(B)\Big) < \frac{2}{3}\Big(\sum_{B\ni x} w(B) + \sum_{B\not\ni x} w(B)\Big) \leqq \frac{2}{3}\Big(\sum_{B\ni x} w(B)+3\Big),$$

i.e. $\sum_{B\ni x} w(B)<6$, and this contradicts (8). All the edges $B_5, \dots, B_{10}$ intersect the set $\{1, 2\}$ in (exactly) one element, hence we get $d_{\mathscr{B}}(1)=d_{\mathscr{B}}(2)=6$. Let us denote the number of outer edges containing the points 1 and 2 by α and β, respectively. By (13) one member of each complementary pair (consisting of one outer and inner set) belongs to $\mathscr{B}$. Thus there are exactly $3-\beta$ outer sets containing 1 belonging to $\mathscr{B}$. Hence $6=d_{\mathscr{B}}(1)=3+\alpha+(3-\beta)$, consequently $\alpha=\beta$.

If $\alpha=\beta=0$, i.e. every edge of $\mathscr{B}$ is inner then we get $\mathscr{H}_2$, the set-system of all 3-tuples of the underlying-set $\{1, 2, \dots, 5\}$ (see Fig. 1).

If $\alpha=\beta=1$ then let the unique outer edge containing the point 1 be B_5 (see Fig. 8a), e.g. $B_5=\{1, 3, 6\}$. By (4) there is an edge B_6 such that $1, 3\notin B_6$, $6\in B_6$, i.e. $B_6=\{2, 4, 6\}$. All the other edges are inner and determined uniquely. Hence we get $\mathscr{H}_3$.

If $\alpha=\beta=2$ then there are only two inner edges (B_5 and B_6, see Fig. 8bc). As $\alpha=\beta$ the edges B_5 and B_6 intersect $\{1, 2\}$ in different points. There are two cases. First case: $|B_5\cap B_6|=2$, e.g. $B_5=\{1, 3, 4\}$, $B_6=\{2, 3, 4\}$, see Fig. 8b. Then the traces of the outer edges of $\mathscr{B}$ on the set $\{1, \dots, 5\}$ are $\{1, 3\}$, $(2, 4\}$, $\{2, 3\}$ and $\{1, 4\}$. Here the first two traces are disjoint, and the last two are disjoint, too. Thus they have a common outer point, i.e. $B_7=\{1, 3, x\}$, $B_8=\{2, 4, x\}$ and $B_9= =\{2, 3, y\}$, $B_{10}=\{1, 4, y\}$. If $x=y$ then we get $\mathscr{H}_3$ again and if $x\neq y$ we get $\mathscr{H}_5(x, y>5)$. Second case: $|B_5\cap B_6|=1$, e.g. $B_5=\{1, 3, 4\}$, $B_6=\{2, 3, 5\}$, see Fig. 8c. The traces of the outer edges on $\{1, \dots, 5\}$ are: $\{1, 3\}$, $\{2, 4\}$, $\{2, 3\}$, $\{1, 5\}$. The outer edges corresponding to these traces are $\{1, 3, x\}$, $\{2, 4, x\}$, $\{2, 3, y\}$, $\{1, 5, y\}$. Hence if x coincides with y we get the set-system $\mathscr{H}_4$, and if they are different $(x>y>5)$ we get again $\mathscr{H}_5$.

If $\alpha=\beta=3$, i.e. all the edges B_i $(5\leqq i\leqq 10)$ are outer then we can order the 6 traces in such a way that any trace is disjoint from its successor e.g. $\{1, 3\}$, $\{2, 4\}$, $\{1, 5\}$, $\{2, 3\}$, $\{1, 4\}$, $\{2, 5\}$ (see Figure 8d). This implies that all B_i $(5\leqq i\leqq 10)$ have a common outer point. This gives $\mathscr{H}_6$.

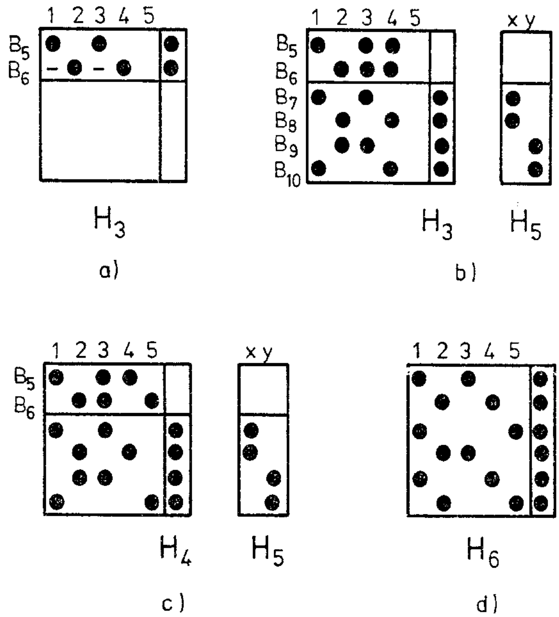

Fig. 8

VII. Proof of Theorem 1 (last part)

The Main Lemma implies that if $|\mathscr{F}|\geqq 10\binom{n-6}{r-3}$ and $n>n_0(r,c)$ then the ν-critical nucleus B of rank 3 of $\mathscr{F}$ is $\mathscr{H}_i$ (for some $1\leqq i\leqq 6$). As $\sum w(B)>9.9$ by (8) we have that $w(B)>0.9$ for each $B\in\mathscr{H}_i$. This means that $|\{F\in\mathscr{F}\colon F\supset B\}|\geqq$ $\geqq|\mathscr{F}_B|>0.9\binom{n-3}{r-3}$. Let us choose $F_0\in\mathscr{F}$ arbitrarily. If $F_0\cap B=\emptyset$ for some $B\in\mathscr{H}_i$ then since $\mathscr{F}$ is intersecting we get

$$0.9\binom{n-3}{r-3}<|\{F\in\mathscr{F}\colon\ F\supset B\}|\leqq\sum_{x\in F_0}\left|\{F\in\mathscr{F}\colon\ F\supset B\cup\{x\}\}\right|\leqq r\binom{n-4}{r-4}.$$

This leads to a contradiction if $n>n_0(r)$. Hence

(14) *If* $B\in\mathscr{H}_i$ and $F\in\mathscr{F}$ *then* $B\cap F\neq\emptyset$.

It is easy to see (it follows from the constructions described in Chapter VI) that if a set F intersects all the edges of $\mathscr{H}_i$ then it contains one of them. (This fact is trivial for $\mathscr{H}_2$. Let $i=1, 3, 4$ or 6, i.e. $\mathscr{H}_i$ is a set-system on 6 points. If F does not contain edges of $\mathscr{H}_i$ then $(\cup\mathscr{H}_i)\setminus F$ meets all the edges of $\mathscr{H}_i$, too. Hence $|F\cap(\cup\mathscr{H}_i)|=|(\cup\mathscr{H}_i)\setminus F|=3$. However one of these two sets is an edge of $\mathscr{H}_i$. This leads to a contradiction. Finally if F meets all the edges of $\mathscr{H}_5$ (see Fig. 1) then $|F\cap\{1,2,3,4\}|\geqq 2$. If $|F\cap\{1,2,3,4\}|>2$ then we are ready, and if

$|F\cap\{1, 2, 3, 4\}|=2$ then $|F\cap\{5, 6, 7\}|\geqq 1$ and it is easy to check that there is a $B\in\mathscr{H}_5$ with $B\subset F$.) Hence

(15) *If* $F\in\mathscr{F}$ *then there exists a* $B\in\mathscr{H}_i$ *such that* $B\subset F$. *That is* $\mathscr{F}\subset\mathscr{F}(\overline{\mathscr{H}_i})$.

Clearly

$$\mathscr{F}(\overline{\mathscr{H}_2})=10\binom{n-5}{r-3}+\binom{5}{4}\binom{n-5}{r-4}+\binom{5}{5}\binom{n-5}{r-5}$$

and

$$\mathscr{F}(\overline{\mathscr{H}_1})=\mathscr{F}(\overline{\mathscr{H}_3})=\mathscr{F}(\overline{\mathscr{H}_4})=\mathscr{F}(\overline{\mathscr{H}_6})=10\binom{n-6}{r-3}+\binom{6}{4}\binom{n-6}{r-4}+\binom{6}{5}\binom{n-6}{r-5}+\binom{6}{6}\binom{n-6}{r-6}$$

and

$$\mathscr{F}(\overline{\mathscr{H}_5})=10\binom{n-7}{r-3}+\left(\binom{7}{4}-10\right)\binom{n-7}{r-4}+\binom{7}{5}\binom{n-7}{r-5}+\binom{7}{6}\binom{n-7}{r-6}+\binom{7}{7}\binom{n-7}{r-7}.$$

An easy computation shows that these three numerical expressions are equal. Thus Theorem 1 is proved.

VIII. Summary, remarks

As a matter of fact we have proved a more general theorem.

THEOREM 3. *Let* $\mathscr{F}$ *be a family of intersecting r-subsets of* X, $|X|=n$. *Suppose that for some* $1/2\leqq c<2/3$ *and for every* $x\in X$, $d_{\mathscr{F}}|x|\leqq c|\mathscr{F}|$, *and that*

$$|\mathscr{F}|\geqq 9\binom{n-3}{r-3}+\frac{20}{(2/3)-c}\,r^{r+1}\binom{n-4}{r-4}.$$

Then $\mathscr{F}\subset\mathscr{F}(\overline{\mathscr{H}_i})$ *for some* $1\leqq i\leqq 6$ *(See Fig. 1).*

If $c=1/2$ then the extremal set-system is $\mathscr{F}(\mathscr{H}_1)$. If c is a little bit greater than $1/2$ then for the extremal set-system $\mathscr{F}$ we get: $\mathscr{F}(\mathscr{H}_1)\subset\mathscr{F}\subset\mathscr{F}(\overline{\mathscr{H}_1})$. However if $c>1/2+r/n$ then the extremal set-system is the whole $\mathscr{F}(\overline{\mathscr{H}_1})$ since $(\max_{x\in X} d_{\mathscr{F}(\overline{\mathscr{H}_1})}(x))/|\mathscr{F}(\overline{\mathscr{H}_1})|<1/2+r/n$ if $n>n_0(r)$. $\mathscr{F}(\overline{\mathscr{H}_1})$ is the unique extremum as long as $c\leqq\frac{3}{5}$, and there are five further extrema only if $c>3/5$.

In fact the proof presented above is a slight improvement of a method due to P. Frankl. The crucial observation is that the nucleus $\mathscr{B}$ of $\mathscr{F}$ is ν-critical in this proof. P. Frankl used a different nucleus which was not ν-critical. Indeed, in Chapters IV—VI we built up the set-systems $\mathscr{H}_1, \dots, \mathscr{H}_6$ using (4). Further results can be found in the paper [7].

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